MOTION OF TWO PLANETS WITH PERIODS COMMENSURABLE IN THE RATIO 2 : 1 SOLUTIONS OF THE HORI AUXILIARY SYSTEM

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ABSTRACT. It is shown that the Hori auxiliary system for the motion of two planets, whose motions around the Sun have commensurable periods in the ratio 2:1, is completely integrable and, an intermediate orbit that includes the effects of the resonance is obtained. The difficulties of classifying some solutions as librations or circulations are discussed.

1. INTRODUCTION

In this paper we study the motion of two planets whose periods around the Sun are commensurable in the ratio 2:1.

This problem is similar to the problem of the motion of two satellites of a planet, whose periods are commensurable in the ratio 2:1, in the particular case where the flattening of the central planet and the influence of the Sun over the system are neglected. Some systems of natural satellites have orbital periods in the same conditions as the problem under consideration here: Io - Europa and Europa-Ganymede among Jupiter's satellites and Mimas-Tethys and Enceladus-Dione among Saturn's satellites.

In the neighbourhood of the conditions where resonance occurs, the general theories of the motion of planetary systems contain small divisors and so the formal convergency of the series giving the solutions is lost. This is the main difficulty in the study of this kind of problem. In order to overcome it, it is necessary to obtain a new intermediate orbit that includes the resonance effect and is closer to the actual motion than a pair of Keplerian orbits.

Most of the published results about orbital resonance in the Solar system deal with problems where only one critical argument appears and in such a case the resulting dynamical system is completely integrable. However, it is well known that the general problem of orbital resonance in a

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system of two planets where the usual methods of Celestial Mechanics are applied, leads to dynamical systems with at least two critical arguments. We show in this paper that the dynamical system obtained when the Hamiltonian is developed in the neighbourhood of the exact commensurability and truncated at the first order in the eccentricities, is separable and completely integrable.

The construction of a new intermediate solution including resonance effects, is the first step in order to built a formal theory of the motion of planetary system where the commensurability 2:1 takes place, as it happens with some pairs of orbits in the Solar System.

The problem is introduced in Section 2; in Section 3 the associated Hori auxiliary system is obtained and in Section 4 its complete integrability is shown. The rest of the paper has two parts to which correspond different approaches. The first one gives the topological structure of the flows in the phase space (Sections 4, 5, 8, and 9). It is shown that by a convenient change of variables, one of the components of the flows in the phase space is reduced to the classical case of the averaged circular restricted problem of three bodies in the case of a resonance (Jefferys, 1966; Message, 1966; Schubart, 1966). The second approach, of a quantitative nature, concerns the study of a polynomial of 4th degree (Sections 6, 7, and Appendix). The explicit solution of this polynomial is necessary when one intends to use the solution of the Hori auxiliary system, to the construction of a formal theory of higher order.

2. EQUATIONS OF MOTION

Consider three mass points S, P_1, P_2 whose masses are M, m', m, respectively. We suppose that m and m' are of the same order of magnitude and much less than M. We introduce Jacobi coordinates and define the vectors

$$\vec{\zeta}_1 = \frac{M}{M + m'}\,\vec{r}\,'; \qquad \vec{\zeta}_2 = \vec{r} - \frac{m'}{M + m'}\,\vec{r}\,',$$

which origin is the center of mass A of S and P_1. The vectors $\vec{r}$, $\vec{r}\,'$ are the heliocentric vectors of position (see Figure 1).

Following Brouwer and Clemence (1961) we introduce 2 sets of Delaunay canonical variables. However instead of referring the semi-major axis of the first planet to the Sun and the semi-major axis of the second planet to the center of mass of the Sun and the first planet, we introduce in the coordinates of the first planet the factor M/(M + m') and thus refer the semi-major axis of both planets to the point A.

$$\begin{aligned} &L_j = m_j\sqrt{\mu_j a_j}, \qquad G_j = L_j\sqrt{1 - e_j^2}, \\ &H_j = G_j \cos I_j; \qquad \ell_j,\ \omega_j,\ \Omega_j, \qquad (j = 1,\ 2) \end{aligned} \tag{1}$$

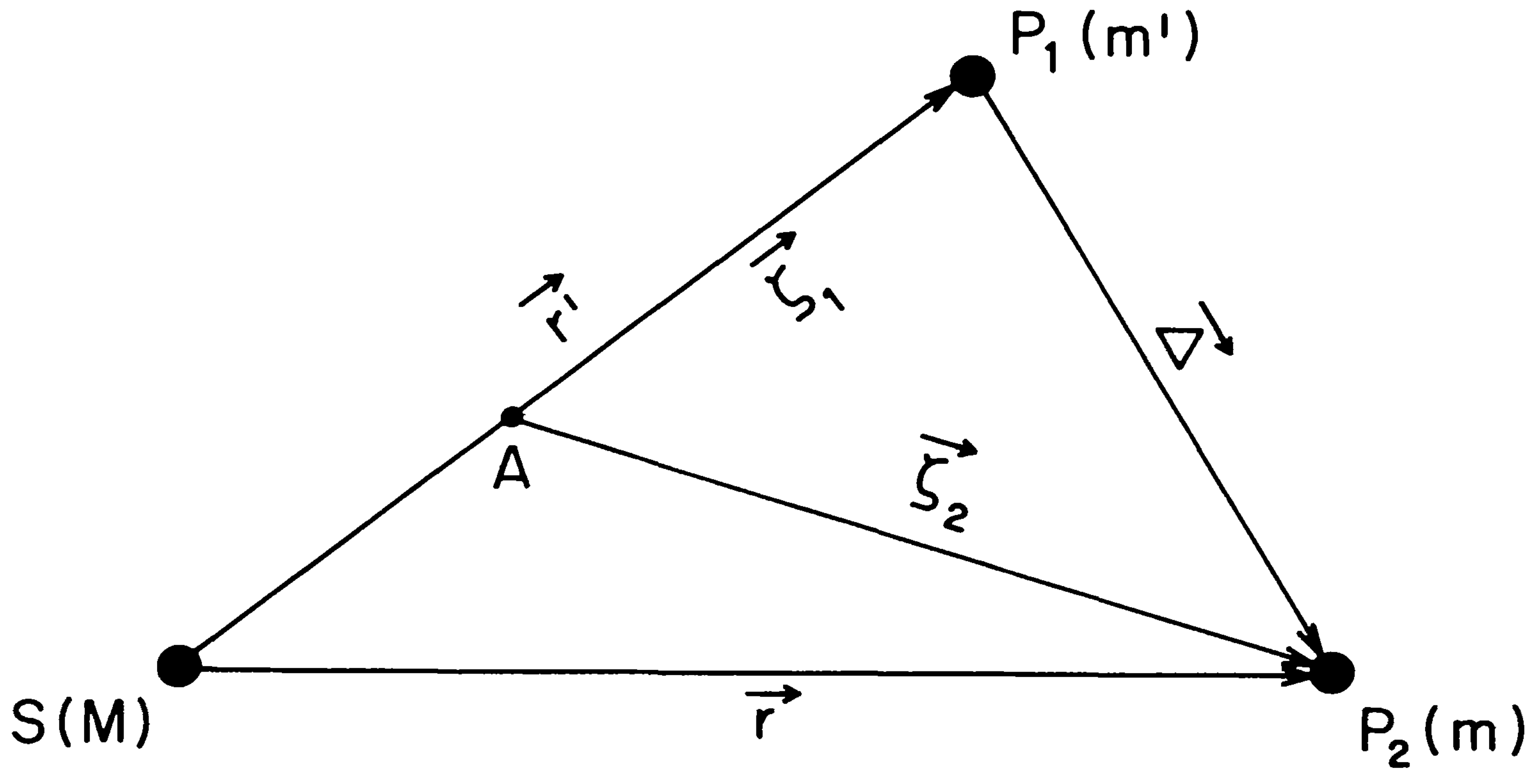

Fig. 1. Mass-points configuration.

where

$$\mu_1 = \frac{k^2 M^3}{(M + m')^2}, \quad \mu_2 = \frac{k^2 (m + m' + M) M}{(m' + M)},$$

$$m_1 = \frac{m'(m' + M)}{M}, \quad m_2 = \frac{m(m' + M)}{(m + m' + M)},$$

and the Keplerian elements a_j, e_j, I_j, ℓ_j, ω_j, Ω_j refer to osculating orbits having focus at A.

In these variables, the canonical equations of motion are

$$\frac{d}{dt}(L_j, G_j, H_j) = \frac{\partial F}{\partial(\ell_j, \omega_j, \Omega_j)},$$
$$\frac{d}{dt}(\ell_j, \omega_j, \Omega_j) = -\frac{\partial F}{\partial(L_j, G_j, H_j)}, \tag{2}$$

$F = F_0 + R$ is the Hamiltonian where

$$F_0 = \sum_{j=1}^{2} \frac{\mu_j^2 m_j^3}{2L_j^2}, \tag{3}$$

$$R = k^2 \left[\frac{mm'}{\Delta} + mM\left(\frac{1}{r} - \frac{1}{\zeta_2}\right)\right]. \tag{4}$$

The mean motions are defined as

$$n_j = -\frac{\partial F_0}{\partial L_j}. \tag{5}$$

The disturbing function R may be expanded in a power series with respect to the small parameter m'/M, and also with respect to the eccentricities and inclinations, assumed to be small. The disturbing function expanded to the first order in the small parameter and, to the second order in the eccentricities and inclinations, may be found in Marsden (1966), or in the classical literature about planetary systems. In the expansions the ratio of the semi-major axes was defined by $\alpha = a_1/a_2 < 1$. That is to say, the orbit of P_1 was assumed as interior with respect to the orbit of P_2. We also assume the elimination of the short periodic terms was performed (see Sessin, 1981).

The following set of canonical variables is introduced

$$\begin{aligned} x_1 &= L_1 + \tfrac{1}{2}L_2, & \lambda_1 &= \ell_1 + \tilde{\omega}_1, \\ x_2 &= -\tfrac{1}{2}L_2, & \theta &= \lambda_1 - 2\lambda_2, \\ y_j &= G_j - L_j = L_j(\sqrt{1-e_j^2} - 1), & \tilde{\omega}_j &= \omega_j + \Omega_j, \\ z_j &= H_j - G_j = -2L_j\sqrt{1-e_j^2}\, s_j^2, & \Omega_j&, \end{aligned} \tag{6}$$

where $\lambda_2 = \ell_2 + \tilde{\omega}_2$ and $s_j = \sin(I_j/2)$.

