TESSERAL RESONANCE EFFECTS ON SATELLITE ORBITS

G. S. GEDEON*
TRW Systems, Redondo Beach, Calif., U.S.A.

(Received 26 May, 1969)

Abstract. Resonance effects on satellite orbits due to tesseral harmonics in the potential field have been studied by many authors. Most of these studies have been restricted to nearly circular 24-hour orbits and to the deep resonance regime, where there is exact commensurability between earth rotation and orbit period. Resonance effects have also been noted, however, on eccentric synchronous and subsynchronous orbits and on orbits with far from commensurate periods. These have received much less attention; the object of this paper is to study the whole spectrum of orbits with respect to resonance effects.

1. Introduction

Synchronous (24-hour) satellites are currently of great utility for communication and navigation purposes. However, the synchronous orbit is just a particular case of satellites with periods commensurate with the earth's rotation; and the special feature of such orbits is that resonances are induced by the longitude-dependent terms in the geopotential. The relevant literature on this subject is extensive [1–11].

The first studies were in connection with the drift of 24-hour satellites under the influence of the dominant longitude-dependent term, namely that associated with the ellipticity of the earth's equator. A brief description of this phenomenon will point up the basic dynamics and will set the stage for the more advanced development that follows.

Consider a satellite launched into a 24-hour circular equatorial orbit, and let us examine the motion in a frame of reference rotating with the earth. If the equatorial cross-section were circular, the force on the satellite would be central, and in a synchronous circular orbit the satellite would always be at the same geographic longitude (geostationary). In the presence of equatorial ellipticity, however, there is also a net transverse force toward the nearest long axis. From symmetry it is clear that this transverse force must vanish on the extensions of the principal axes of the equatorial ellipse, and that these constitute equilibrium positions.

For a satellite launched in a synchronous orbit at some arbitrary longitude the effective acceleration will be opposite in direction to the force, and the satellite will move toward the minor axis. This is another case of the 'satellite paradox' which is well known for drag-perturbed satellites. The drag-perturbed satellites accelerate (because they fall in radially), the synchronous satellites decelerate under the forces F shown in Figure 1a because of an outward movement. The resulting motion will be a long-period (2 years and up) libration of the satellite about the nearest stable

* Senior Scientist, Software and Information Systems Division.

equilibrium position (Figure 1b). In an inclined orbit the satellite's apparent motion to an earth-fixed observer will be a diurnal figure-eight pattern, symmetrical with respect to the equator; while the node will exhibit the long-period motion in longitude as displayed in Figure 1b.

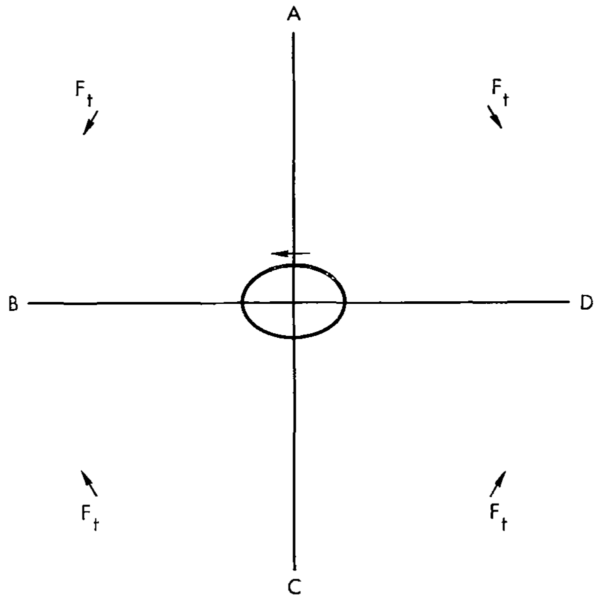


Fig. 1a. Transverse force (F_t) on satellite, stable equilibrium position at A and C, unstable ones at B and D (equatorial ellipticity exaggerated). Coordinate system rotating with earth.

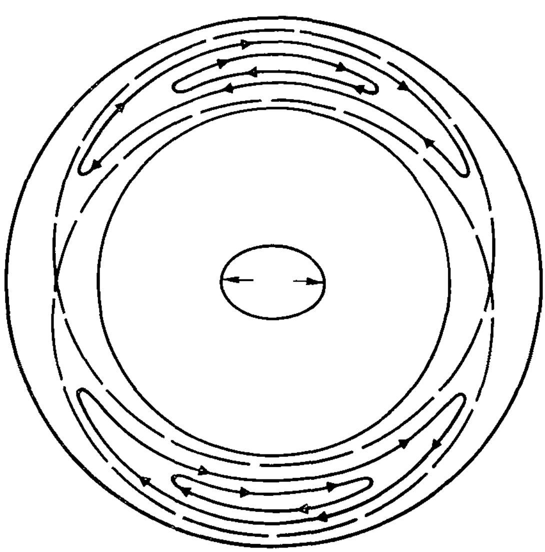


Fig. 1b. Paths of geostationary orbits in coordinate system rotating with earth (equatorial ellipticity exaggerated).

In cases where the commensurability is not exact, the longitude of the ascending node has a secular rate upon which oscillatory resonance effects are impressed. This regime has received much less attention and only Garfinkel [10] has presented a unified treatment for the whole range of resonance. He established that the full resonance solution is valid in all cases, with the classical solution matching the resonance solution asymptotically outside the resonance band. The boundaries of this resonance band are defined by Garfinkel as orbit periods for which the two solutions differ by a quantity $0(J_{lm})$, where J_{lm} is the tesseral causing the resonance.

This paper describes a different and physically more meaningful method of analyzing resonance effects and applies the method to investigation of the entire spectrum of orbit periods. In general, the method consists of developing a second-order differential equation for the variation of a longitude dependent quantity, the 'stroboscopic' mean node (defined in Section 'Commensurability') and using this equation with Lagrange's Planetary Equations to obtain the solution by starting at a point far from commensurability and gradually approaching it.

The present paper is a comprehensive study based on previous publications coauthored by my colleagues M. P. Francis, B. C. Douglas, M. P. Palmiter, and O. L. Dial [12–14]. For more technical detail on certain of the points presented, these references may be consulted.

