LIBRATION POINTS IN THE GENERALISED ELLIPTIC RESTRICTED THREE BODY PROBLEM

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Abstract. In this paper the existence of collinear as well as equilateral libration points for the generalised elliptic restricted three body problem has been studied distinct from Kondurar and Shinkarik (1972) where a study has been made for the generalised circular restricted three body problem. Here the coordinates of the libration points have been found to be functions of time *t*.

1. Introduction

This paper is a generalisation of the paper of Kondurar and Shinkarik (1972) on the libration points in the generalised circular restricted three body problem. Here we have assumed that the finite bodies which are each of spherical shape, are revolving in elliptic orbits and the body with the infinitesimal mass is taken of the shape of a spheroid. Under these assumptions we have studied here the existence of libration points both collinear and triangular.

In this paper the coordinates of the libration points will be functions of time whereas for the circular case or in the classical problem the coordinates were seen to be constants. Secondly for the existence of libration points we shall assume that $\ddot{\psi} + \ddot{v} = \dot{\psi} + \dot{v} = 0$ different from the assumption for the circular case where Shinkarik assumed that $\ddot{\psi} = \dot{\psi} = 0$.

In the Section 2 we have formed the equations of motion. In the Section 3 we have studied the existence of collinear libration points under the various cases of the motion and in the Section 4 we have studied the existence of triangular libration points. Here these libration points form an isosceles triangles with the finite bodies similar to the case for the circular generalised restricted problem and different from the case of the classical restricted problem of three bodies where these points form equilateral triangles with the finite masses.

2. Equations of Motion

Consider three bodies M_1, M_2 and M with masses $m_1 \geq m_2 \gg m$ and assume that M_1 and M_2 are homogeneous spheres or bodies with spherical structure or mass points and M is a dynamically symmetrical satellite with an equatorial plane of symmetry.

Let $\bar{A} = \bar{B}$ and $\bar{C}$ denote the equatorial and polar moments of inertia of the body M.

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Furthermore, let r_i $(i = 1, 2)$ be the distances between the centres of mass of the bodies M_i and the centre of mass of the body M and v_i be the cosine of the angle between the radius vector and the axis of the satellite.

As in the classical elliptic restricted problem we shall assume that the body M_2 is describing an ellipse with M_1 for a focus. Let us adopt a right-handed barycentric coordinate system $Ox'y'z'$ with the origin at the centre of mass O of the bodies M_1 and M_2. x' and y' axes will be assumed to rotate with the angular velocity $\dot{v}$ about the z'-axis orthogonal to the orbital plane of the finite bodies where v is the true anomaly of M_1 or M_2.

With this frame of reference the position of the centre of mass of the satellite will be specified by the Nechvil's coordinates x, y, z [Duboshin, 1964] and its orientation by the Eulerian angles ψ, θ, ϕ where ψ is measured from the rotating x-axis which will be taken to be directed along the line of centres of the bodies M_1 and M_2. The units are so chosen that $m_1 + m_2 = 1$, the gravitational constant $= 1$ and the parameter p of the elliptic orbit $= 1$.

The equations of translational-rotational motion of the satellite in the system of rotating axes are represented by the system of equations which are as follows:

$$\left.\begin{aligned} x'' - 2y' &= \frac{\partial\Omega}{\partial x}, \quad y'' + 2x' = \frac{\partial\Omega}{\partial y}, \quad z'' = \frac{\partial\Omega}{\partial z}, \\ \psi'' &= -(\psi' + 1)\frac{\ddot{v}}{\dot{v}^2} - 2(\psi' + 1)\theta' \operatorname{ctg}\theta + \frac{C}{A}\bar{r}\frac{\theta'}{\sin\theta} + \frac{1}{A\dot{v}^2\sin^2\theta}\frac{\partial\Omega}{\partial\psi} \\ \theta'' &= -\theta'\frac{\ddot{v}}{\dot{v}^2} + (\psi' + 1)^2\cos\theta\sin\theta - \frac{C}{A}\bar{r}(\psi' + 1)\sin\theta + \frac{1}{A\dot{v}^2}\frac{\partial\Omega}{\partial\theta} \end{aligned}\right|, \tag{1}$$