The canonical equations of motion then become

$$\begin{aligned} \frac{d}{dt}(x_1, x_2, y_j, z_j) &= \frac{\partial F}{\partial(\lambda_1, \theta, \tilde{\omega}_j, \Omega_j)}, \\ \frac{d}{dt}(\lambda_1, \theta, \tilde{\omega}_j, \Omega_j) &= -\frac{\partial F}{\partial(x_1, x_2, y_j, z_j)}. \end{aligned} \tag{7}$$

The Hamiltonian is

$$F = F_0 + R(x_1, x_2, y_j, z_j, -, \theta, \tilde{\omega}_j, \Omega_j), \tag{8}$$

where

$$F_0 = \frac{\mu_1^2 m_1^3}{2(x_1 + x_2)^2} + \frac{\mu_2^2 m_2^3}{8x_2^2}. \tag{9}$$

Since the Hamiltonian F is independent of λ_1, x_1 is a constant and Equations (7) may be reduced to five degrees of freedom. The 2:1 commensurability condition now takes the form

$$\frac{\partial F_0}{\partial x_2} = \frac{-\mu_1^2 m_1^3}{(x_1 + x_2)^3} - \frac{\mu_2^2 m_2^3}{4x_2^3} = -n_1 + 2n_2 \simeq 0. \tag{10}$$

3. RESONANT TERMS

Consider the canonical equations (7), where x_1 is a constant. In order to study the resonant terms we expand the Hamiltonian F in the neighbourhood of the conditions for exact commensurability. Let

$$x = x_2 - x_{20}, \tag{11}$$

where x_{20} is the value of x_2 at the exact resonance, such that

$$\left(\frac{\partial F_0}{\partial x_2}\right)_{x=0} = 0. \tag{12}$$

Further it is assumed that

$$\left(\frac{x}{x_{20}}\right) = \mathcal{O}(\sqrt{m'/M}). \tag{13}$$

At the exact resonance we have

$$L_{10} = x_1 + x_{20}, \quad L_{20} = -2x_{20}, \quad \alpha_0 = \frac{m_2^2\mu_2}{m_1^2\mu_1}\left(\frac{L_{10}}{L_{20}}\right)^2,$$
$$e_{j0}^2 = 1 - \left(1 + \frac{y_j}{L_{j0}}\right)^2, \quad s_{j0}^2 = \frac{-z_j}{2(y_j + L_{j0})}. \tag{14}$$

We also assume that the variables y_j and z_j are of the order of the square root of the small parameter, that is

$$\left(\frac{y_j}{x_{20}}\right) = \mathcal{O}(\sqrt{m'/M}), \quad \left(\frac{z_j}{x_{20}}\right) = \mathcal{O}(\sqrt{m'/M}), \tag{15}$$

and therefore

$$e_{j0}^2 = \mathcal{O}(\sqrt{m'/M}), \quad s_{j0}^2 = \mathcal{O}(\sqrt{m'/M}). \tag{16}$$

The expansion of the Hamiltonian to the order 3/2 in the small quantities leads to

$$F_1 = F_{02}\left(\frac{x}{x_{20}}\right)^2 + \frac{m'}{M} \times$$
$$\times\ [P_{00} - P_{30}e_1' \cos(\theta + \tilde{\omega}_1) + P_{40}e_2' \cos(\theta + \tilde{\omega}_2)], \tag{17}$$

$$F_{3/2} = -F_{03}\left(\frac{x}{x_{20}}\right)^3 + \frac{m'}{M}\left[-P_{01}\left(\frac{x}{x_{20}}\right) + \right.$$
$$+ \frac{1}{2} P_{10}(e_1'^2 + e_2'^2) - \frac{1}{4} Q_{10}(s_1'^2 + s_2'^2) +$$
$$+ P_{31}\left(\frac{x}{x_{20}}\right) e_1' \cos(\theta + \tilde{\omega}_1) + \frac{1}{8} P_{30}e_1'^3 \cos(\theta + \tilde{\omega}_1) -$$
$$- P_{41}\left(\frac{x}{x_{20}}\right) e'^2 \cos(\theta + \tilde{\omega}_2) - \frac{1}{8} P_{40}e_2'^3 \cos(\theta + \tilde{\omega}_2) +$$
$$+ \frac{1}{2} P_{50}e_1'^2 \cos 2(\theta + \tilde{\omega}_1) + \frac{1}{2} P_{60}e_2'^2 \cos 2(\theta + \tilde{\omega}_2) -$$

$$- P_{70}e_1'e_2' \cos(2\theta + \tilde{\omega}_1 + \tilde{\omega}_2) + P_{80}e_1'e_2' \cos(\tilde{\omega}_1 - \tilde{\omega}_2) +$$

$$+ \frac{1}{4} Q_{30}s_1'^2 \cos 2(\theta + \Omega_1) + \frac{1}{4} Q_{30}s_2'^2 \cos 2(\theta + \Omega_2) -$$

$$- \frac{1}{2} Q_{30}s_1's_2' \cos(2\theta + \Omega_1 + \Omega_2) +$$

$$+ \frac{1}{2} Q_{10}s_1's_2' \cos(\Omega_1 - \Omega_2)\Big], \tag{18}$$

where

$$e_1' = \left(\frac{-2C_2Y_1}{x_{20}}\right)^{1/2}, \quad e_2' = \left(\frac{Y_2}{x_{20}}\right)^{1/2},$$
$$s_1' = \left(\frac{-2C_2z_1}{x_{20}}\right)^{1/2}, \quad s_2' = \left(\frac{z_2}{x_{20}}\right)^{1/2}, \tag{19}$$

and the time scale has been changed by the factor $(\mu_2^2 m_2^3/8x_{20}^2)$. The constants that appear in (17), (18), and (19) are

$$x_{20} = \frac{-m_2\mu_2^{2/3}}{m_1(2\mu_1)^{2/3} + m_2(\mu_2)^{2/3}} x_1, \quad \alpha_0 = \left(\frac{\mu_1}{4\mu_2}\right)^{1/3},$$

$$C_1 = (1 - C_2), \qquad C_2 = -\frac{m}{m'}\left(\frac{(M + m')^2}{4M(M + m + m')}\right)^{1/3},$$

$$F_{02} = 3(1 - C_2), \qquad F_{03} = 4(1 - C_2^2),$$

$$P_{00} = A_0^{(0)}, \qquad P_{01} = 2\left(1 + C_1\alpha_0 \frac{d}{d\alpha_0}\right) A_0^{(0)},$$

$$P_{10} = [A_1^{(0)} + A_2^{(0)}], \quad P_{30} = [4A_0^{(2)} + A_1^{(2)}],$$

$$P_{31} = \left[(2 + \tfrac{1}{2}C_2) + 2C_1\alpha_0 \frac{d}{d\alpha_0}\right] [4A_0^{(2)} + A_1^{(2)}],$$

$$P_{40} = 3A_0^{(1)} + A_1^{(1)} - 4\alpha_0,$$

$$P_{50} = 22A_0^{(4)} + 7A_1^{(4)} + A_2^{(4)},$$

$$P_{41} = \left(\frac{5}{2} + 2C_1\alpha_0 \frac{d}{d\alpha_0}\right) (3A_0^{(1)} + A_1^{(1)}) - 4\alpha_0 \left(\frac{9}{2} - 2C_2\right),$$

$$P_{60} = 19A_0^{(2)} + 7A_1^{(2)} + A_2^{(2)},$$

$$P_{70} = 21A_0^{(3)} + 7A_1^{(3)} + A_2^{(3)},$$

$$P_{80} = A_0^{(1)} - A_1^{(1)} - A_2^{(1)}, \quad Q_{10} = B_0^{(1)},$$
$$Q_{30} = B_0^{(3)},$$

where the $A_p^{(i)}$ and $B_p^{(i)}$ are given functions of the Laplace coefficients

$$A_p^{(i)} = \frac{\alpha_0^p}{p!} \frac{d^p}{d\alpha_0^p} b_{1/2}^{(i)}(\alpha_0), \quad B_p^{(i)} = \frac{\alpha_0^{p+1}}{p!} \frac{d^p}{d\alpha_0^p} b_{3/2}^{(i)}(\alpha_0).$$

The most important critical arguments in Equations (7) are $(\theta + \tilde{\omega}_1)$ and $(\theta + \tilde{\omega}_2)$ since they appear in the first order of the small parameter. The critical arguments involving the ascending nodes appear in the order 3/2 of the small parameter. Therefore, in this paper we consider only the resonance of the eccentricity type. The method of Hori (1966) applied to this system leads to the auxiliary system of equations

$$\frac{d}{d\tau}(x, y_j, z_j) = \frac{\partial F_1}{\partial(\theta, \tilde{\omega}_j, \Omega_j)}, \quad \frac{d}{d\tau}(\theta, \tilde{\omega}_j, \Omega_j) = -\frac{\partial F_1}{(x, y_j, z_j)}. \qquad (20)$$

The solution of the Hori auxiliary system generates a new intermediate orbit that includes the effects of the resonance of eccentricity type.

4. THE INTEGRABILITY OF THE HORI AUXILIARY SYSTEM

Since F_1 is independent of z_j, Ω_j, it follows that

$$z_j = \text{constant}, \quad \Omega_j = \text{constant}, \quad (j = 1, 2) \qquad (21)$$

and the system may be reduced to three degrees of freedom. Note that besides the energy integral

$$F_1 = \text{constant}, \qquad (22)$$

we also have the first integral

$$x - y_1 - y_2 = -(G_1 + G_2) + x_1 + \tfrac{1}{2}L_{20} = \text{constant}, \qquad (23)$$

where G_1, G_2 are the angular momenta (of the Keplerian motions) defined by Equations (1).

In order to study this reduced system in the neighbourhood of $y_j = 0$, we introduce a set of non-singular variables defined by

$$h_j = e_j' \sin(\theta + \tilde{\omega}_j), \quad k_j = e_j' \cos(\theta + \tilde{\omega}_j), \qquad (24)$$

$$\xi = \frac{x}{x_{20}}, \quad \Lambda = \frac{F_{02}}{x_{20}}(\tau - \tau_0), \qquad (25)$$

and the constants

$$D_1 = \frac{m'}{M}\frac{P_{30}C_2}{F_{02}}, \quad D_2 = \frac{m'}{M}\frac{P_{40}}{2F_{02}}. \qquad (26)$$

It must be emphasized that $D_1 \neq 0$ and $D_2 \neq 0$ except if $m' = 0$ or $m = 0$.

This new set of variables is not canonical, and the Hori auxiliary system of equations becomes

$$\frac{d\xi}{d\Lambda} = \frac{D_1}{C_2} h_1 - 2D_2 h_2, \qquad \frac{d\theta}{d\Lambda} = -2\xi,$$
$$\frac{dh_j}{d\Lambda} = -D_j - 2\xi k_j, \qquad \frac{dk_j}{d\Lambda} = 2\xi h_j, \qquad (j = 1, 2). \tag{27}$$

From this system it is easy to obtain the integrals

$$D_2 h_1 - D_1 h_2 = \rho \sin(\theta + \theta_1), \qquad D_2 k_1 - D_1 k_2 = \rho \cos(\theta + \theta_1), \tag{28}$$

where ρ, θ_1 are two new constants of integration.