2. Gravitational Potential in Terms of Kepler Elements

The gravitational potential V must satisfy Laplace's equation: $\nabla^2 V=0$. The standard expression of earth potential which satisfies the above equation is based on geographic polar coordinates. This form, however, is not a convenient one when applied to analytic perturbation techniques. Since Lagrange's planetary equations will be used for the study of the resonance problems (where the partials of the disturbing function with respect to the Keplerian elements are required), it would be advantageous to use a potential function which is expressed in terms of the Kepler elements. Such an expression is available in [15] where the standard potential function is transformed into the following form:

$$V = \frac{\mu}{r} + \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{q=-\infty}^{\infty} V_{lmpq} = \frac{\mu}{r} + R\,, \tag{1}$$

where

$$V_{lmpq} = \frac{\mu}{a}\left(\frac{a_e}{a}\right)^l F_{lmp}(i)\, G_{lpq}(e)\, J_{lm} \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix}_{(l-m)\ \mathrm{odd}}^{(l-m)\ \mathrm{even}}, \tag{2}$$

and

$$\psi = [(l-2p)\,\omega + (l-2p+q)\,M + m(\Omega - \theta - \lambda_{lm})]\,. \tag{3}$$

In the above equation a, e, i, Ω, ω, M are the osculating elements, $F_{lmp}(i)$ is the inclination function and $G_{lpq}(e)$ the eccentricity function. Expressions for these functions (which can be incorporated into a digital computer program) are given by Kaula in

[15]. For ease of analytical investigation, however, these are also tabulated in the same reference for $0 \leqslant l \leqslant 4$ and $-2 \leqslant q \leqslant 2$. The function $G_{lpq}(e)$ is of order $e^{|q|}$ therefore the summation for q need be over only a few values near zero for orbits of low or moderate eccentricity. The rest of the symbols appearing in Equations (1), (2) and (3) are the following: μ = gravitational constant times mass of earth; a_e = mean equatorial radius; r = position radius; θ = right ascension of Greenwich; J_{lm}, λ_{lm} = coefficient and the longitude of major axis of symmetry of the (l, m) spherical harmonic; ψ = argument of the trigonometric expressions.

LAGRANGE'S PLANETARY EQUATIONS

According to Lagrange, the rates of change of the osculating orbital elements for a disturbing potential R may be written as follows:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na} \frac{\partial R}{\partial M}, \tag{4a}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{1 - e^2}{na^2 e} \frac{\partial R}{\partial M} - \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial R}{\partial \omega}, \tag{4b}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{na^2 \sqrt{1 - e^2} \sin i} \frac{\partial R}{\partial i}, \tag{4c}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{\cot i}{na^2 \sqrt{1 - e^2}} \frac{\partial R}{\partial \omega} - \frac{1}{na^2 \sqrt{1 - e^2} \sin i} \frac{\partial R}{\partial \Omega}, \tag{4d}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = - \frac{\cot i}{na^2 \sqrt{1 - e^2}} \frac{\partial R}{\partial i} + \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial R}{\partial e}, \tag{4e}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = n - \frac{1 - e^2}{na^2 e} \frac{\partial R}{\partial e} - \frac{2}{na} \frac{\partial R}{\partial a}. \tag{4f}$$

See e.g., [15].

In Equation (4f) the mean motion n may be related to a by

$$\mu = n^2 a^3 . \tag{5}$$

3. Commensurability

Resonance is due to longitude dependent tesserals and to commensurability between orbit and earth rotation periods. Its effect can be observed on all Kepler elements, though these are small. The most pronounced effect of resonance shows up on a longitude dependent quantity, the 'stroboscopic* mean node' which will be introduced now.

* The name was suggested by Dr. Boris Garfinkel.

If a number s is defined by the equation

$$s(\dot{\theta} - \dot{\Omega}) = \dot{M} + \dot{\omega}, \tag{6}$$

it is then essentially the number of node to node satellite revolutions per earth rotation. Let $s = s_0 + \Delta s$, where $s_0 = P/Q$, the ratio of two integers and $\Delta s < 1$, Equation (6) can then be rewritten as

$$\frac{1}{s_0}(\dot{M} + \dot{\omega}) - (\dot{\theta} - \dot{\Omega}) = \dot{\lambda}_N, \tag{7}$$

with

$$\dot{\lambda}_N \equiv \frac{\Delta s}{s_0}(\dot{\theta} - \dot{\Omega}). \tag{8}$$

Integration of Equation (7) leads to

$$\lambda_N = \frac{1}{s_0}(M + \omega) - (\theta - \Omega). \tag{9}$$

If $\dot{\lambda}_N = 0$, λ_N remains a constant defined by the initial conditions.* The physical interpretation of λ_N is the following. Introduce the 'mean satellite', and assume that at $t=0$, $M+\omega=0$. 'Illuminate' earth with a stroboscopic flash light and you will find the mean satellite above the equator at $\lambda_N = \Omega - \theta$ longitude. Q days later repeat the same, then if $\dot{\lambda}_N = 0$ the mean satellite will be at the same longitude, if $\dot{\lambda}_N \neq 0$ it will be $\int \dot{\lambda}_N \, dt$ away. Recapitulating above; though Equation (9) defines λ_N as a continuous function, physically λ_N can be interpreted only at integral number times Q days, thus the name stroboscopic mean node.

Since the rates $\dot{\Omega}$, $\dot{\omega}$, and $\dot{M}$ do not remain constant under the action of the tesseral harmonics, it is necessary to consider the acceleration

$$\ddot{\lambda}_N = \ddot{\Omega} + \frac{1}{s_0}(\ddot{M} + \ddot{\omega}). \tag{10}$$

Designating the Keplerian elements by α_i, and their rate of change by $\dot{\alpha}_i$, where $i = 1, \ldots, 6$, the acceleration in any of the elements can be written as

$$\ddot{\alpha} = \sum_{j=1}^{6} \frac{\partial \dot{\alpha}_i}{\partial \alpha_j} \frac{d\alpha_j}{dt} + \frac{\partial \dot{\alpha}_i}{\partial t}. \tag{11}$$

Equation (11) can be rewritten as

$$\ddot{\alpha} = \frac{\partial \dot{\alpha}_i}{\partial \psi} \dot{\psi} + \sum_{\alpha_j = a, e, i} \frac{\partial \dot{\alpha}_i}{\partial \alpha_j} \dot{\alpha}_j. \tag{12}$$

* The groundtrace (initially) repeats itself.

The order of the products in the summation is $0(J_{lm}^2)$ with the notable exception of $(\partial\dot{M}/\partial a)\,\dot{a}$. This partial contains a zero order term

$$\frac{\partial\dot{M}}{\partial a}=\frac{\partial n}{\partial a}=-\frac{3}{2}\frac{n}{a}+0(J_{lm}), \tag{13}$$

which is multiplied by $\dot{a}$ as given in Lagrange's Planetary Equation. Thus

$$\ddot{M}=\frac{\partial\dot{M}}{\partial\psi}\dot{\psi}-\frac{3}{a^2}\frac{\partial V_{lmpq}}{\partial M}+0(J_{lm}^2), \tag{14}$$

and

$$\ddot{\omega}=\frac{\partial\dot{\omega}}{\partial\psi}\dot{\psi}+0(J_{lm}^2),$$

$$\ddot{\Omega}=\frac{\partial\dot{\Omega}}{\partial\psi}\dot{\psi}+0(J_{lm}^2). \tag{15}$$

A note for those who are interested in deep resonance only: they can assume that $\dot{\psi}=0$ thus only the second term of Equation (14) will appear in Equation (10). This leads immediately to Equation (22) without the term $A(m\dot{\lambda}_{\mathrm{N}}-q\dot{\omega})/n$.