where $mA = \bar{A}$, $mC = \bar{C}$

$$U_i = rm_i\left[\frac{1}{r_i} + \frac{A\sigma}{2r_i^3r^2}(1 - 3v_i^2)\right] \qquad (i = 1, 2)$$

$$\sigma = (C - A)A^{-1}(-1 \le \sigma \le +1)$$

$$\Omega = r\{\tfrac{1}{2}(x^2 + y^2) - \tfrac{1}{2}e\cos vz^2 + U_1 + U_2\}$$

$$\frac{\partial\Omega}{\partial x} = rx - r\sum_{i=1}^{2}\left[m_i\left\{\frac{x - x_i}{r_i^3} + \tfrac{3}{2}A\sigma\frac{x - x_i}{r_i^5}\frac{1}{r^2}(1 - 3v_i^2) + \right.\right.$$
$$\left.\left. + \frac{3Av_i\sigma}{r^2r_i^3}\left(-\frac{v_i(x - x_i)}{r_i^2} + \frac{a_{13}}{r_i}\right)\right\}\right], \tag{2}$$

$$\frac{\partial\Omega}{\partial y} = ry - r\sum_{i=1}^{2}\left[m_i\left\{\frac{y}{r_i^3} + \tfrac{3}{2}A\sigma\frac{y}{r_i^5}\frac{1}{r^2}(1 - 3v_i^2) + \right.\right.$$
$$\left.\left. + \frac{3A\sigma v_i}{r_i^3r^2}\left(-\frac{v_iy}{r_i} + a_{23}\right)\right\}\right], \tag{3}$$

$$\frac{\partial\Omega}{\partial z} = -re\cos vz - r\sum_{i=1}^{2} m_i\left\{\frac{z}{r_i^3} + \tfrac{3}{2}A\sigma\frac{z}{r_i^5 r^2}(1-3v_i^2) + \right.$$
$$\left. + \frac{3A\sigma v_i}{r_i^3 r^2}\left(-\frac{v_i z}{r_i} + a_{33}\right)\right\}, \tag{4}$$

$$\frac{\partial\Omega}{\partial\psi} = -r\sum_{i=1}^{2} m_i\frac{3A\sigma v_i}{r_i^4 r^2}[-a_{23}(x-x_i) + ya_{13}], \tag{5}$$

$$\frac{\partial\Omega}{\partial\theta} = -r\sum_{i=1}^{2} m_i\frac{3A\sigma v_i}{r_i^4 r^2}[(x-x_i)\sin\psi\cos\theta - y\cos\psi\cos\theta - z\sin\theta], \tag{6}$$

$$v_i = \frac{1}{r_i}[(x-x_i)a_{13} + ya_{23} + za_{33}], \qquad (i = 1, 2) \tag{7}$$

$$\left.\begin{aligned} r_i^2 &= (x-x_i)^2 + y^2 + z^2 \\ x_1 &= -\frac{m_2}{m_1+m_2},\ x_2 = \frac{m_1}{m_1+m_2} \end{aligned}\right|, \tag{8}$$

$$\left. r = \frac{1}{1+e\cos v},\ p = a(1-e^2) = 1,\ \frac{\ddot{v}}{\dot{v}^2} = -\frac{2e\sin v}{1+e\cos v}\right|, \tag{9}$$

$$e = \text{the eccentricity of the elliptic orbit,} \tag{10}$$

$$a_{13} = \sin\psi\sin\theta,\ a_{23} = -\cos\psi\sin\theta,\ a_{33} = \cos\theta, \tag{11}$$

$$\bar{r} = \{(\psi'+1)\cos\theta + \phi'\}\dot{v} = \text{constant.} \tag{12}$$

3. Conditions for the Existence of Collinear Libration Points

For a libration point we must have

$$\frac{\partial\Omega}{\partial x} = \frac{\partial\Omega}{\partial y} = \frac{\partial\Omega}{\partial z} = \frac{\partial\Omega}{\partial\psi} = \frac{\partial\Omega}{\partial\theta} = 0. \tag{13}$$