By introducing the variables

$$H = \frac{D_1}{C_2} h_1 - 2D_2 h_2, \qquad K = \frac{D_1}{C_2} k_1 - 2D_2 k_2, \tag{29}$$

and the positive constant (since $C_2 < 0$)

$$D = -\frac{D_1^2}{C_2} + 2D_2^2 = \left(\frac{m'}{M}\right)^2 \frac{P_{40}^2 - 2P_{30}^2 C_2}{2F_{02}^2}, \tag{30}$$

Equations (27) become

$$\frac{d\xi}{d\Lambda} = H, \qquad \frac{d\theta}{d\Lambda} = -2\xi, \qquad \frac{dH}{d\Lambda} = D - 2\xi K, \qquad \frac{dK}{d\Lambda} = 2\xi H, \tag{31}$$

and the first integrals (22) and (23) become

$$\xi^2 - K = E, \qquad H^2 + K^2 = 2D(\xi + G), \tag{32}$$

where E is the energy constant and

$$G = \frac{1}{x_{20}} (G_1 + G_2) - \frac{x_1}{x_{20}} + \frac{1}{DC_2} \rho^2 + 1,$$

is a constant related to the angular momenta of the undisturbed Keplerian motions.

The dynamical system defined by Equations (31) has only one critical argument, the polar angle σ associated to the rectangular coordinates H, K (see Equation (38)), while the dynamical system defined by Equation (20) had two critical arguments: $\theta + \tilde{\omega}_1$ and $\theta + \tilde{\omega}_2$.

An easy manipulation of Equations (31) and (32) leads to the differential equation

$$\frac{d\xi}{d\Lambda} = \pm \sqrt{-P(\xi)}, \tag{33}$$

where $P(\xi)$ is a 4th degree polynomial

$$P(\xi) = \xi^4 - 2E\xi^2 - 2D\xi + (E^2 - 2DG). \tag{34}$$

Once, Equation (33) is solved, the solution of the auxiliary system (31) is obtained through

$$K = \xi^2 - E, \quad H = \pm\sqrt{-P(\xi)}, \quad \theta = \theta_0 - 2\int_0^{\Lambda} \xi \, d\Lambda, \tag{35}$$

with four integration constants E, G, θ_0, τ_0. The solutions of the auxiliary system (27) obtained from Equations (28) and (29) are given by

$$h_j = S_j H + T_j \sin(\theta + \theta_1), \quad k_j = S_j K + T_j \cos(\theta + \theta_1), \tag{36}$$

where

$$S_j = -\frac{D_j}{D}, \quad T_1 = 2\frac{D_2}{D}\rho, \quad T_2 = \frac{D_1}{DC_2}\rho, \tag{37}$$

and have six integration constants E, G, ρ, θ_0, θ_1, τ_0. This completes the solution of the Hori auxiliary system (20) in non-singular variables.

Because of the definitions of D_j and D, the quantities S_j and T_j are of the order of the inverse of the small parameter and, apparently the solutions (36) have small divisors. However, from the hypotheses (13) and (15), ξ and $e_j'^2$ are quantities of the order of the square root of the small parameter; as a consequence E is of the order of the small parameter, G is of the order of the square root of the small parameter, and ρ is of the order of the small parameter raised to the power 3/2. Therefore, h_j and k_j are finite quantities without small divisors.

5. THE MOTION IN THE PLANE (k, h)

A new intermediate orbit can be obtained from the Hori auxiliary system, which includes the resonance effects due to the most important terms in the disturbing function. This orbit is likely to be a better intermediate orbit than a pair of Keplerian orbits, for the construction of a formal theory of the motion of a 2-planet system where the 2:1 commensurability takes place.

In the plane (k, h) these orbits are similar for either $j = 1$ or $j = 2$, except for the values of the constants S_j and T_j; as a consequence both cases may be analysed together and the subscripts are dropped out from now on.

The orbits in the plane (k, h) are a composition of the orbits in the plane (K, H), given by Equation (35), and of the circles $\{T\cos(\theta + \theta_1), T\sin(\theta + \theta_1)\}$, cf. Equation (36). It is worthwhile mentioning that the orbits (SK, SH) are the same given by Equation (35) multiplied by the scale factor $|S|$ and reversed with respect to axis $K = 0$ when $S < 0$. We always have $S \neq 0$ since $S = 0$ implies $D_1 = 0$ or $D_2 = 0$, which we have ruled out in Section 4. Therefore, in the subsequent analysis we will consider S as equal to 1. The composition of the orbits is shown in Figure 4.

We emphasize that while S is a function of the physical parameters (masses), T is an integration constant.

We may introduce the polar coordinates

$$K = e'_0 \cos \sigma, \quad H = e'_0 \sin \sigma, \tag{38}$$

associated with the cartesian coordinates (K, H). When $T = 0$ the motion in the plane (k, h) reduces to the first component and we have $e' = e'_0$ and $\sigma = \theta + \tilde{\omega}$.

The other oscillation is given by a circle of radius T and does not present any special feature to be emphasized. $\theta(\Lambda)$ will be composed of a progressive part with half mean motion equal to the average of $\xi(\Lambda)$, plus periodic oscillations. Therefore the angle $\theta + \theta_1$ will always circulate.

6. THE ROOTS OF THE POLYNOMIAL $P(\xi)$

In order to solve the Hori auxiliary system one must solve Equation (33). Since $P(\xi)$ is a fourth-degree polynomial with two arbitrary constant E and G, the solution of Equation (33) will depend essentially on the value of E and G.

The theory about the roots of an algebraic equation of the 4th degree, developed in the XVIth Century by Ludovico Ferrari, may be applied to the study of the polynomial $P(\xi)$. This study is rather cumbersome while not involving any theoretical difficulty. The main results (Sessin, 1981) are listed in the Appendix. These results may be described taking into account Figure 2. In that figure the plane (E, G) is divided in three regions. The nature of the roots of $P(\xi) = 0$ is different for each region of the plane (E, G) and the transition from one to the other occurs at the transition curves ε_i $(i = 1, 2, 3)$ where multiple roots take place.

Once the nature of the roots of $P(\xi) = 0$ is known, the explicit form of the solutions may be obtained for every region of the plane (E, G). When $P(\xi)$ has only single roots and elliptic integrals are needed, they could be obtained directly from Byrd and Friedman (1971). Otherwise, when multiple roots occurs and common integrals are needed, any usual handbook of integrals could be used to obtain these solutions.

For the sake of simplicity, in Figure 2, we have normalized the value of G by taking

$$G^{*3} = \frac{27D}{32},$$

as the unit.

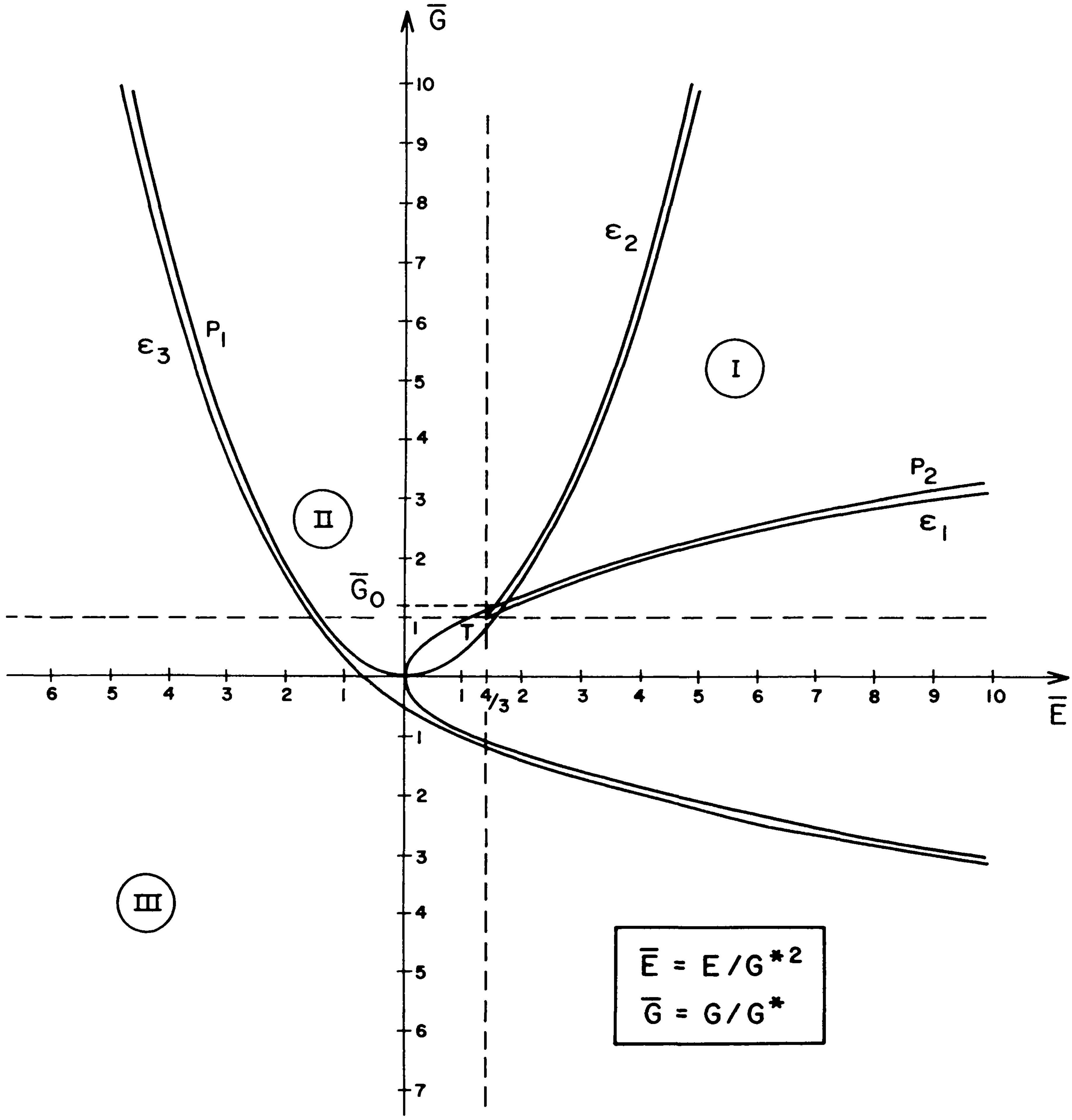

Fig. 2. Regions of the plane (E, G).

Let us now summarize the main results about the nature of the roots of $P(\xi) = 0$ and of the solutions of Equation (33).