Now from Lagrange's Planetary Equation for $\dot{M}$, $\dot{\omega}$, $\dot{\Omega}$ the required partials are

$$\frac{\partial\dot{M}}{\partial\psi}=\frac{1}{na^2}\left[-\frac{1-e^2}{e}\frac{G'}{G}+2(l+1)\right]V'_{lmpq}, \tag{15a}$$

$$\frac{\partial\dot{\Omega}}{\partial\psi}=\frac{1}{na^2}\frac{1}{(1-e^2)^{1/2}\sin i}\frac{F'}{F}V'_{lmpq}, \tag{15b}$$

$$\frac{\partial\dot{\omega}}{\partial\psi}=\frac{1}{na^2}\left[-\frac{\cot i}{(1-e^2)^{1/2}}\frac{F'}{F}+\frac{(1-e^2)^{1/2}}{e}\frac{G'}{G}\right]V'_{lmpq}, \tag{15c}$$

where

$$G'=\frac{\partial G_{lpq}(e)}{\partial e}\quad F'=\frac{\partial F_{lmp}(i)}{\partial i}\quad V'_{lmpq}=\frac{\partial V_{lmpq}}{\partial\psi}.$$

But ψ and $\dot{\psi}$ can be expressed in terms of λ_{N} if Equation (9) is introduced into Equation (3)

$$\psi=\left(l-2p+q-\frac{m}{s_0}\right)(M+\omega)+m(\lambda_{\mathrm{N}}-\lambda_{lm})-q\omega. \tag{16}$$

Terms resonating with an orbit are those for which

$$l-2p+q=m/s_0. \tag{17}$$

These are called critical terms and for these

$$\psi=m(\lambda_{\mathrm{N}}-\lambda_{lm})-q\omega. \tag{18}$$

Next, we introduce the expressions

$$P_{lmpq}=\frac{1}{m}\left[\frac{(a/a_e)^l}{3\,|F_{lmp}(i)\,G_{lpq}(e)|\,J_{lm}}\right]^{1/2}, \tag{19}$$

and

$$\bar{\lambda}_{lm} = \lambda_{lm} + \frac{2\pi k}{m} - \left\{ \begin{array}{ll} (1+\xi)\dfrac{\pi}{2m} & (l-m) \text{ even} \\ \xi\dfrac{\pi}{2m} & (l-m) \text{ odd} \end{array} \right\} + \frac{q}{m}\,\omega, \tag{20}$$

where

$$\xi = \left\{ \begin{array}{c} +1 \\ -1 \end{array} \right\} \quad \text{if} \quad F_{lmp}(i)\, G_{lpq}(e) \left\{ \begin{array}{c} > \\ < \end{array} \right\} 0. \tag{21}$$

It will be shown that when $n = 2\pi s_0$ (Libration regime) then P_{lmpq} is the ratio of the period of small amplitude libration and that of earth rotation, $\bar{\lambda}_{lm}$ is the longitude of a stable node. Note that due to the $(q\omega/m)$ term, the stable node in the general case rotates with angular velocity $(q\dot{\omega}/m)$ in the equatorial plane.

Figures 2 and 3 show small amplitude libration periods for several circular commensurate orbits.*

Now substituting $\ddot{\omega}$, $\ddot{\Omega}$ and $\ddot{M}$ into Equation (10) and making use of Equations (18) through (21), after some manipulation one obtains:

$$\ddot{\lambda}_N = -\frac{n^2}{s_0^2} \sum_{\text{crit.}} \left[1 + A \cdot \frac{m\dot{\lambda}_N - q\dot{\omega}}{n} \right] \frac{1}{mP_{lmpq}^2} \sin m(\lambda_N - \bar{\lambda}_{lm}) + 0(J_{lm}^2) \tag{22}$$

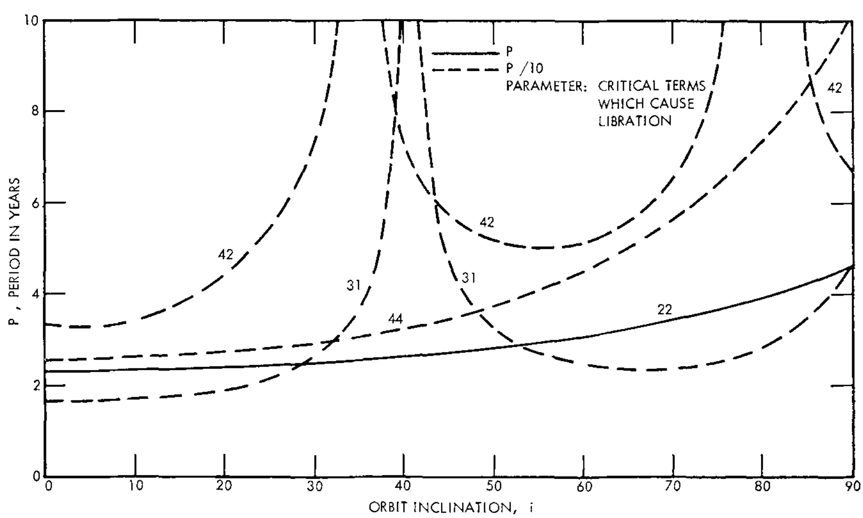


Fig. 2. Small-amplitude libration periods of synchronous circular orbits.

* Note that these and the following graphs up till Figure 9 are based on Guier and Newton's (1965) tesseral coefficients. Figures 10 and 11. however, are based on the 1966 Smithsonian coefficients.

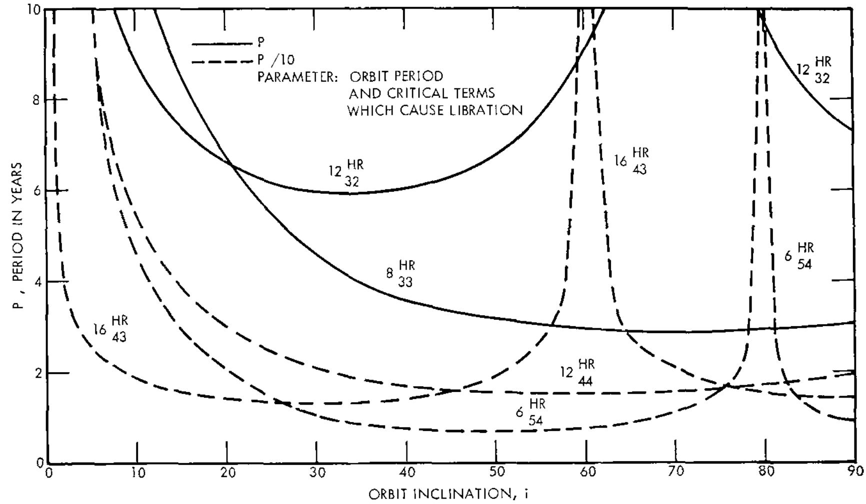


Fig. 3. Small-amplitude libration periods of circular subsynchronous orbits.

with

$$A = -\frac{s_0}{3m}\left[2(l+1) + \frac{e(1-e^2)^{1/2}}{1+(1-e^2)^{1/2}}\frac{G'}{G} + \frac{s_0 - \cos i}{(1-e^2)^{1/2}\sin i}\frac{F'}{F}\right]. \tag{23}$$

No general solution to Equation (22) is known, but several approximate solutions are available for varying degrees of closeness to commensurability. The following discussion will begin with cases that are far from commensurability and gradually approach the point of strict commensurability.