It is not difficult to see that the Equations (13) admit a plane motion. We find that $\partial\Omega/\partial z = 0$ is satisfied by $z = 0$ if $a_{33} = \cos\theta = 0$ or when $\sin\theta = 0$ which gives $v_i = \pm z/r_i$. Thus we find that a plane motion is admissible when $\theta = \pi/2$ as well as when $\theta = 0$.

Next let us take up the condition $\partial\Omega/\partial y = 0$ which will be satisfied by $y = 0$ if $\theta = \pi/2$, $\psi = \pi/2, 3\pi/2$ and also if $\theta = 0$, $\psi = 0, \pi$ because in the latter case

$$v_i\left(-\frac{v_i y}{r_i} + a_{23}\right) = \frac{z}{r_i}\left(-\frac{zy}{r_i^2}\right) = 0$$

identically with $z = 0$. The case when $\theta = 0$, $\psi = \pi/2, 3\pi/2$ gives $y = 0$ will be dealt with the preceding case. Lastly let us take up the case $\theta = \pi/2$, $\psi = 0, \pi$. In this case $v_i = \pm y/r_i$ and so $\partial\Omega/\partial y = 0$ is satisfied by $y = 0$ identically and $v_i = 0$.

Thus we find that collinear libration points are possible under the following conditions:

$$\text{(i) } \theta = \frac{\pi}{2}, \psi = \frac{\pi}{2}, \frac{3\pi}{2}, \nu_i = \pm \frac{x - x_i}{r_i} = \pm 1,$$

$$\text{(ii) } \theta = 0, \ \psi = 0, \pi, \quad \nu_i = 0,$$

$$\text{(iii) } \theta = \frac{\pi}{2}, \psi = 0, \pi, \quad \nu_i = 0,$$

$$\text{(iv) } \theta = 0, \ \psi = \frac{\pi}{2}, \frac{3\pi}{2}, \nu_i = 0.$$

It remains to be examined if under these conditions $\partial\Omega/\partial\psi = \partial\Omega/\partial\theta = 0$. Let us take up these cases one by one.

(i) Under the conditions $\theta = \pi/2$, $\psi = \pi/2, 3\pi/2$, we find that the expression under the square bracket in the expression for $\partial\Omega/\partial\psi$ is identically zero. Coming to $\partial\Omega/\partial\theta$ we find here that the expression under square bracket is equal to

$$(x - x_i) \sin\psi \cos\theta - y \cos\psi \cos\theta - z \sin\theta = 0.$$

Thus it is seen that $\theta = \pi/2$, $\psi = \pi/2, 3\pi/2$, gives collinear libration points if $\partial\Omega/\partial x = 0$ gives some real roots.

(ii) $\theta = 0$, $\psi = 0, \pi$ and $\theta = 0$, $\psi = \pi/2, 3\pi/2$.

Since on the right hand side of θ'' in the Equations (i) we find $\sin\theta$ in the denominator and hence it gives rise to singularity. To avoid the presence of singularity the equations for θ'' and ψ'' need modification. For this we shall introduce new variables by substitutions

$$p = \sin\psi \sin\theta, \ q = -\cos\psi \sin\theta, \ w = \cos\theta = \sqrt{1 - p^2 - q^2}.$$