Region III. $P(\xi) = 0$ has only complex roots. Therefore, no motion is possible for (E, G) in this region.

Curve ε_3. $P(\xi) = 0$ has one double real root and two complex roots. Equation (33) has just the stationary solution $\xi = \bar{\xi}_1 = \text{constant}$, which corresponds to a stable periodic orbit of the Hori auxiliary system.

Region II. $P(\xi) = 0$ has two simple real roots and two complex roots. The solutions of Equation (33) are periodic and oscillate in the interval $\xi_2 \leq \xi \leq \xi_1$.

Curve ε_2. $P(\xi) = 0$ has two simple real roots and one double real root. Equation (33) has a stationary solution $\xi = \bar{\xi}_2 = \text{constant}$, which corresponds to an unstable periodic orbit of the Hori auxiliary system, and two asymptotic solutions that tend towards $\bar{\xi}_2$ when Λ tends to $\pm\infty$. These asymptotic solutions are defined in the intervals $\xi_4 \leq \xi \leq \bar{\xi}_2 \leq \xi \leq \xi_1$.

Region I. $P(\xi) = 0$ has four simple real roots. Equation (33) has two families of periodic solutions oscillating in the intervals $\xi_4 \leq \xi \leq \xi_3$ and $\xi_2 \leq \xi \leq \xi_1$, respectively.

Curve ε_1. $P(\xi) = 0$ has two simple real roots and one double real root. Equation (33) has a stationary solution $\xi = \bar{\xi}_3 = \text{constant}$, which correspond to a stable periodic orbit of the Hori auxiliary system, and a periodic solution $\xi = \xi(\Lambda)$ that oscillates in the interval $\xi_2 \leq \xi \leq \xi_1$.

Point T. Curves ε_2 and ε_1 meet at the point T where $P(\xi) = 0$ has one simple real root and one triple real root. At this point Equation (33) has the stationary solution $\xi = \bar{\xi}_2 = \bar{\xi}_3 = \text{constant}$, which corresponds to an unstable periodic orbit of the Hori auxiliary system, and an asymptotic solution defined in the interval $\bar{\xi}_2 = \bar{\xi}_3 \leq \xi \leq \xi_1$ that tends towards $\bar{\xi}_2 = \bar{\xi}_3$ when Λ tends to $\pm\infty$.

The period of the periodic solutions may be obtained immediately from Byrd and Friedman (1971) or tables of definite integrals, as functions of the roots of $P(\xi)$, that is, as functions of the constants E and G.

An atlas of the polynomial $P(\xi)$ is shown in Figure 3. For the sake of showing all distinct possibilities every region is subdivided by the straight lines $E = E^* = \frac{4}{3}G^{*2}$ (where inflexion points of $P(\xi)$ occur) and $G = G^*$, both passing through the point T.

7. THE TRAJECTORIES H(K)

The trajectories H(K) are given by

$$H(K) = \pm \sqrt{Q(K)}, \tag{39}$$

where

$$Q(K) = -P(\xi) = -K^2 \pm 2D\sqrt{K+E} + 2DG. \tag{40}$$

To Q(K) there corresponds two polynomials: $Q_+(K)$ and $Q_-(K)$, given by the positive and negative determination of $\xi = \pm\sqrt{K+E}$.

The roots of the function Q(K) are roots of the function $R(K) = Q_+(K) \cdot Q_-(K)$ and obviously they are related with the roots of $P(\xi)$. Thus, an analysis similar to that made for $P(\xi)$ in the Appendix can be made for Q(K). Also, in the Appendix, the peculiarities of Q(K) and their implications in the trajectories H(K) are pointed out.

The real roots of Q(K) (when they exist) have the following signs: $K_1 > 0 > K_3 > K_2$ while K_4 may be positive, negative or zero $(K_1 > K_4 > K_3)$. The sign of K_4 is determined by the equation $G^2 = E$, represented by the parabola P_2 in Figure 2. K_4 is positive outside and negative inside the parabola P_2. The sign of K_4 has special interest in the analysis of the types of motion

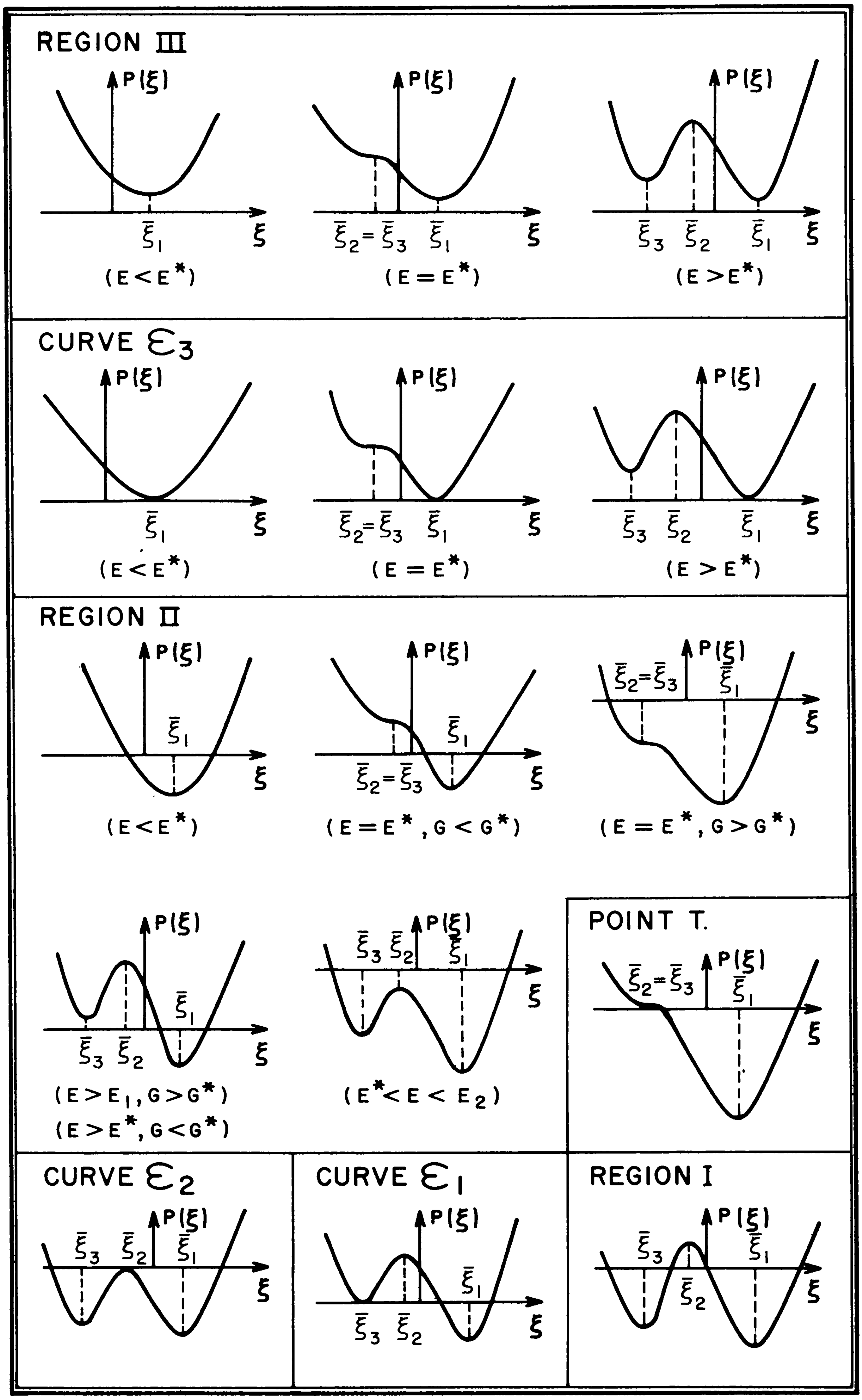

Fig. 3. Atlas of the polynomial P(ξ), (Note: the sign of ξ_2 has not been take into account.)

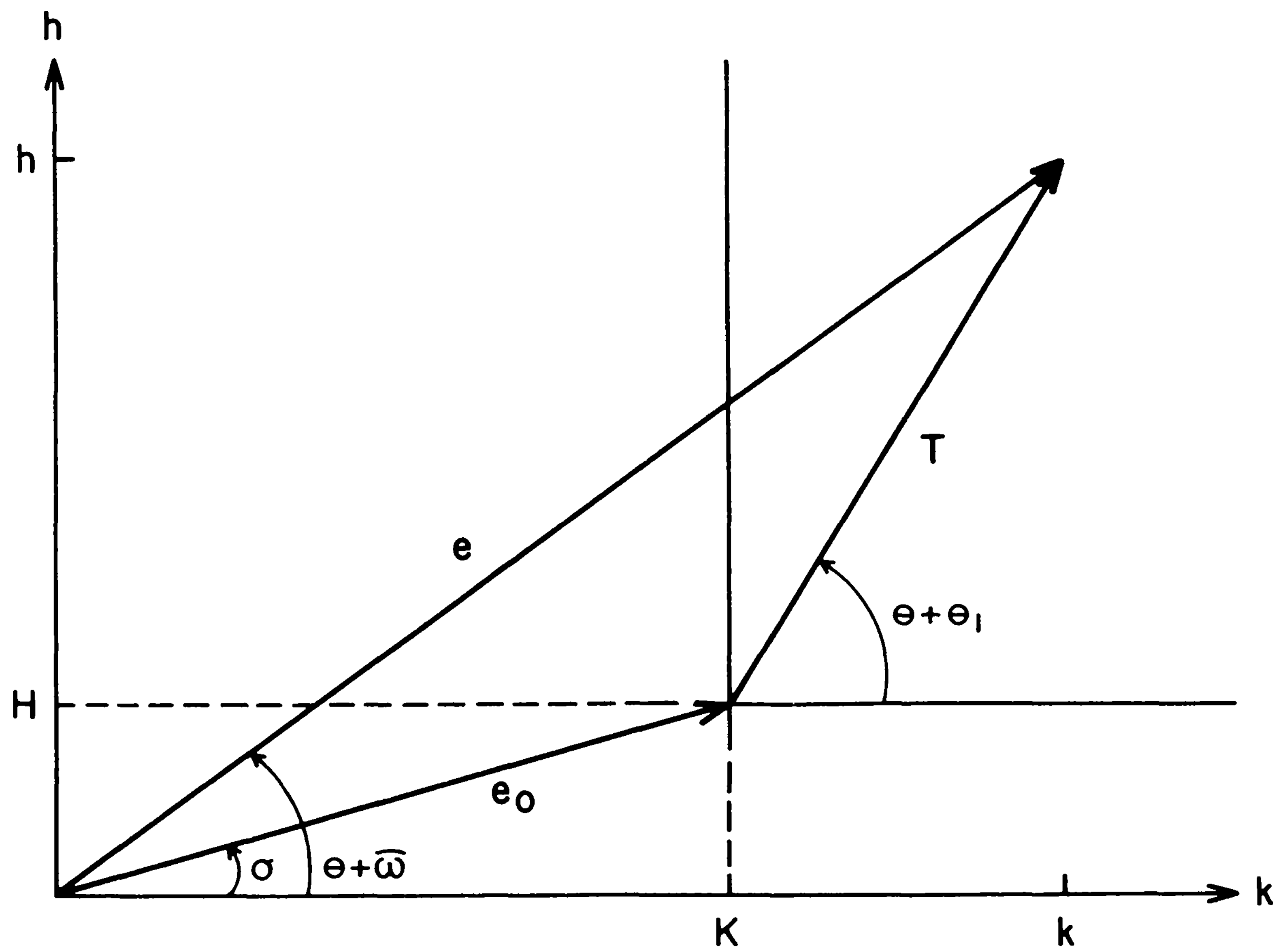

Fig. 4. Composition of the motions in the plane (k, h).

of the trajectories H(K). Note that the parabola P_2 intersect the curve ε_2. The value of G for which this intersection occurs is denoted, in Figure 2, by $\bar{G}_0$ and it is used to subdivide the plane (E, G) in order to show all possible kinds of motion.