4. The Classical Solution Regime

We begin with the six first-order differential equations (Equation 4) for the rates of change of the orbital elements, plus one second-order differential equation (Equation 22) for the stroboscopic mean node. In order to integrate Equations (4) we introduce the expression

$$\mathrm{d}t = (1/\dot{\psi})\,\mathrm{d}\psi \tag{24}$$

and if $\dot{\psi}$ is a constant then integration can be executed in closed form (see Equation (3.76) of [15]). This is the definition of the classical solution. Now $\dot{\psi}$ usually is constant if the only variations considered are the secular rates $\dot{M}$, $\dot{\omega}$, $\dot{\Omega}$ due to oblateness and the rotation rate $\dot{\theta}$ of the earth. The resultant perturbations of the elements are sinusoidal, with periods that are submultiples of the periods of orbital motion, earth rotation, the rotation of the line of nodes, the apsides, and any combinations of these.

Consideration of Equation (18), however, reveals that the

$$\dot{\psi} = m\dot{\lambda}_{\mathrm{N}} - q\dot{\omega} \tag{25}$$

must also be constant for a term V_{lmpq} that satisfies Equation (17). Thus new periods must be added to those already defined. These are the circulation periods *, defined by

$$P_c = 2\pi/(m\dot{\lambda}_N - q\dot{\omega}). \tag{26}$$

We next determine those conditions under which $\dot{\lambda}_N$ may be considered constant. Examination of Equation (22) reveals that when $\dot{\lambda}_{N0}$, as defined by Equation (7), is large, the rapid variation of the sine function yields negligible mean acceleration. Therefore in cases that are far from commensurability, where the circulation periods are not considerably longer than the orbit period, the variation of $\dot{\lambda}_N$ can safely be neglected.

This regime can be qualified numerically by introducing the ratio of the circulation period to the orbit period, which Garfinkel calls the resonance parameter R. For circular orbits there is only one circulation period, that associated with $q=0$. This is identical with the time it takes for the mean node to drift from one stable node to the next. Using Equation (8), the circulation period becomes

$$P_c = \frac{2\pi/m}{\dfrac{\Delta s}{s_0}(\dot{\theta} - \dot{\Omega})}. \tag{27}$$

If we define the $m=s_0$ case as 'fundamental' resonance, then $m=2s_0$ is its 'first overtone'. Higher 'overtones' have not yet been observed; even the first overtone is very weak, although it has yielded coefficients of 27th and 28th order tesserals (see [16]).

Relating this circulation period to the fundamental resonance and writing the orbit period as

$$P \cong \frac{2\pi}{(s_0 + \Delta s)(\dot{\theta} - \dot{\Omega})}, \tag{28}$$

the ratio of the two periods (the resonance parameter) becomes

$$R = P_c/P = 1 + s_0/\Delta s. \tag{29}$$

Since $\Delta s < 1$, it can be seen that for orbits corresponding to many revolutions per day, the ratio R is at least in the low tens, while for low values of s_0 the ratio can be less than 10.

This result explains the physical fact that resonance effects can be observed for all satellites making more than about 10 revolutions per day, while for those making fewer than about 10 revolutions per day it is possible to observe resonance only when

* Also often called: beat periods.

the orbit period is nearly commensurate (i.e., Δs is small). It follows that when the resonance parameter R is less than 10, resonance effects can be ignored.*

When R reaches values in the tens, however, the perturbation due to a critical tesseral builds up to appreciable amplitudes. This is the beginning of the so-called 'shallow' resonance regime where the classical small divisor problem appears (i.e., $\dot{\psi}$ in Equation (24) is small). As long as $\dot{\psi}$ remains constant, however, it is still possible to integrate Lagrange's Planetary Equations analytically, although resonance effects begin to appear as forced along-track oscillations impressed on the motion of the secularly processing ellipse. These along-track oscillations can be calculated from

$$\Delta \cong a(\Delta\omega + \Delta\Omega \cos i + \Delta M). \tag{30}$$

Evaluating $\Delta\omega$, $\Delta\Omega$ and ΔM with Equation (3.76) of [15], above becomes

$$\Delta = \frac{an}{\dot{\psi}}\left(\frac{a_e}{a}\right)^l F_{lmp}(i)\, G_{lpq}(e)\, J_{lm}\left\{2(l+1) + \frac{e(1-e^2)^{1/2}}{1+(1-e^2)^{1/2}} \cdot \frac{\partial G_{lpq}(e)/\partial e}{G_{lpq}(e)} - 3\frac{n}{\dot{\psi}}\frac{m}{s_0}\right\} \sin\psi. \tag{31}$$

Equation (31) is written for one V_{lmpq} and is directly applicable for circular orbits, since for $e=0$, q must be zero because $G_{lpq}(0)=0$ for $q \neq 0$. The quantity m is also fixed; for fundamental resonance, it is s_0, the closest integer to s, or for overtones, its multiple.

For eccentric orbits there are several sets of $lmpq$'s which produce resonance. For a single order m, a solution can be obtained using a procedure similar to that described later, with Equations (34) and (35). More detail is presented in [14].

It should be pointed out here that satellites in eccentric, non-critically inclined orbits in the shallow resonance region are ideal for geodesy purposes because of the frequency split present in the terms of Equation (31). That is, for any $\dot{\lambda}_{N0}$ the circulation periods depend on q. This was verified in [17], where those circulation periods that produce perturbations greater than 50 meters are tabulated for a great number of satellites.