The modified equations are given as

$$\begin{aligned} p'' = {} & pw\left(w - \bar{r}\frac{C}{A}\right) - \left(-2 + \bar{r}\frac{C}{A}w^{-1}\right)[q' - p(pq' - p'q)] + \\ & + pw^{-2}[(q'p - p'q)^2 - (p'^2 + q'^2)] - \\ & - 2e \sin vr(q - p') + r^4[(1 - p^2)A^{-1}\Omega p - pqA^{-1}\Omega q] \end{aligned} \tag{14}$$

$$\begin{aligned} q'' = {} & qw\left(w - \bar{r}\frac{C}{A}\right) + \left(-2 + \bar{r}\frac{C}{A}w^{-1}\right)[p' + q(q'p - p'q)] + \\ & + qw^{-2}[(q'p - p'q)^2 - (p'^2 + q'^2)] + 2e \sin vr(p + q') + \\ & + r^4[(1 - q^2)A^{-1}\Omega q - pqA^{-1}\Omega q], \end{aligned} \tag{15}$$

$$\frac{\partial\Omega}{\partial p} = -r \sum_{i=1}^{2} m_i \frac{3A\sigma\nu_i}{r_i^4 r^2}[(x - x_i) - zpw^{-1}], \tag{16}$$

$$\frac{\partial \Omega}{\partial q} = -r \sum_{i=1}^{2} m_i \frac{3A\sigma v_i}{r_i^4 r^2} (y - zqw^{-1}). \tag{17}$$

In both the cases we find that $p = 0 = q$ which clearly reduces the right hand side of p'' and q'' to zero gives rise to collinear libration point.

(iii) Under the conditions $\theta = \pi/2$, $\psi = 0$, π, $v_i = 0$ as well we find that $\partial\Omega/\partial\psi$ and $\partial\Omega/\partial\theta$ are identically zero.

Hence we can conclude that under all the four conditions collinear libration points will exist provided $\partial\Omega/\partial x = 0$ gives real roots. These conditions are classified to represent the motion of the types 'spoke', 'level', 'arrow' and 'float' respectively.

Now it remains to investigate the existence of real roots for $\partial\Omega/\partial x = 0$ which will only confirm the existence of collinear libration points in the various cases. We have two cases namely $v_i = \pm 1$ or $v_i = 0$. We may write $\partial\Omega/\partial x = 0$ as

$$x - \sum_{i=1}^{2} m_i \left\{ \frac{x - x_i}{r_i^3} + \tfrac{3}{2}\sigma \frac{x - x_i}{r_i^5} \frac{1}{r^2} (1 - 3v_i^2) + \right.$$
$$\left. + \frac{1}{r^2} \frac{3A\sigma v_i}{r_i^3} \left(-v_i \frac{x - x_i}{r_i^2} + \frac{a_{13}}{r_i} \right) \right\} = 0,$$

i.e.,

$$x - \left\{ \frac{m_1(x - x_1)}{[(x - x_1)^2]^{3/2}} + \frac{m_2(x - x_2)}{[(x - x_2)^2]^{3/2}} \right\} +$$
$$+ k \frac{\sigma}{r^2} \sum_{i=1}^{2} m_i \frac{x - x_i}{[(x - x_i)^2]^{5/2}} = 0,$$

where $k = -3$ when $v_i = \pm 1$ and $k = +1.5$ when $v_i = 0$.

i.e.,

$$f(x) + \frac{k\sigma}{r^2} \sum_{i=1}^{2} m_i \frac{x - x_i}{[(x - x_i)^2]^{5/2}} = 0, \tag{18}$$

where

$$f(x) = x - \left\{ \frac{m_1(x - x_1)}{[(x - x_1)^2]^{3/2}} + \frac{m_2(x - x_2)}{[(x - x_2^2)^2]^{3/2}} \right\}.$$

It may be noticed that in either cases (i.e. $v_i = \pm 1$ and 0) if in (18) we put $\sigma = 0$, then the resulting equation coincides with similar equations for classical points of libration (Duboshin, 1968).