It must be emphasized that $Q_+(K)$ corresponds to positive values of ξ and $Q_-(K)$ to negative values; the transition from Q_+ to Q_- (when it exists) occurs for $\xi = 0$, i.e., $K = -E$. The points where this transition occurs are branch points (they have a vertical tangent) and exist inside the parabola P_1 of Figure 2, determined by the equation $E^2 = 2DG$. The sign of the real roots of $P(\xi)$ (when they exist) are: $\xi_1 > 0 > \xi_3 > \xi_4$ while ξ_2 may be positive, negative or zero ($\xi_1 > \xi_2 > \xi_3$). The sign of ξ_2 is determined by the parabola P_1 and it is positive outside and negative inside this parabola. Note that at the branch point we have $\xi = 0$, that is to say, the exact commensurability of the mean motions.

The trajectories H(K) exist only when $Q(K) \geqq 0$ (i.e., $P(\xi) \leqq 0$) and are different following the values of E and G. They are shown in Figures 5 to 8. In each of these figures, all possible trajectories are obtained keeping G fixed while E varies from left to right crossing the different regions of Figure 2.

Region III. No motion is possible since Equation (33) has no real solution.

Curve ε_3. The motion only exists at the stationary solution of Equation (33) and the corresponding curve is one point, the center S_1.

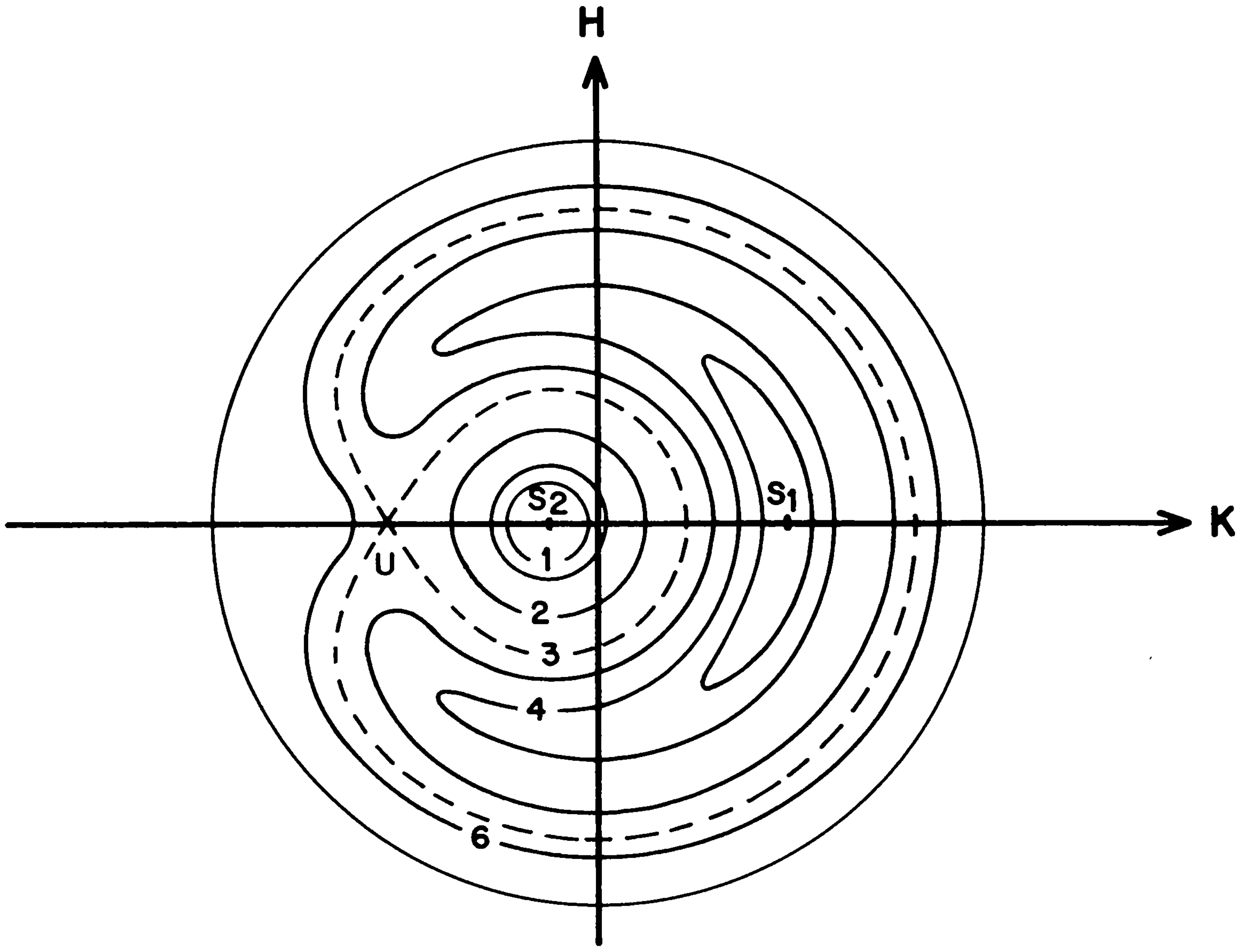

Fig. 5. $G > G_0$.

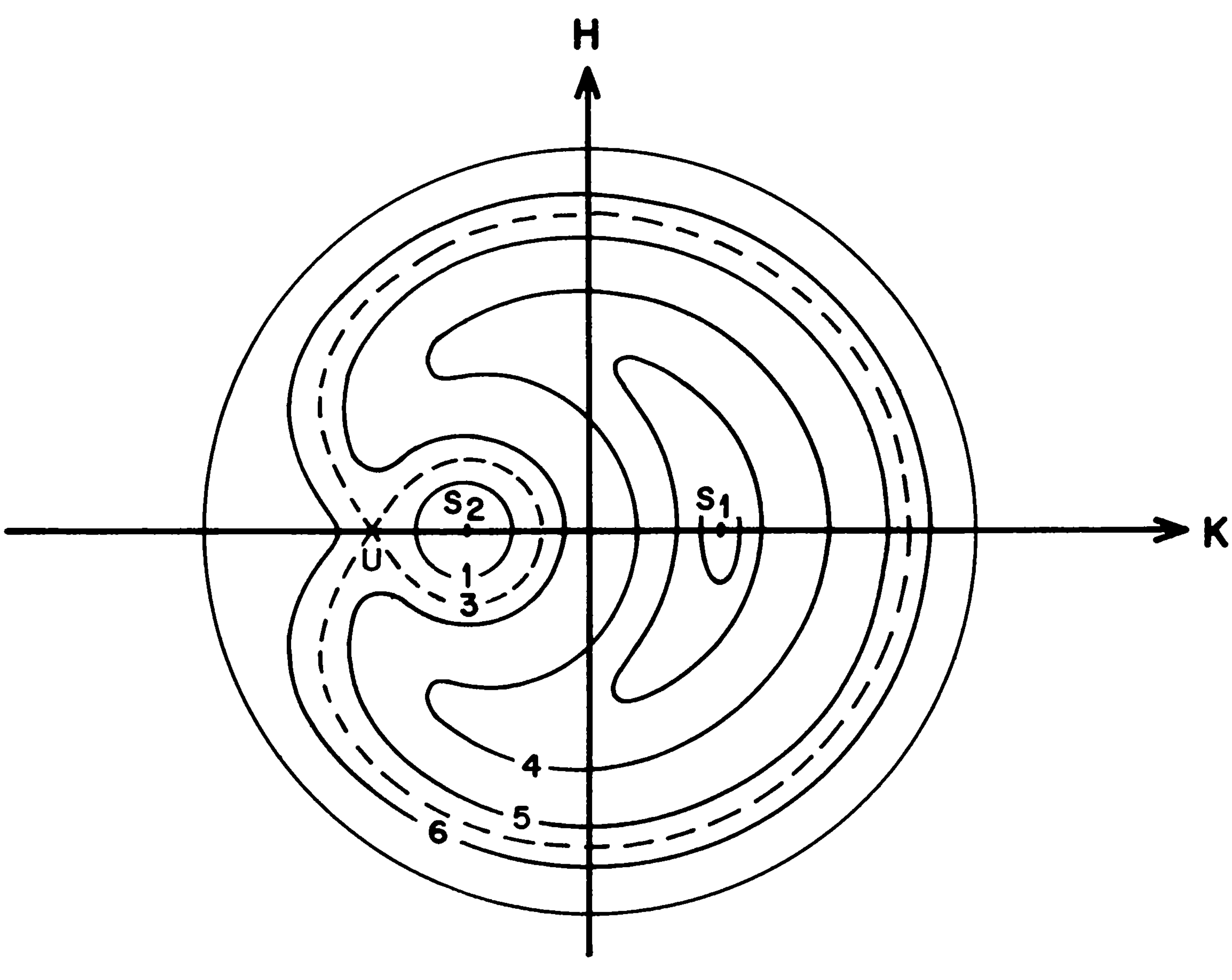

Fig. 6. $G_0 \geq G > G^*$.