5. Solution in the Circulation Regime

When the resonance parameter R is in the high tens or in the hundreds, circulation periods become considerably longer than the orbit period and the variation of $\dot{\lambda}_N$, as described by Equation (22), should not be ignored if an accurate solution is desired. For a closed-form solution to Equation (22), however, certain assumptions must be

* Incidentally the drift period expressed in days has the same numerical value as the resonance parameter. This is due to the fact that the drift period is defined as $P_d = 2\pi/\dot{\lambda}_N$, thus $P_d = mP_c$. If we express the orbit period as $P \cong P_E/s_0$ where P_E is the earth rotation period, then

$$R = P_c/P = \frac{P_d}{m} \div \frac{P_E}{s_0} = P_d/P_E.$$

made. First we note that the resonance parameter R, which was assumed to be high in this regime, can be written as

$$R = n/(m\dot{\lambda}_{\mathrm{N}} - q\dot{\omega}) . \tag{32}$$

Since the constant A in Equation (22) is small, the term

$$A(m\dot{\lambda}_{\mathrm{N}} - q\dot{\omega})/n = \frac{A}{R} \tag{33}$$

in Equation (22) can be neglected. It will be shown later that this term becomes important only in the transition from the circulation to the libration regime; i.e., for the case of a nearly 'stalled' satellite.

The next task is to remove the summation sign from Equation (22), which will reduce it to the differential equation for an m-fold pendulum. For purposes of a mathematical analysis this could be done arbitrarily (see [13]) and the solution obtained even if $\dot{\lambda}_{lm}$ is not stationary. (In the non-stationary case the system can be made autonomous by transformation into a coordinate system rotating with angular velocity $q\dot{\omega}/m$ in the equatorial plane. The solution then turns out to be an oscillation about the mean motion of λ_{N}.) Though such solutions are quite interesting, in this paper we confine ourselves to cases which relate to the 'real world'. Analytic solutions to Equation (22) apply in two real cases which follow here.

First for the case of negligible eccentricity the number of critical terms is low, since the eccentricity function reduces the solution to the $q=0$ terms. Likewise, the stable nodes are stationary.

Second, for the case of orbits at the critical inclination all $\bar{\lambda}_m$ are also stationary. Thus, in both cases terms of the same order (m) can be combined. The period P of a small amplitude oscillation and the longitude of the stable node λ_0 for the resulting motion are then given by

$$P = \frac{1}{\left\{\left[\sum_{l,q} \frac{1}{P_{l,q}^2} \sin m\bar{\lambda}_{lm}\right]^2 + \left[\sum_{l,q} \frac{1}{P_{l,q}^2} \cos m\bar{\lambda}_{lm}\right]^2\right\}^{1/2}} \tag{34}$$

and

$$\tan m\lambda_0 = \frac{\sum_{l,q} \frac{1}{P_{l,q}^2} \sin m\bar{\lambda}_{lm}}{\sum_{l,q} \frac{1}{P_{l,q}^2} \cos m\bar{\lambda}_{lm}} . \tag{35}$$

Since one order (m) dominates in both cases; the analytic solution is realistic.

In order to avoid the complexity of the m-fold pendulum problem we will return to the variable ψ, but it must be kept in mind that the value of ψ considered will be that defined by Equations (17) and (20) as $m(\lambda_{\mathrm{N}} - \bar{\lambda}_{lm})$. Ignoring $\ddot{\omega} \ll \ddot{\lambda}_{\mathrm{N}}$, we can write

$$\ddot{\psi} + \kappa^2 \sin\psi = 0 , \tag{36}$$

where

$$\kappa = \frac{n/s_0}{P_{lmpq}} \tag{37}$$

is the circular frequency of motion.

Equation (36) has a first integral

$$\dot{\psi}^2 = C + 2\kappa^2 \cos\psi . \tag{38}$$

In the circulation regime $\dot{\psi} > 0$, hence $C > 2\kappa^2$. Equation (38) can be integrated to yield

$$t = \int \frac{\mathrm{d}\psi}{[C + 2\kappa^2 \cos\psi]^{1/2}} + \text{const.} \tag{39}$$

Equation (39) will be evaluated in two different ways. In the rapid circulation regime bordering the regime where the classical solution is applicable, an approximation for ψ may be used. In [18], ψ is expanded as

$$\psi = (ut + \varepsilon) + \frac{\kappa^2}{u^2} \sin(ut + \varepsilon) + \frac{\kappa^4}{8u^4} \sin 2(ut + \varepsilon) + \cdots, \tag{40}$$

where u, the secular rate, is related to the period P by

$$P = \frac{2\pi}{u} = \int_0^{2\pi} \frac{\mathrm{d}\psi}{[C + 2\kappa^2 \cos\psi]^{1/2}}, \tag{41}$$

and ε is a phase angle. In the rapid circulation regime $u \gg \kappa$, so that only the first two terms in Equation (40) are needed.

It is also possible to find u and ε from the initial conditions by setting Equation (18) equal to Equation (40):

$$(ut + \varepsilon) + \frac{\kappa^2}{u^2} \sin(ut + \varepsilon) = m(\lambda_{\mathrm{N}} - \bar{\lambda}_{lm}). \tag{42}$$

For convenience we can choose initial conditions such that the right-hand side of this equation is zero at $t = 0$, with the result that $\varepsilon = 0$ at the same time.

Now, differentiating Equation (42) and applying the initial conditions, we have

$$u + \frac{\kappa^2}{u} \cos(ut) \cong m\dot{\lambda}_{\mathrm{N}0} - q\dot{\omega} = \dot{\psi}_0 . \tag{43}$$

Recalling that $u \gg \kappa$, we can write $u \cong \dot{\psi}_0$. With these constants, the solution becomes

$$\psi = ut + \frac{\kappa^2}{u^2} \sin(ut). \tag{44}$$

For cases where this first approximation holds, the amplitude B of the variation in ψ is

$$B = \frac{\kappa^2}{u^2} = \left[\frac{n/s_0}{P_{lmpq}\dot{\psi}_0}\right]^2 \cong \left(\frac{P_c}{P_{lmpq}}\right)^2 . \tag{45}$$

If we return to the integration of Equations (4) and consider Equation (44), then the trigonometric term becomes

$$\sin\psi = \sin ut + \frac{\kappa^2}{2u^2}\sin 2ut + 0\left(\frac{\kappa^4}{u^4}\right). \tag{46}$$

This can be integrated to yield

$$\int \sin\psi \, \mathrm{d}t = -\frac{1}{u}\left[\cos ut + \frac{\kappa^2}{4u^4}\cos 2ut\right]. \tag{47}$$

We have already seen that u is almost the same as $m\dot{\lambda}_{\mathrm{N0}} - q\dot{\omega}$. The basic amplitude therefore does not change, but a higher frequency motion appears and produces a sharper change around the stable node and a less sharp change around the unstable node. The greatest deviation from the classical solution will occur midway between the nodes, when the displacement calculated by the circulation method is

$$\frac{1}{2}\left(\frac{P_{\mathrm{c}}}{P_{lmpq}}\right)^2 A,$$

where A is the amplitude of the Small Divisor solution.