Let us now show that for $\sigma \neq 0$ as well the Equation (18) has three distinct real roots each of which tends to coincide with the classical points of libration when $\sigma \to 0$. Really the equation (18) may be reduced to the form

$$f(x)\,(x - x_1)^4 (x - x_2)^4 + \sigma Q(x) = 0, \tag{19}$$

where $Q(x)$ is polynomial of the fourth degree w.r. to x for all values of x distinct from x_1 and x_2.

For $\sigma = 0$, this equation as it is well known has a real root $x = b_i$ $(i = 1, 2, 3)$ in each of the intervals

$$-\infty < x < x_1, x_1 < x < x_2, x_2 < x < +\infty. \tag{20}$$

As the left-hand side of the Equation (19) is a polynomial for all the intervals (20), then in each of them it can be expanded in a convergent series in powers of $x - b_i$, σ. In this expansion the independent term will not be present and the coefficient with $x - b_i$ is not equal to zero as the derivative of the left-hand side w.r. to x for $x = b_i$, $\sigma = 0$ by virtue of monotonic character of $f(x)$ in the intervals (20) is not equal to zero.

Consequently by virtue of the theorem on implicit function (Goursat, 1936) Equation (19) in each of the intervals (20) will have unique root which tends to the corresponding classical one when $\sigma \to 0$.

Thus if σ is sufficiently small in the modulus, then for oblate as well as for prolate body of rotation, to each value of σ there will correspond three collinear points in which either the axis of satellite coincides with the line of centres or is perpendicular to it, i.e., for $-1 \leq \sigma < 0$ and $0 < \sigma \leq 1$ and when $k = -3$ or 1.5, the Equation (19) has distinct real roots.

Really for above values of k and σ the Equation (18) represents a monotonic continuous function for each of the intervals

$$[-1/\varepsilon, x_1 - \varepsilon], [x_1 + \varepsilon, x_2 - \varepsilon], [x_2 + \varepsilon, 1/\varepsilon],$$

where ε is a very small positive number.

Thus for each oblate body ($\sigma < 0$), there will correspond three distinct collinear points of libration, in which the axis of the satellite is directed along the line of centres of the bodies or when it is perpendicular to the line of centres of the bodies and for each prolate body ($\sigma > 0$) three distinct collinear points of libration in which the axis lies in the plane normal to the line of centres or when the axis is directed along the line of centres.

Thus we may represent the collinear libration points as

$$L_j^{(s)}: x_0 = \alpha_{js}, y_0 = 0, z_0 = 0, \qquad (j = 1, 2, 3; s = 1, 2, 3, 4),$$

where it will be assumed that

$s = 1$ corresponds to 'spoke',
$s = 2$ 'level',
$s = 3$ 'arrow',
$s = 4$ 'float'.

4. Conditions of the Existence of Triangular Libration Points

(a) Let us now pass on to find out the triangular points of libration for which $\nu_1 = 0 = \nu_2$, $y \neq 0$. In this case the equations $\partial\Omega/\partial\psi = 0 = \partial\Omega/\partial\theta$ turn out to be identities and the equations for x and y assume the following form:

$$\left.\begin{aligned} x - \sum_{i=1}^{2}\left[m_i\left\{\frac{x-x_i}{r_i^3} + \tfrac{3}{2}A\sigma\frac{x-x_i}{r_i^5}\frac{1}{r^2}\right\}\right] = 0 \\ 1 - \sum_{i=1}^{2}\left[m_i\left\{\frac{1}{r_i^3} + \tfrac{3}{2}A\sigma\frac{1}{r_i^5}\frac{1}{r^2}\right\}\right] = 0 \end{aligned}\right\} . \tag{21}$$

If $r_1 = r_2 = \tilde{r}$, then multiplying the second equation by x and subtracting it from the first equation, we have

$$\frac{1}{\tilde{r}^3}(m_1x_1 + m_2x_2) + \tfrac{3}{2}A\sigma\frac{1}{r^2\tilde{r}^5}(m_1\dot{x}_1 + m_2x_2) = 0$$

i.e.