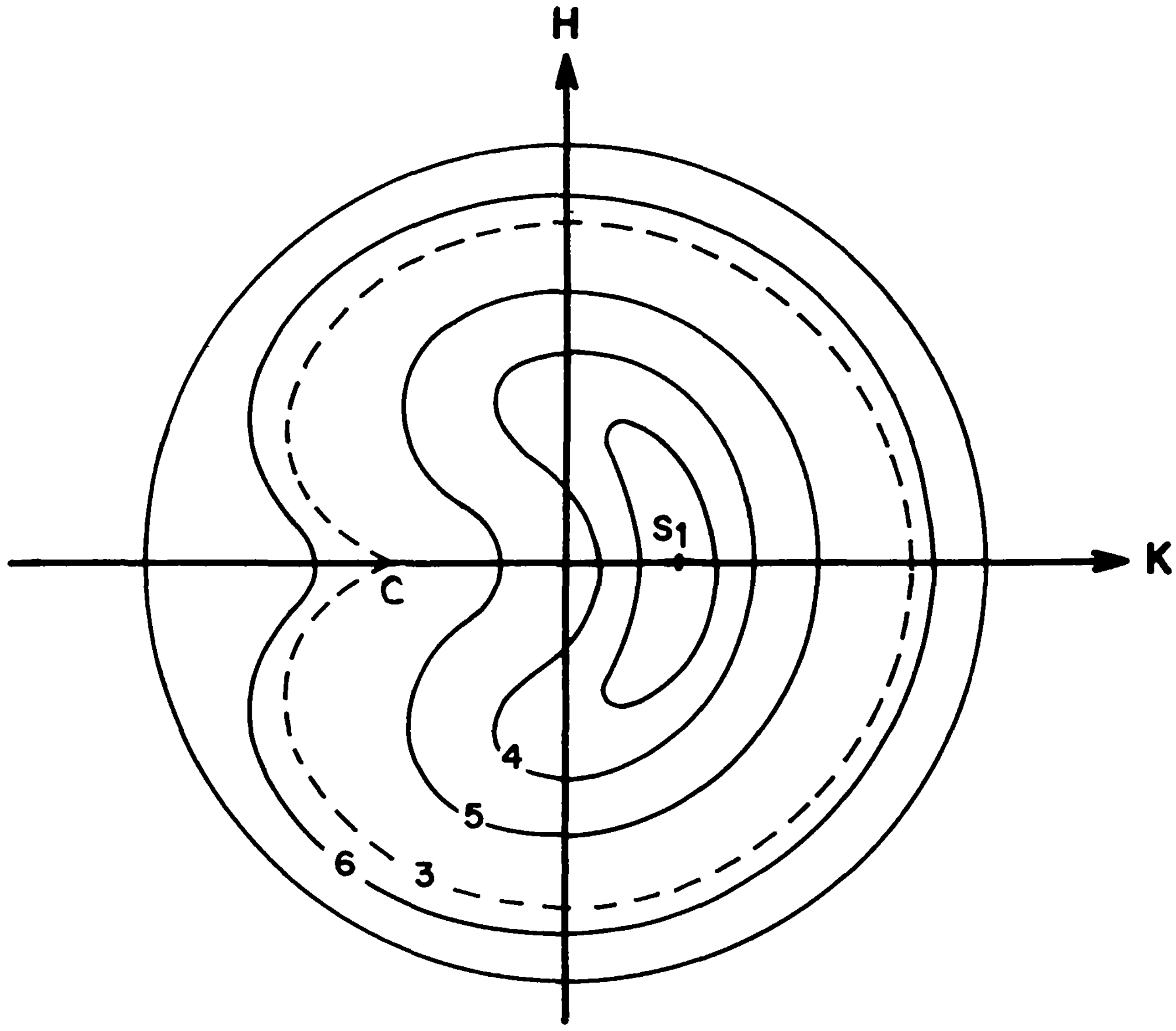

Fig. 7. G = G*.

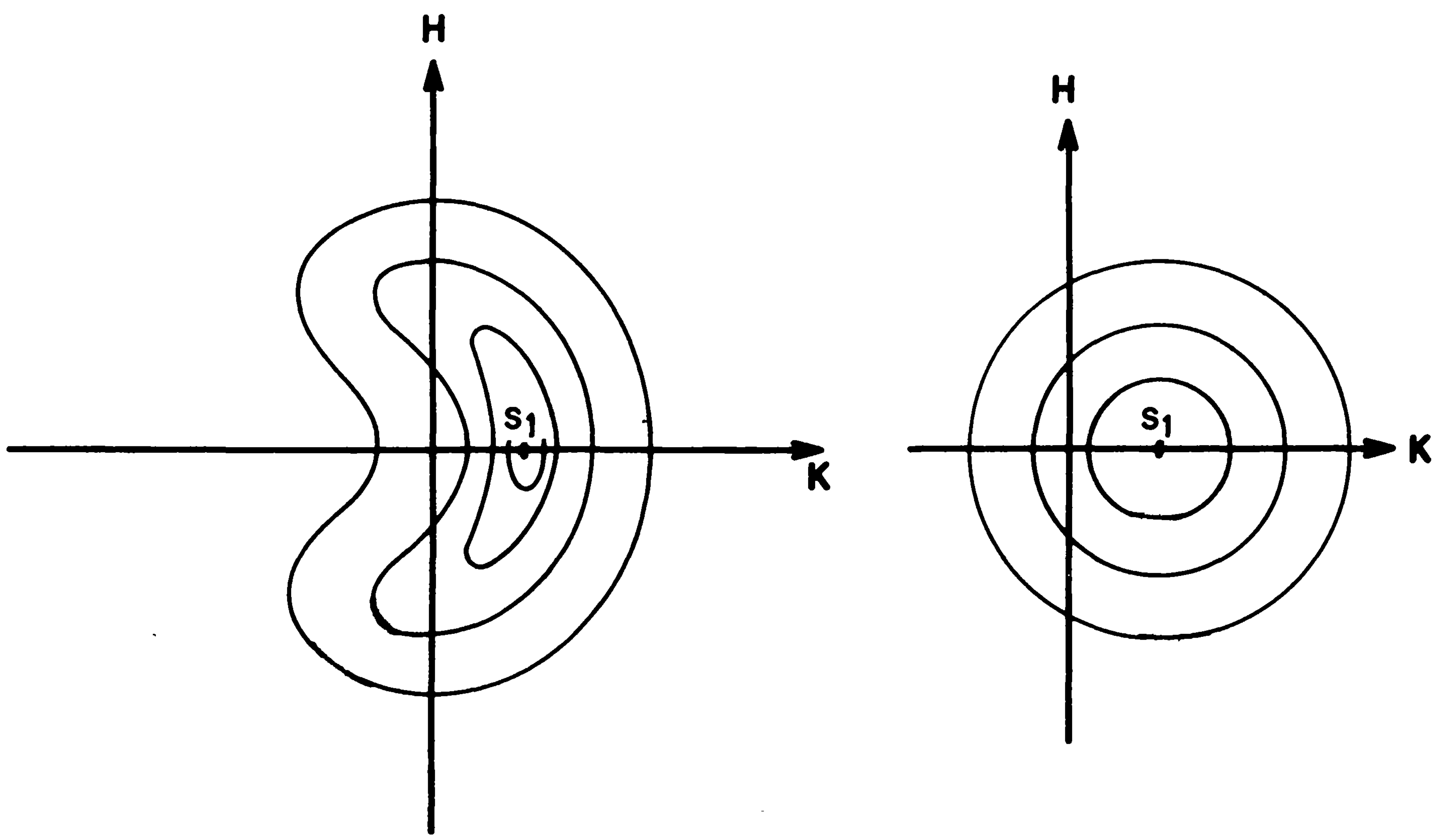

Fig. 8. G < G*.

Region II. There is only one interval in which Equation (33) has real solution. When $G \geqq G^*$ and the point (E, G) lies between curves ε_3 and ε_2, the corresponding trajectories are curves 4 of Figure 5 and curves 4 and 5 of Figures 6 and 7. But, if the point (E, G) lies at the right of the curve ε_1, the corresponding trajectories are the curves 6 of Figures 5, 6, and 7. When $G < G^*$, the trajectories are the curves shown in Figure 8. It must be emphasized that the curves that have branch points are generated inside the parabola P_1 and at the branch points the exact commensurability of mean motions occurs. Therefore, the exact commensurability does not occur neither in a neighbourhood of the center S_1 nor for curves generated by $G \leqq 0$ (as in Figure 8), but it may occur for curves of type 6 (Figures 5, 6, 7). Another noticeable feature of this region is the existence of curves of type 5 in Figures 6 and 7. Such a kind of curves exists for $G^* \leqq G \leqq \bar{G}_0$ because the straight line $G = \text{constant}$ intersect first parabola P_2, then curve ε_2, and so the real root K_4 of $Q(K)$ is negative.

Curve ε_2. There are two intervals in which Equation (33) has real solution. But these two intervals have a common limit that is the stationary solution $\bar{\xi}_2$ and it corresponds to the saddle point U in Figures 5 and 6. Inside these two intervals the trajectories are the limaçon-like separatrix curves 3. These two curves generate a motion asymptotic to the saddle point U, to which it tends as Λ tends to $\pm\infty$. Note that curve ε_2 is inside parabola P_1, therefore the external separatrix has a branch point where the exact commensurability occurs.

Region I. There are two independent intervals in which Equation (33) has real solutions. The corresponding trajectories are the curves 2, 1, and 6 of Figures 5 and 6. Curves 2 and 1 are situated inside the separatrix curve 3 and curves 6 are situated outside it. Region I has a thin subregion inside the parabola P_1; therefore, some of the curves 6 have branch points where the exact commensurability occurs.

Curve ε_1. One of the two intervals of region I becomes a point, which corresponds to a stationary solution of Equation (33). Thus, the inner curve is now one point, the center S_2. The outer curve continues as such, and the exact commensurability may exist over it. It is noteworthy that to the right of ε_1 lies a part of region II where $G \geqq G^*$ and to which corresponds only curves of type 6.

Point T. Curves ε_1 and ε_2 meet at this point and there we have $G = G^*$. The features of curves ε_1 and ε_2 act together to give rise to a separatrix with a cusp (Figure 7). Over this separatrix the exact commensurability occurs since T is inside the parabola P_1.

The difference among Figures 5 to 8 is related to the separatrix curves 3, determined by the critical curve ε_2. In Figure 5, the separatrix has two parts. When G decreases, the inner separatrix decreases. It crosses the origin and finally, for $G = G^*$, becomes a point (C). The outer separatrix still exists and has a singularity at the point C. When $G < G^*$, this singularity disappears and the separatrix ceases of being a separatrix and subsists as a regular curve.

8. TYPES OF MOTION IN THE PLANE (K, H)

The points S_1 and S_2 are centers and U is a saddle point in the plane (K, H). Therefore they correspond to equilibrium points of the motion defined in this plane and to periodic solutions of Equations (27). The point C, in Figure 7, is also an equilibrium point. It is unstable and gives rise to an unstable periodic solution of Equations (27).

The usual nomenclature used to classify the various types of motion of Figure 5 is imprecise. Here, various cases in Figures 5 and 6 will be considered and the imprecisions will be discussed.

