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# Miscellany

## Twitching Weak Dictators

By

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### 1. Introduction

In Arrow's version of his impossibility theorem, Collective Rationality requires that social choices are determined by a transitive social preference. This attracted much criticism and there is now an extensive literature on weakening this condition. The result of this line of inquiry is that Dictators disappear but Weak Dictators (vetoers) and Oligarchies appear.[1] In achieving these results Arrow's condition of independence of irrelevant alternatives is invariably maintained.

The purpose of this note is to show that the same effect may be obtained by maintaining Collective Rationality and weakening the Independence of Irrelevant Alternatives. In the particular weakening considered here, the social preference over a particular pair of alternatives is allowed to change a little, that is to twitch, while the Independence of Irrelevant Alternatives would insist on no change at all. Weak Dictators replace Dictators as a result of this weakening. It is somewhat intriguing that weakening a condition that was confused with one aspect of Collective Rationality[2] should have much the same effect as weakening Collective Rationality itself.

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[1] The literature is extensive; see Sen (1986) for references.

[2] Ray (1973). The particular aspect in question is known as condition $\alpha$, the Chernof condition and several other names.

Section 2 develops the notation and the concepts that are required to obtain the result in Section 3. Brief conclusions are presented in Section 4.

## 2. The Formal Framework

The formal framework set out in this section is thoroughly conventional.

Let I denote a constant finite set of individuals such that $|\mathrm{I}| \geq 2$ and let $A$ denote a finite set of alternatives such that $A \geq 3$. A preference $R$ is a binary relation on $A$ that is complete, reflexive and transitive. The asymmetric component (strict preference) of $R$ will be written $P$ and the symmetric component (indifference) will be written $I$. $\mathscr{R}$ will denote the set of all preferences. The preference of any individual $i \in \mathrm{I}$ will be written $R_i$. For any subset $C \subseteq \mathrm{I}$ of individuals, if all individuals in $C$ have the same preference, that preference will be written $R_C$. The restriction of any preference $R$ to a subset $B \subseteq A$ of alternatives will be written $R|B$. A profile is an $|\mathrm{I}|$-tuple of preferences in which a preference is assigned to each individual. $\underline{\mathscr{R}}$ will denote the set of all profiles. An $|\mathrm{I}|$-tuple of preferences each of which is restricted to $B \subseteq A$ will be written $\underline{\mathscr{R}}|B$.

A *social welfare function* $f\colon D \to \mathscr{R}$ where $D \subseteq \underline{\mathscr{R}}$, assigns a social preference to every $\underline{R} \in D$. A social welfare function satisfies *Unrestricted Domain* ($U$) if the domain of $f$ is $D = \underline{\mathscr{R}}$; it satisfies the *Weak Pareto Principle* (WP) if, for all $x, y \in A$, $xP_iy$ for all $i \in \mathrm{I}$ implies $xPy$ where $R = f(\underline{R})$. An individual $i \in \mathrm{I}$ is *decisive* on $\{x, y\}$, $x, y \in A$, if $xP_iy$ implies $xPy$ and $yP_ix$ implies $yPx$ where $R = f(\underline{R})$, for all $\underline{R} \in \underline{\mathscr{R}}$. An individual is *semi decisive* on $\{x, y\}$, $x, y \in A$, if $xP_iy$ implies $xRy$ and $yP_ix$ implies $yRx$ where $R = f(\underline{R})$, for all $\underline{R} \in \underline{\mathscr{R}}$. An individual $i \in \mathrm{I}$ is a *Dictator (Weak Dictator)* if $i$ is decisive (semi decisive) over all pairs of alternatives.

Although weaker than Arrow's condition of Independence of Irrelevant Alternatives, it is now well recognized that the following binary version is sufficient for the impossibility result.[3]

($BI$) A social welfare function $f$ satisfies *Binary Independence of Irrelevant Alternatives* if: For all $x, y \in A$ and all $\underline{R}, \underline{R}' \in \underline{\mathscr{R}}$, $\underline{R}|\{x, y\} = \underline{R}'|\{x, y\}$ implies $f(\underline{R})|\{x, y\} = f(\underline{R}')|\{x, y\}$.

This is an invariance condition. If two profiles are the same on a pair of alternatives, the social preference on that pair must be the same. A weakening of $BI$ would allow a social strict preference to

[3] Sen (1986, p. 1077).

remain the same or change to indifference but not to be completely reversed to the opposite strict preference. This is embodied in the following condition.