$$m_1x_1 + m_2x_2 = 0$$

which reduces to an identity. Thus the only condition to be satisfied by $\tilde{r}$ is

$$1 - \frac{m_1 + m_2}{\tilde{r}^3} - \tfrac{3}{2}A\sigma\frac{1}{r^2\tilde{r}^5}(m_1 + m_2) = 0,$$

i.e.,

$$1 - \frac{1}{\tilde{r}^3} - \tfrac{3}{2}A\sigma\frac{1}{r^2\tilde{r}^5} = 0, \tag{22}$$

since $m_1 + m_2 = 1$.

This case corresponds to the case $a_{13} = a_{23} = 0 = v_i$, i.e., to the motion of the type of 'float'.

The coordinates of the triangular points of libration in this case are given by the formula

$$\left.\begin{aligned} x &= \frac{x_1 + x_2}{2} = \frac{m_1 - m_2}{2(m_1 + m_2)} = \frac{m_1 - m_2}{2}, \\ y^2 &= \tilde{r}^2 - \tfrac{1}{4}. \end{aligned}\right\} \tag{23}$$

If the satellite is a homogeneous sphere or poseseses spherical structure, then in this case $U_i = fm_i/r_i$ and we find that $r_1 = r_2 = r$ which shows that the triangular points of libration coincides in this case with that in the classical case.

(b) Next let us consider the problem of the existence of the triangular points of libration for which $v_1 = v_2 \neq 0$. This corresponds to the motion of the type 'arrow' and hence the libration points in this case will be called of the type 'arrow'. Here $\partial U_i/\partial v_i \neq 0$ and the planar motion is possible only when $\theta = \pi/2$.

Here it follows that $v_1 = v_2$ when $r_1 = r_2 = \tilde{r}$, $\psi = 0, \pi$ and hence

$$v = v_1 = v_2 = \pm\frac{y}{r}. \tag{24}$$

If we take into consideration the relations

$$\frac{\partial U_1}{\partial r_1} = \frac{m_1}{m_2}\frac{\partial U_2}{\partial r_2}, \quad \frac{\partial U_1}{\partial v_1} = \frac{m_1}{m_2}\frac{\partial U_2}{\partial v_2}, \tag{25}$$

then we find that the derivative $\partial\Omega/\partial\psi$ will be given as

$$\frac{\partial\Omega}{\partial\psi} = r\frac{\partial U_1}{\partial v}\left(\frac{\partial v_1}{\partial\psi} + \frac{m_2}{m_1}\frac{\partial v_2}{\partial\psi}\right), \tag{26}$$

where

$$\frac{\partial v_i}{\partial\psi} = -\frac{a_{23}}{\tilde{r}}(x - x_i).$$

Then (26) may be written as

$$-r\frac{\partial U_1}{\partial v}\frac{a_{23}}{\tilde{r}}\left[x - x_1 + \frac{m_2}{m_1}(x - x_2)\right] = 0,$$

whence we find that $x = m_1x_1 + m_2x_2 = 0$ and from (23)

$$m_1 = m_2, x_1 = -x_2. \tag{27}$$

It is clear that $\partial\Omega/\partial\theta = 0$ and $\partial\Omega/\partial x = 0$ identically under the conditions (27) and (25).

The ordinates y of the triangular points of libration are defined by the second Equation (23) where $\tilde{r}$ is given by

$$y - \sum_{i=1}^{2}\left[m_i\left\{\frac{y}{\tilde{r}^3} + \tfrac{3}{2}A\sigma\frac{y}{\tilde{r}^5}\frac{1}{r^2}(1 - 3v^2) + \right.\right.$$
$$\left.\left. + \frac{3A\sigma v}{\tilde{r}^4 r^2}\left(-\frac{vy}{\tilde{r}} + a_{23}\right)\right\}\right] = 0$$