The polar coordinates (e_0', σ) defined in Equation (38) are used and the classification is founded on the behaviour of the polar angle σ.

The curves of type 6 are called circulations since they correspond to motions that circulate around the origin of the plane (K, H). Froeschlé and Scholl (1977) call them 'outer circulations' since they are external with respect to the separatrix.

The curves of the type 2 are also circulations and are called 'inner circulations' since they are internal with respect to the separatrices.

The curves of the type 4 correspond to motions that, in non-singular resonant problems, are called librations. In these motions the angular variable σ oscillate around $\sigma = 0$ with a bounded interval of variation.

The curves of type 1 are oscillations around $\sigma = \pi$. Franklin _et al._ (1975) call them 'apocentric librations' since they oscillate around π. However, the word apocentric is not a good choice since these oscillations became pericentric when $S < 0$ (for instance in the system Io-Europa the signs of S_1 and S_2 are opposite; see Ferraz-Mello, 1979, p. 89). We call them _paradoxal librations_. Indeed, notwithstanding the fact in these curves the angle σ behaves like a libration, this set of curves is an analytic continuation of the inner circulations and form with them only one family of structurally stable curves.

Poincaré (1902) analysing a figure similar to Figure 5, that appears in the study of the motion of Hecuba, did not consider curves of type 1 as true librations. For these curves, e_0' may become zero or very close to zero and $\dot{\tilde{\omega}}$ may take finite values. Therefore, the frequency $\dot{\sigma} = \dot{\theta} + \dot{\tilde{\omega}}$ may be close to zero for finite values of $\dot{\theta} = -2\xi$ and in this situation the commensurability of mean motions is destroyed (ξ is not in a neighbourhood of zero). Poincaré considers as true librations only the curves of type 4.

Indeed, as it was discussed in Section 7, the exact commensurability of frequencies takes place at the branching of the two determinations of Q(K) (points outside the H-axis having a vertical tangent). Thus the exact commensurability never occurs for curves of type 1 (only for curves of types 4 and 6).

The curves of type 5 (Figures 6 and 7) correspond to motion where σ circulates. However they belong to the same topological family of the curves of type 4. We call them _paradoxal circulations_.

The main difference between the two types of oscillatory motions called librations in these figures is the existence or non existence of a bifurcation. For curves of the types 1 and 2 of Figure 5 (as for curves of types 4 and 5 of Figures 6 and 7) no bifurcation exists between trajectories with oscillatory or circulatory behaviour and they form a single family of trajectories. On the other hand, the oscillatory motions of the type 4 in Figure 5 and of type 1 in Figure 6, are isolated from circulation (motions of type 6) by a bifurcation and form distinct families of trajectories. Wisdom (1980) considers the curves that pass throught the origin also as separatrices; these separatrices isolate different kinds of motion inside the same continuous family of trajectories.

When a bifurcation separating libration and circulation does not exist, these two types of motion differ by elements that are not intrinsic to the curves themselves, that is to say, the origin of the coordinate system. However a translation in the coordinate system that alters the oscillatory or circulatory character of a given trajectory is usually not considered since the angular variable σ is a natural variable of the problem.

In Figure 8 the structure induced by the resonance disappear completely. No separatrix exists whatever is the value of E. The family of curves shown there, is homeomorphic to that of the undisturbed problem when no resonance exists; when $G^* > G > 0$, the exact commensurability occurs (branch points exist).

Figure 7 shows the catastrophe that separates the family of curves shown in Figures 5 and 6 from those shown in Figure 8. The inner branch of the separatrix disappeared and the center S_2 and the saddle U coalesced into the cusp C.

Sessin and Tsuchida (1983) made an application of the theory developed here to the Uranus-Neptune system. For this system, they calculated the periods of all types of motion that appear in Figures 5 to 8. Their results show period similarity for the motions around the centers S_1 and S_2.

9. TYPES OF MOTION IN THE PLANE (k, h)

The complete solution given by Equations (36) may now be analysed. Let ρ be different from zero, i.e., $T \neq 0$. First of all, let it be said that in general the motion given by Equations (36) is not periodic. Indeed the period of the motion in the plane (K, H) is given by

$$P_1 = 2 \int_{R_2}^{R_1} \frac{d\xi}{\sqrt{-P(\xi)}}, \tag{41}$$

where $R_2 \leq \xi \leq R_1$ and R_1, R_2 are two simple real roots of $P(\xi)$. For each region of the plane (E, G) of Figure 2, this period may be obtained explicitly as a function of the constants E and G. Besides this period, there is also the period of $\theta(\Lambda)$ in the solutions of the Hori auxiliary system.

It is given by

$$P_2 = \frac{\pi}{<\xi>}, \tag{42}$$

where $<\xi>$ is the average of ξ over the period P_1. These two periods are generally not equal and as a consequence the resulting solutions of Equations (36) are not periodic.

The type of motions in the plane (k, h) will depend on the relative magnitude of $e_0'(\Lambda)$ and T. Three possibilities may occur for each trajectory in the plane (K, H). We consider $T \neq 0$, since for $T = 0$ the trajectories in the plane (k, h) are the same as in the plane (K, H).

If $e_0'(\Lambda) > T$, for all Λ, the trajectory (k, h) is contained in a strip of width 2T around the curve (K, H). Therefore, the resulting trajectory (k, h) has the same behaviour as the trajectory (K, H).

If $e_0'(\Lambda)$ may be greater, less or equal to T depending on the value of Λ, the resulting trajectory (k, h) may have a different behaviour and the alternation between motions that have a libratory-like behaviour and motions that have a circulatory-like behaviour is possible. It must be remarked that this behaviour is due to an harmonic oscillation around a motion of a fixed type, allowing the trajectory to pass in either sides of the origin of the coordinate system. It is not to be confounded with similar results found by Sinclair (1972) and by Froeschlé and Scholl (1977). In these papers perturbations not taken into account in the construction of the Hori auxiliary system, were considered. Indeed when these perturbations are considered G is no more constant with respect to Λ (Sessin, 1983), and its variation may allow the separatrix to be crossed by the actual trajectories that will then be allowed to have the libratory-circulatory behaviour found by these authors.

If $e_0'(\Lambda) < T$, for all Λ, the behaviour of the trajectory (k, h) is determined by the circle of radius T. In this case, the resulting trajectory (k, h) will be always circulatory whatever is the type of motion of the angle σ, since the angle $\theta + \theta_1$ always circulates.

10. CONCLUSIONS

It was shown that the problem of the motion of two planets around the Sun with periods commensurable in the ratio 2:1, in the case of small eccentricities and inclinations, studied by Hori's perturbation method, leads to a completely integrable Hori auxiliary system and therefore to a new intermediate orbit. This intermediate orbit includes the resonant effects and therefore it is better than just a pair of Keplerian orbits as a first approximation to the study of this problem. These orbits are plane (not necessary coplanar) and show two components. One of them may behave as circulation or libration following the values of two main integration constants E and G. These solutions were classified in the plane (E, G).

All discussions in this paper were limited to the functions $H(\Lambda)$, $K(\Lambda)$ or $h(\Lambda)$, $k(\Lambda)$. (Λ is the independent variable in Hori's system.) In fact the complete solution of the Hori auxiliary system given by Equations (31) needs the solution of the remaining equations

$$\frac{d\xi}{d\Lambda} = H, \quad \frac{d\theta}{d\Lambda} = -2\xi. \tag{43}$$

However, since $H(\Lambda)$ is known, these two equations may be solved by quadrature. It is noteworthy that $H(\Lambda)$ is periodic, except in the case of the asymptotic motion of the separatrices (bifurcations), and has zero average. Therefore $\xi(\Lambda)$ is also periodic and $\theta(\Lambda)$ will be composed of a progressive part with half mean motion equal to the average $\xi(\Lambda)$, plus a periodic oscillation. In each case the functions $\xi(\Lambda)$ and $\theta(\Lambda)$ may be obtained easily.

ACKNOWLEDGEMENTS

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APPENDIX

A. Introduction

In order to construct Figure 2 and obtain the results shown in Section 6 it is necessary to study the polynomial $P(\xi)$. In this appendix we give a short description of this study, as well as some properties of the functions $Q(K)$ used in Section 7, and of the trajectories $H(K)$. This study is rather cumbersome (see Sessin, 1981) while not involving any theoretical difficulty thus, we give only the main results without detailled proof. They are obtained using the classical formulae (Pincherle, 1906) for the roots of the fourth degree polynomial.

B. The Polynomial $P(\xi)$

Consider the polynomial

$$P(\xi) = \xi^4 - 2E\xi^2 - 2D\xi + (E^2 - 2DG). \tag{44}$$

The roots of the algebraic equation $P(\xi) = 0$ are given functions of the cubic resolvent

$$H(y) = y^3 + 4Ey^2 + 8DGy + 4D^2 = 0, \tag{45}$$

or of its cubic transform

$$z^3 + \left(8DG - \frac{16}{3} E^2\right) z + \left(\frac{128}{27} E^3 - \frac{32}{3} DGE + 4D^2\right) = 0, \tag{46}$$

where $y = z - 4E/3$. The roots of the cubic transform depend on the sign of the discriminant

$$\Delta_z = -1024g(E), \tag{47}$$

where

$$g(E) = E^3 - G^2E^2 - \frac{9}{4} DGE + \left(2DG^3 + \frac{27}{64} D^2\right). \tag{48}$$

We have to consider, thus, the roots of the equation $g(E) = 0$ whose discriminant is

$$\Delta_E = 8D(G^3 - G^{*3})^3, \tag{49}$$

where

$$G^{*3} = \frac{27D}{32}. \tag{50}$$

The equation $g(E) = 0$ has at least one real root given by

$$E_3 = \begin{cases} \dfrac{G^2}{3} + \frac{2}{3}[G(G^3 + 8G^{*3})]^{1/2} \cos \frac{1}{3}(\Phi + 2\pi), & \text{if} \quad G > G^*, \\ -\frac{5}{3}G^{*2}, & \text{if} \quad G = G^*, \\ \dfrac{G^2}{3} + \frac{1}{3}\{[G^6 - 20G^3G^{*3} - 8G^{*6} + \frac{3}{2}\sqrt{-3\Delta_E}]^{1/3} + \\ \quad + [G^6 - 20G^3G^{*3} - 8G^{*6} - \frac{3}{2}\sqrt{-3\Delta_E}]^{1/3}\}, & \text{if} \quad G < G^*, \end{cases} \tag{51}$$

and whose sign is opposite to the sign of $(G^3 + G^{*3}/4)$. Two others positive real roots exist when $G > G^*$, which are given by

$$\begin{aligned} E_1 &= \frac{G^2}{3} + \frac{2}{3} [G(G^3 + 8G^{*3})]^{1/2} \cos \tfrac{1}{3}\Phi, \\ E_2 &= \frac{G^2}{3} + \frac{2}{3} [G(G^3 + 8G^{*3})]^{1/2} \cos \tfrac{1}{3}(\Phi - 2\pi), \end{aligned} \tag{52}$$

with $E_3 < E_2 < E_1$. For $G = G^*$, we have the double root

$$E^* = E_1 = E_2 = \tfrac{4}{3}G^{*2}. \tag{53}$$

The angle Φ is defined by the equations

$$\cos\Phi = \frac{G^6 - 20G^3G^{*3} - 8G^{*6}}{[G(G^3 + 8G^{*3})]^{3/2}},$$
$$\sin\Phi = \frac{2048}{9}\left[\frac{G^{*3}(G^3 - G^{*3})}{G(G^3 + 8G^{*3})}\right]^{3/2} \tag{54}$$

the sign of sin Φ is chosen to be positive in order to obtain $E_3 < E_2 < E_1$.