In the slow circulation regime the period of the motion is defined by Equation (39), which can be rewritten as

$$P = 2\int_0^{\pi} \frac{\mathrm{d}\psi}{[C + 2\kappa^2 - 4\kappa^2 \sin^2\psi/2]^{1/2}}. \tag{48}$$

Introducing $\phi = \psi/2$ and

$$k = \left[\frac{4\kappa^2}{C + 2\kappa^2}\right]^{1/2}, \tag{49}$$

which is always less than unity since $C > 2\kappa^2$, the period becomes

$$P = \frac{2k}{\kappa}\int_0^{\pi/2} \frac{\mathrm{d}\phi}{[1 - k^2 \sin^2\phi^2]^{1/2}}, \tag{50}$$

or

$$P = \frac{\kappa}{\pi} P_{lmpq} K(k), \tag{51}$$

where $K(k)$ is a complete elliptic integral of the first kind. From the period the secular rate is given by

$$\bar{\dot{\phi}} = \tfrac{1}{2}\, 2\pi/P. \tag{52}$$

To find the amplitude of the periodic part of the overturning pendulum, Equation (33) should be written in the form

$$t - t_0 = \frac{k}{2\pi} P_{lmpq} \int_0^{\phi} \frac{\mathrm{d}\phi}{[1 - k^2 \sin^2 \phi]^{1/2}}. \tag{53}$$

Assuming that the inverse of the above is $\phi = f(t)$, the periodic part is

$$\Delta\phi = f(t) - \bar{\dot{\phi}} t, \tag{54}$$

which has a maximum when

$$\frac{\mathrm{d}(\Delta\phi)}{\mathrm{d}t} = f(t) - \bar{\dot{\phi}} = 0. \tag{55}$$

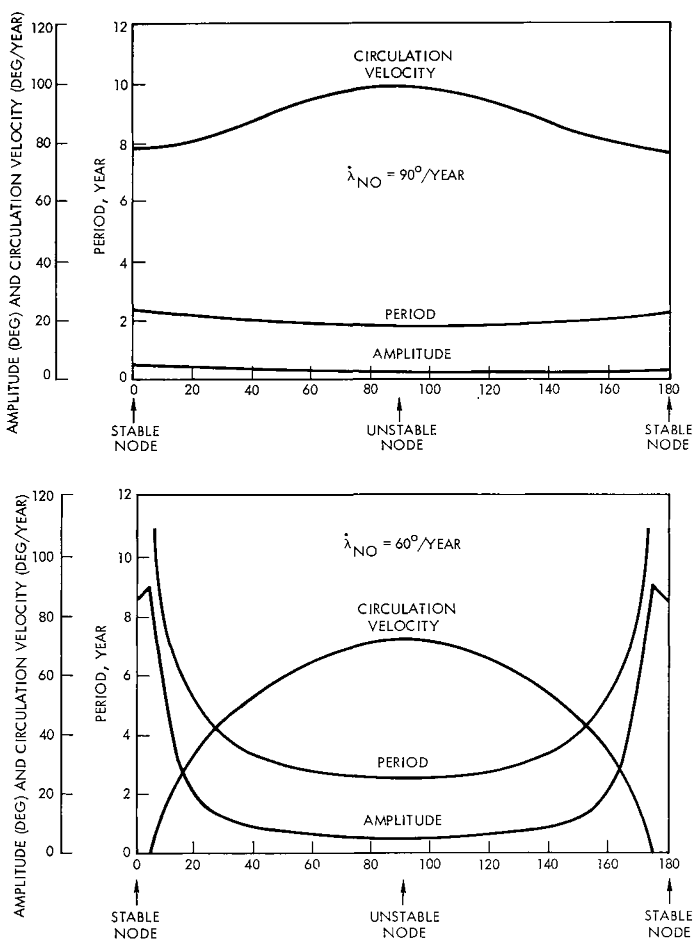


Fig. 4. Theoretical values of amplitudes, periods, and Circulation velocities of 12 hour, 30° inclined circular orbits due to V_{3210}.

Thus

$$f(t) = \frac{1}{\mathrm{d}t/\mathrm{d}\phi} = \frac{2\pi}{kP_{lmpq}}\,[1 - k^2 \sin^2 \phi_m]^{1/2} = \pi/P\,, \tag{56}$$

from which

$$\sin \phi_m = \frac{1}{k}\left\{1 - \left[\frac{\pi/2}{K(k)}\right]^2\right\}^{1/2}. \tag{57}$$

The corresponding amplitude is obtained from

$$\Delta\phi_m = \phi_m - \bar{\dot{\phi}} t_m\,. \tag{58}$$

Figure 4 shows examples of periods, amplitudes, and secular rates calculated from the above equations.

Note that in this regime (and in the libration regime to be treated next) we concentrate on the stroboscopic mean node and do not mention the variation of the other elements. This treatment is due to the fact that $\dot{\psi}$ is no longer constant and it is therefore no longer possible to integrate Lagrange's Planetary Equations analytically. If more information is required, the equations of motion must be numerically integrated. If tesserals are included, resonance effects are automatically accounted for.

Numerical integration is also required in cases of eccentric orbits with noncritical inclinations, where no analytical solution is available even for the stroboscopic mean node.

6. Solution in the Libration Regime

When the integration constant of Equation (38) is smaller than $2\kappa^2$, the satellite begins to librate. In this case $\dot{\psi}$ will vanish, as in the case of a pendulum, at its extreme elongations. For this case,

$$C + 2\kappa^2 \cos\alpha = 0\,, \tag{59}$$

where α is the amplitude. Expressing C with κ and α, Equation (38) becomes

$$\dot{\psi}^2 = 4\kappa^2(\sin^2 \alpha/2 - \sin^2 \psi/2)\,. \tag{60}$$

If we introduce the auxiliary variable ϕ' by

$$\sin\phi' = (\sin\psi/2)/(\sin\alpha/2)\,, \tag{61}$$

then Equation (60) can be integrated between fixed limits to yield

$$P = \frac{4}{\kappa}\int\limits_0^{\pi/2} \frac{\mathrm{d}\phi'}{(1 - k'^2 \sin^2 \phi')^{1/2}}\,, \tag{62}$$

where

$$k' = \sin\frac{\alpha}{2} = \left[\frac{C + 2\kappa^2}{4\kappa^2}\right]^{1/2} = \frac{1}{k}\,. \tag{63}$$

Equation (62) can also be written if $n/s_0 = 2\pi/\text{day}$ as

$$P = \frac{2}{\pi} P_{lmpq} K(k') \text{ days}, \tag{64}$$

where $K(k')$ is a complete elliptic integral of the first kind. For small amplitudes k' is small and $K(k') \simeq \pi/2$. Thus P_{lmpq} truly represents the period of small amplitude librations.

The borderline between circulation and libration is at the point where $C = 2\kappa^2$, corresponding to a 'stalled' pendulum. At this point and in its vicinity the term A/R should not be neglected, since it has an effect similar to that of drag or thrust and can modify enormously the behavior of the pendulum in this regime. No closed-form solution is available for this case, and the equations of motion must be numerically integrated.