(*WBI*) A social welfare function $f$ satisfies *Weak Binary Independence of Irrelevant Alternatives* if: For all $x, y \in A$ and all $\underline{R}$, $\underline{R}' \in \mathscr{R} | \{x, y\} = \underline{R}' \{x, y\}$ implies $[(xPy \Rightarrow xR'y)$ and $(yPx \Rightarrow yR'x)]$ where $R = f(\underline{R})$ and $R' = f(\underline{R}')$.

*WBI* is strictly weaker than *BI*. While *BI* requires the social preference to remain constant on a pair, *WBI* permits a twitch.

What justifications are there for *WBI*? An obvious justification is that given in Sen (1984, p. 14), "the questioning of independence may come from recognizing that combining independence with other apparently mild conditions produces quite unacceptable results, and it can be, thus, reasonably argued that something or other 'has to give'". A somewhat positive justification may be offered along the following lines. While Arrow's explicit intention was to rule out interpersonal comparisons of cardinal utility his Independence of Irrelevant Alternatives even in the binary form used here, does more than that. It rules out interpersonal comparisons of ordinal utility as well. Such comparisons are made by the Borda Rule in particular, and more generally by any positional voting function. It will become clear that even *WBI* rules out procedures like the Borda Rule. However it is a small step in the direction of allowing social choice to depend on ordinal comparisons of utility. The effect of replacing *BI* by *WBI* is considered in the next section.

## 3. Results

As in conventional proofs of Arrow's theorem, a "field expansion lemma" is required. Sen (1984) has recently provided an improved proof in that it does not require the popular concept of almost decisiveness. Since the proof here follows exactly Sen's proof, with semi decisiveness replacing decisiveness, it will only be sketched.

*Lemma:* *For any social welfare function that satisfies U, WP, WBI, any subset of individuals that is semi decisive over one pair of alternatives is semi decisive over all pairs of alternatives.*

*Proof:* Consider any subset $C \subseteq I$ and any profile $\underline{R} \in \mathscr{R}$ such that $xP_Cy$ & $yP_Cz$ and $yP_{I-C}z$ where $x, y, z \in A$. If C is semi decisive over $\{x, y\}$ then $xRy$. $yPz$ follows from *WP*.

Therefore, $xPz$ follows from transitivity. For all such profiles, $xRz$ now follows from *WBI*. A similar argument shows that if $C$ is semi decisive over $\{x, z\}$ then $C$ is semi decisive over $\{x, y\}$. Therefore, $C$ is semi decisive over $\{x, y\}$ if and only if it is semi decisive over $\{x, z\}$ and let this be expressed by writing $\{x, y\} E \{x, z\}$. In general it is clear that $E$ is an equivalence relation on $A^2$. Thus, transitivity of $E$ extends semi decisiveness on $\{x, y\}$ to all pairs of alternatives. For example, for any $x, y, w, z \in A$, $\{x, y\} E \{w, y\}$ & $\{w, y\} E \{w, z\}$ imply $\{x, y\} E \{w, z\}$.

The main result may now be proved by showing the existence of a familiar contraction property on semi decisive subsets of **I**.

*Theorem: If a social welfare function satisfies U, WP & WBI then it is weakly dictorial.*

*Proof:* From *WP* there exists a decisive subset of I and therefore a semi decisive subset. Since I is finite, there exists a smallest semi decisive subset $C' \subseteq \mathrm{I}$. Assume $|C'| > 1$. Consider any subset $C \subseteq C'$ and any profile $\underline{R} \in \underline{\mathscr{R}}$ such that $xP_C y$ & $xP_C z$, $xP_{C'-C} y$ & $zP_{C'-C} y$ and $xP_{\mathrm{I}-C'} y$. $xPy$ follows from *WP*. If $zPy$ then the semi decisiveness of $C' - C$ on $\{y, z\}$ would follow from *WBI*. Then $C' - C$ would be semi decisive, by the lemma. Since $C'$ is the smallest semi decisive subset this cannot be the case. Therefore, completeness requires $yRz$. Since $xPy$, transitivity therefore implies that $xPz$. But the semi decisiveness of $C$ over $\{x, z\}$ now follows from *WBI* and, using the lemma, $C$ is semi decisive. Since this is also contrary to assumption, it follows that $|C'| = 1$. That is, there is a weak dictator.

## 4. Conclusion

The result of the previous section is both simple and straightforward. It therefore needs no summary. What requires comment is the generality of the result.

It may appear that in choosing to consider weakening the Independence axiom in the context of social welfare functions, the result is somewhat narrowly based. However, social welfare function results can be extended to the more general social choice functions by means of the base relation and some consistency conditions. In fact, "The impossibility results following from Arrow's work are robust enough to surface in widely different formulations of the problem of consistency of social choice".[4] It is for this reason that the framework chosen was the simplest, namely the social welfare function.

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[4] Sen (1986, p. 1103).