The roots are represented by the curves ε_1, ε_2, ε_3 in the plane (E, G) of Figure 2 and they divide the plane (E, G) in three regions. For each region the discriminant Δ_z has the opposite sign of the polynomial g(E). Therefore, it is easy to determine the nature of the roots of the cubic transform (46) and consequently of the cubic resolvent $H(y) = 0$. However, the real roots of the algebraic equation $P(\xi) = 0$ only exist if all real roots of the cubic resolvent $H(y) = 0$ are negative (except in the case of double real roots of $H(y) = 0$). The cubic resolvent $H(y) = 0$ has always a negative real root; let it be y_1. According to Hurwitz's Theorem, the necessary and sufficient condition for the real part of y_2, y_3 to be negative is $G > 0$ and $E > D/8G$. Since the hyperbola $8EG = D$ has only one branch because of the condition $G > 0$, it lies completely in region II of Figure 2. Therefore, the sign of y_2, y_3 (when real), may be easily determined. Let us summarize these results according to the nomenclature of Figure 2.

Region III. $\Delta_z > 0$ and the cubic resolvent $H(y) = 0$ has three simple real roots $y_1 < 0$, $y_2 > 0$, $y_3 > 0$. Therefore, the equation $P(\xi) = 0$ has four complex roots.

Curve ε_3. $\Delta_z = 0$ and the cubic resolvent $H(y) = 0$ has one simple real root $y_1 < 0$ and one double real root $y_2 = y_3 > 0$. Therefore, the equation $P(\xi) = 0$ has one double real root and two complex roots.

Region II. $\Delta_z < 0$ and the cubic resolvent $H(y) = 0$ has one simple real root $y_1 < 0$ and two complex roots y_2, y_3. Therefore, the equation $P(\xi) = 0$ has two simple real roots and two complex roots.

Curves ε_2 and ε_1. $\Delta_z = 0$ and the cubic resolvent $H(y) = 0$ has one simple real root $y_1 < 0$ and one double real root $y_2 = y_3 < 0$. Therefore, the equation $P(\xi) = 0$ has two simple real roots and one double real root.

Point T. $\Delta_z = 0$ and the cubic resolvent $H(y) = 0$ has one triple real root $y_1 = y_2 = y_3 = -\tfrac{4}{3}E^* < 0$. Therefore, the equation $P(\xi) = 0$ has one simple and one triple real root.

Region I. $\Delta_z > 0$ and the cubic resolvent $H(y) = 0$ has three simple negative real roots y_1, y_2, y_3. Therefore, the equation $P(\xi) = 0$ has four simple real roots.

The sign of the roots of $P(\xi)$ (when they are real) are: $\xi_1 > 0$, $\xi_4 < \xi_3 < 0$; ξ_2 has the same sign as $(E^2 - 2DG)$. The parabola $E^2 = 2DG$ is denoted by P_1 in Figure 2. ξ_2 is positive outside this parabola and negative inside.

In order to construct the atlas of the polynomial $P(\xi)$ that is shown in Figure 3, we need to know the stationary points of $P(\xi)$. These points are the solutions of the algebraic equation $\partial P/\partial\xi = 0$, i.e., of the cubic equation

$$\xi^3 - E\xi - \frac{D}{2} = 0, \tag{55}$$

whose discriminant is

$$\Delta_\xi = 4(E^3 - E^{*3}), \tag{56}$$

E^* is defined in Equation (53). This equation has at least one positive real root, point of minimum of $P(\xi)$, given by

$$\bar{\xi}_1 = \begin{cases} \left(\dfrac{D}{4}\right)^{1/3} \{[1 + \sqrt{1 - (E/E^*)^3}]^{1/3} + \\ \qquad + [1 - \sqrt{1 - (E/E^*)^3}]^{1/3}\}, & \text{if } E < E^*, \\ (2D)^{1/3}, & \text{if } E = E^*, \\ \left(\dfrac{4E}{3}\right)^{1/2} \cos \frac{1}{3}\eta, & \text{if } E > E^*, \end{cases} \tag{57}$$

where

$$\eta = \cos^{-1}\left(\frac{E^*}{E}\right)^{3/2}. \tag{58}$$

For $E = E^*$, $P(\xi)$ has a point of inflexion

$$\bar{\xi}_2 = \bar{\xi}_3 = -\left(\frac{D}{4}\right)^{1/3} < 0. \tag{59}$$

It is noteworthy to mention that $P(\bar{\xi}_2 = \bar{\xi}_3)$ has the same sign as $(G^* - G)$. This fact determines the position of the point of inflexion in the plane $(\xi, P(\xi))$ for different values of G. At Point T $(E = E^*, G = G^*)$ of Figure 2, the point of inflexion is a triple root of $P(\xi)$. When $E > E^*$, there appear two others stationary points of $P(\xi)$. The point of maximum is

$$\bar{\xi}_2 = \left(\frac{4E}{3}\right)^{1/2} \cos \tfrac{1}{3}(\eta - 2\pi) < 0, \tag{60}$$

and the other point of minimum is

$$\bar{\xi}_3 = \left(\frac{4E}{3}\right)^{1/2} \cos \tfrac{1}{3}(\eta + 2\pi) < 0. \tag{61}$$

It must be emphasized that the angular coefficient of the line that joins the points $(\bar{\xi}_1, P(\bar{\xi}_1))$ and $(\bar{\xi}_3, P(\bar{\xi}_3))$ is negative and this shows that the first point of minimum is deeper than the second one.

C. The Function Q(K)

Consider the function

$$Q(K) = -P(\xi) = -K^2 \pm 2D\sqrt{K+E} + 2DG, \tag{62}$$

where

$$\xi = \pm\sqrt{K+E}. \tag{63}$$

To Q(K) there corresponds two polynomials: $Q_+(K)$ and $Q_-(K)$, given by the positive and negative determination of ξ respectively. The transition point from one polynomial to the other occurs for $\xi = 0$.

The roots of the function Q(K) are roots of the 4th degree polynomial $R(K) = Q_+(K) \cdot Q_-(K)$ and, obviously, they are related to the roots of $P(\xi)$. Therefore, a similar analysis to that made for $P(\xi)$ is made for Q(K). We will only point out some peculiarities.

The function Q(K) has double real roots $K_1' > 0$, $K_2' < 0$, $K_3' < 0$ at the curves ε_3, ε_2, ε_1, respectively, with $K_2' < K_3'$. At Point T, it has a triple real root $K_2' = K_3' < 0$. The real roots of Q(K) have monotonic behaviour for increasing values of E with fixed value of G according to the order

$$K_1 \geq K_1' \geq K_4 \geq K_3' \geq K_3 \geq K_2' \geq K_2.$$

We recall that in Region III no real root exist. In Region II, only $K_1 = (\xi_1^2 - E)$ and $K_4 = (\xi_2^2 - E)$ are real roots and in Region I, all roots $K_1 = (\xi_1^2 - E)$, $K_2 = (\xi_2^2 - E)$, $K_3 = (\xi_3^2 - E)$, $K_4 = (\xi_4^2 - E)$ are real. Note that, only K_4 may change its sign according to the values of E and G. K_4 has the same sign as $(G^2 - E)$. The parabola $G^2 = E$ is denoted by P_2 in Figure 2 and K_4 is positive outside and negative inside it. The parabola P_2 intersect the curve ε_2 for $G = G_0$ such that $E_2(G_0) = G_0^2$.

The positive and negative determination of Q(K) only exist when the real root ξ_2 is negative, i.e., inside the parabola P_1. Outside it, Q(K) is given only by one of its determinations (the positive determination in Region II and in Region I by the positive determination for $\xi_2 \leq \xi \leq \xi_1$ and by the negative for $\xi_3 \leq \xi \leq \xi_4$).

The stationary points of Q(K) are also determined by Equation (55) that gives the stationary points of $P(\xi)$. They will be also stationary points of the curves

$$H(K) = \pm\sqrt{Q(K)}. \tag{64}$$

$\bar{K}_1 = (\bar{\xi}_1^2 - E)$ is always a point of maximum of $Q_+(K)$ and, since $Q_+(\bar{K}_1) > 0$ for $E > E_3$, than it is also point of maximum of the curve H(K). Other stationary points of Q(K) only exist for $G > G^*$. $\bar{K}_2 = \bar{K}_3 = (\bar{\xi}_2^2 - E^*) = (\bar{\xi}_3^2 - E^*)$ is a point of inflexion of $Q_-(K)$ and, since $Q_-(\bar{K}_2 = \bar{K}_3) > 0$, it is a point of inflexion of the curves H(K). Note that for $G = G^*$, the point of inflexion is the triple real root $K_2' = K_3'$ of Q(K), and consequently also of the curves H(K). $\bar{K}_2 = (\bar{\xi}_2^2 - E)$ is point of minimum of $Q_-(K)$ and, since $Q_-(\bar{K}_2) > 0$ for $E < E_2$, it is point of minimum of the curves H(K) for $E^* < E < E_2$. $\bar{K}_3 = (\bar{\xi}_3^2 - E)$ is point of maximum of $Q_-(K)$ and, since $Q_-(\bar{K}_3) > 0$ for $E < E_1$, it is point of maximum of the curves H(K) for $E^* < E < E_1$. The stationary points $\bar{K}_1$, $\bar{K}_2$, $\bar{K}_3$ become double roots of Q(K) and consequently of the curves H(K) at the curves ε_3, ε_2, ε_1, respectively.

The tangents to the curves H(K) calculated at the point $(0, K_s)$ where K_s is a simple root of H(K) and at the branch point (H(-E), -E) are perpendicular to the K-axis. They are oblique and have equal inclination at $(0, K_2')$, where K_2' is a double real root of H(K). They become parallel to the K-axis at this point (cusp) when $K_2' = K_3'$ (triple real root of H(K)).

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