Typical examples of libration in circular orbits are shown in Figure 5. Figure 6

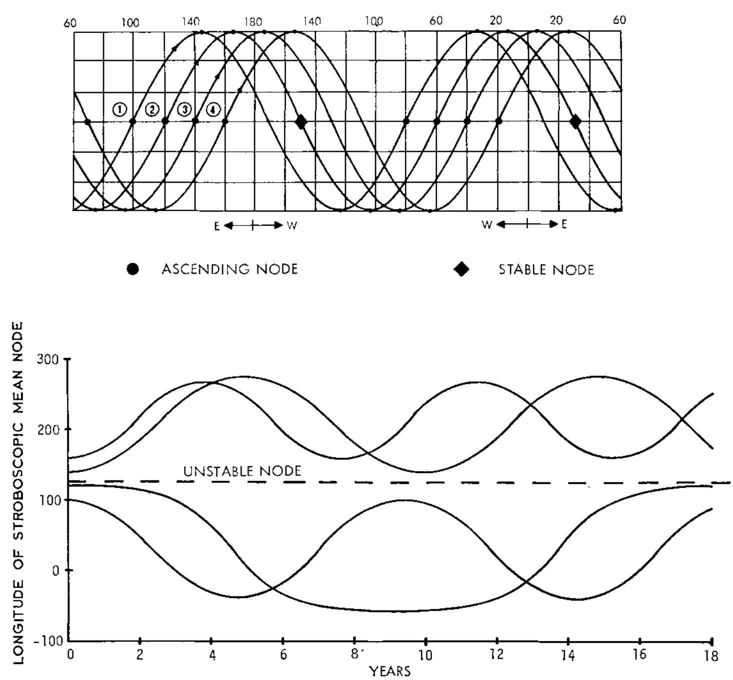


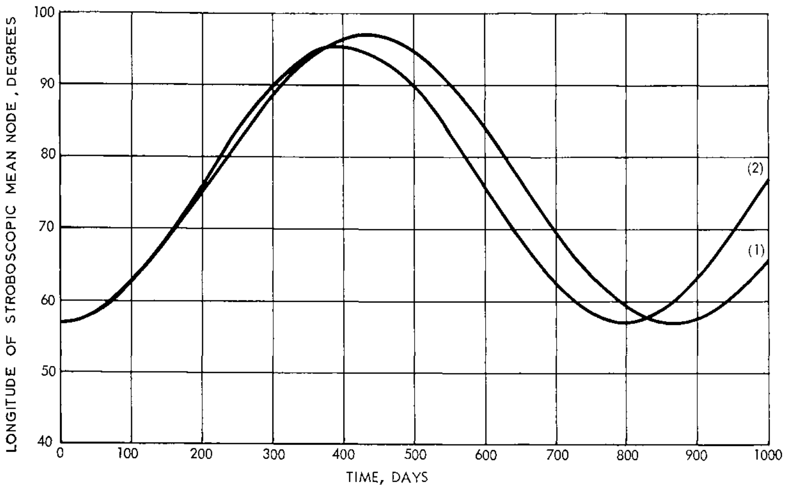
Fig. 5. Examples of libration of 12-hour 30° circular orbits.

shows the libration of a geostationary satellite under the influence of one critical tesseral and under that of three critical tesserals. The latter result was obtained by numerical integration, since the three critical tesserals were not of the same order. The principal term is much more dominant in cases of other circular resonant orbits.

Resonance effects on eccentric orbits are shown in Figure 7. The three upper curves represent librations with respect to a stable node rotating with angular velocity $+\dot{\omega}$. As soon as the integration was started, the satellites began their librating motion.

For the two lower curves of this set, this starting 'jolt' of nodal rotation was enough to carry the satellites past the unstable node and into the circular regime. Note that curve labelled 0° represents a satellite which was on the stable node, but began to librate because of the motion of the node.

For cases of more than one critical tesseral, resonance effects were calculated by numerical integration, using a high-speed computer program devised for resonance



a = 6.61105
e = 10^{-4}
i = 1°

1.) $J_{20} + J_{22}$
2.) $J_{20} + J_{22} + J_{31} + J_{33}$

Fig. 6. Libration of a geostationary satellite.

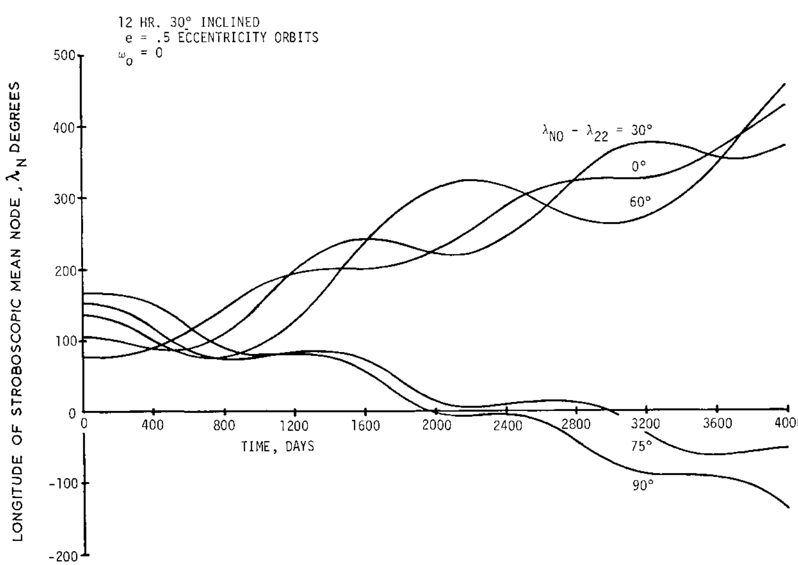


Fig. 7. Libration due to V_{2211}.

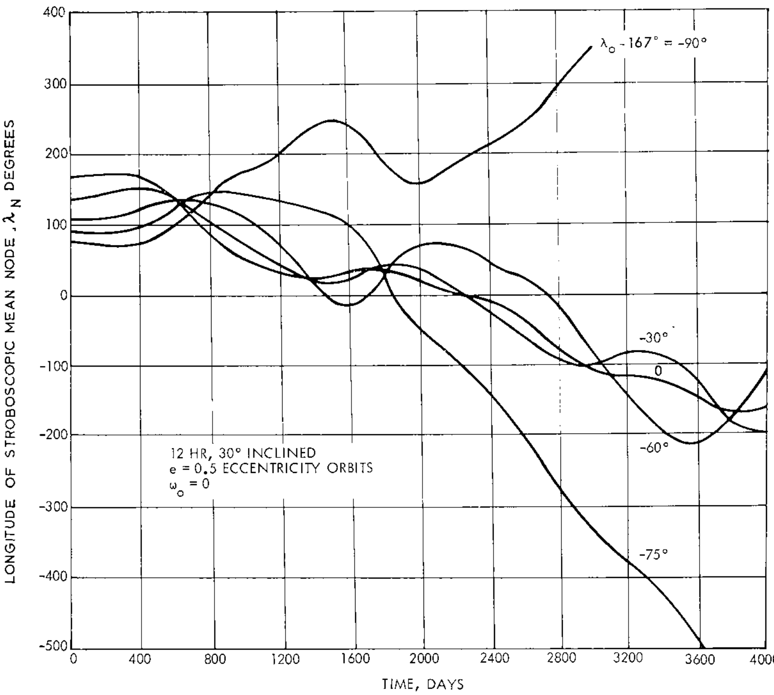


Fig. 8. Libration due to all critical tesserals.