i.e.,
$$y - 2m\left\{\frac{y}{\tilde{r}^3} + \tfrac{3}{2}A\sigma\frac{y}{\tilde{r}^5 r^2}\left(1 - 3\frac{y^2}{\tilde{r}^2}\right) + \frac{A\sigma}{\tilde{r}^4 r^2}\left(-\frac{3y^3}{\tilde{r}^3} + \frac{3y}{\tilde{r}}\right)\right\} = 0,$$

i.e.,
$$y\left[1 - \frac{2m}{\tilde{r}^3} - \frac{9A\sigma m}{\tilde{r}^5 r^2} + 15A\sigma m\frac{y^2}{\tilde{r}^7 r^2}\right] = 0,$$

i.e.,
$$\left[1 - \frac{2m}{\tilde{r}^3} - \frac{9A\sigma m}{\tilde{r}^5 r^2} + 15A\sigma m\frac{1}{\tilde{r}^7 r^2}(\tilde{r}^2 - \tfrac{1}{4})\right] = 0,$$

i.e.,
$$1 - \frac{2m}{\tilde{r}^3} + \frac{6A\sigma m}{\tilde{r}^5 r^2} - \frac{15}{4}\frac{A\sigma m}{\tilde{r}^7 r^2} = 0. \tag{28}$$

(c) Thus we find that the existence of triangular points of libration now depends on the existence of real roots of the Equation (23) in the case of 'float' and the Equation (28) in the case of 'arrow'. Firstly let us take up the Equation (23), i.e.,

$$\tilde{r}^5 - \tilde{r}^2 - \tfrac{3}{2}A\sigma\frac{1}{r^2} = 0. \tag{29}$$

For $\sigma = 0$, the Equation (29) has a unique real root equal to unity distinct from zero. The left-hand side of this equation can be expanded in convergent series in powers of $\tilde{r} - 1$, σ for all r and in this series there will be no constant term and the coefficient with $\tilde{r} - 1$ is not equal to zero. Hence by the implicit function theorem the Equation (29) for $\sigma \neq 0$ as well but sufficiently small, will have a unique root reducing to 1 for $\sigma = 0$.

From the Equation (29) it is clear that $\sigma > 0$ gives $\tilde{r} > 1$ and $\sigma < 0$ gives $\tilde{r} < 1$. Hence it follows that for prolate bodies the triangular points of libration (for $y > 0$) lie above the vertex of the corresponding equilateral triangle and for oblate bodies it will be below.

Secondly we shall consider the case of the triangular libration point for the motion of the type 'arrow'. The coordinates in this case are given as

$$L_j^{(4)}: x_0 = 0, \; y_0 = \pm\sqrt{\tilde{r}^2 - \tfrac{1}{4}}, \; z_0 = 0,$$

where $\tilde{r}$ is given by the equation

$$1 - \frac{1}{\tilde{r}^3} + \frac{3A\sigma}{\tilde{r}^5 r^2} - \frac{15}{8}\frac{A\sigma}{\tilde{r}^7 r^2} = 0$$

i.e.,

$$\tilde{r}^7 - \tilde{r}^4 + \frac{3A\sigma}{8r^2}(8\tilde{r}^2 - 5) = 0. \tag{30}$$

For $\sigma = 0$ the preceding equation has unique real root 1 distinct from zero and the derivative of the left-hand side of the equation w.r. to $\tilde{r}$ is not equal to zero for $\tilde{r} = 1$ and so it follows from the implicit function theorem that for $\sigma \neq 0$ but sufficiently small the Equation (30) under consideration will have a unique root reducing to 1 for $\sigma = 0$.

5. Conclusion

Taking into considerations the various types of motion we find that the collinear libration points and the triangular libration points exist in all the various cases depending only on $v_i = 0$ or $v_1 = v_2$. Difference lies in the fact that the coordinates of these libration points are not constant as it has been found for the circular case. Here they are seen to be functions of the true anomaly of the primary bodies moving in ellipses with the common centre of mass for a common focus.

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