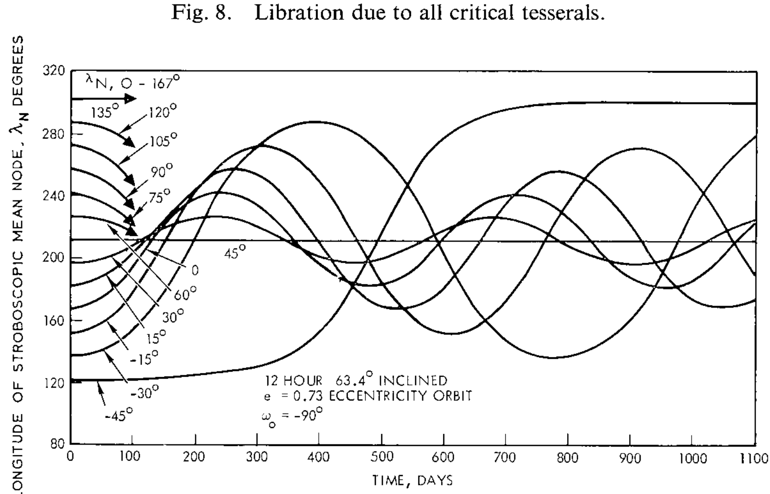


Fig. 9. Libration at the critical inclination due to V_{2211}, V_{2201}, and V_{3210}.

investigations. The results are plotted in Figure 8, where again three of the satellites (0°, 30°, and 60° behind the node) librate and the other two (at 75° and 90° behind the node) circulate. In this case the motion is analytically intractable. An exception is the case of critical inclination and for criticals with same order m, plotted in Figure 9. It is seen that the librations are regular, due to the absence of rotating stable nodes.

7. Boundaries of the Resonance Regimes

Garfinkel [10] defines a resonance band within which the solution is accurate to within quantities of order J_{lm}. From the point of view of practical applications, however, this definition is too narrow because it leads to overlapping resonance bands calculated for neighboring commensurate orbits. Admittedly it would be difficult to set up practical criteria sufficiently accurate in the general case to suit all purposes. More useful than such arbitrary criteria would be the use of the appropriate equations for the particular regime under investigation and comparison of the results to assess the accuracy of a simpler calculation.

8. Station Keeping

Equation (38) can be also written as

$$\tfrac{1}{2}\dot{\psi}^2 = \kappa^2 [\cos\psi - \cos\psi_0], \tag{65}$$

with

$$\kappa = \frac{n/s_0}{P_{lmpq}} \cong \frac{2\pi}{P_{lmpq}} \tag{66}$$

for deep resonance where station keeping might be required. Since station keeping restricts the satellite to a small bandwidth, it is possible to consider more than one critical term. Thus, Equation (65) is generalized to

$$\tfrac{1}{2}\dot{\lambda}_{\mathrm{N}}^2 = \sum \left(\frac{2\pi/m}{P_{lmpq}}\right)^2 [\cos m(\lambda_{\mathrm{N}} - \bar{\lambda}_{lm}) - \cos m(\lambda_{\mathrm{N},0} - \bar{\lambda}_{lm})], \tag{67}$$

where summation is on all critical terms considered.

The libration rate at the top of the deadband is zero, and the libration rate at the bottom, $\dot{\lambda}_{\mathrm{N,B}}$, can be written as

$$\dot{\lambda}_{\mathrm{N,B}} = 4\pi \left\{ \sum \frac{1}{m^2 P_{lmpq}^2} \sin\frac{m}{2} [(\lambda_{\mathrm{N,T}} - \bar{\lambda}_{lm}) + (\lambda_{\mathrm{N,B}} - \bar{\lambda}_{l,m})] \times \sin\frac{m}{2}(\lambda_{\mathrm{N,T}} - \lambda_{\mathrm{N,B}}) \right\}^{1/2}. \tag{68}$$

If the deadband width is $\pm\Delta\lambda$ and the distance of the deadband center from a stable node is β_{lm}, then the libration velocity change $\delta\dot{\lambda}_{\mathrm{N}} = \dot{\lambda}_{\mathrm{N,B}}$ can be expressed as

$$\delta\dot{\lambda}_{\mathrm{N}} = 4\pi \left[\sum \frac{1}{m^2 P_{lmpq}^2} \sin m\beta_{lm} \sin m\Delta\lambda \right]^{1/2}. \tag{69}$$

Taking the variation of Equation (7) which defines the libration rate

$$\delta\dot{\lambda}_{\mathrm{N}} \cong \delta n/s_0 \tag{70}$$

we see that it is caused by a variation of the mean motion.

Now the mean motion is defined by Equation (5), thus

$$\delta n = -\tfrac{3}{2} n\, \delta a / a \tag{71}$$

and from the vis-viva integral

$$\delta V = \frac{1}{2} \frac{V_c^2}{V} \frac{\delta a}{a}, \tag{72}$$

where V_c is the circular orbit velocity, and V is the velocity at the point where correction is made. Combining Equations (70) through (72) we get

$$\delta \dot{\lambda}_N = -\, 6\pi \left(\frac{V}{V_c}\right)^2 \frac{\delta V}{V}. \tag{73}$$

Applying a δV would only stop the libration rate, but twice the amount will make the orbit librate to the top of the deadband. Thus, introducing the inplane deadband halfwidth, $\pm \Delta\phi = s_0 \Delta\lambda$, the required velocity correction is

$$\frac{\delta V}{V} = \frac{4}{3}\left(\frac{V_c}{V}\right)^2 \left[\sum \frac{1}{m^2 P_{lmpq}^2} \sin m\beta_{lm} \sin \frac{m}{s_0}\, \Delta\phi\right]^{1/2}. \tag{74}$$

To find the period of the limit cycle we can use the average acceleration of the stroboscopic node from the simplified Equation (22) written at the deadband center for all critical terms as

$$|\ddot{\lambda}_{N,\,AV}| = \frac{M^2}{s_0^2} \sum \frac{1}{m P_{lmpq}^2} \sin m\beta_{lm}. \tag{75}$$

Using this acceleration to 'fall' $2\Delta\lambda$, the half period is obtained. Thus the full period is

$$T = 2\left[\frac{4\Delta\phi/s_0}{\ddot{\lambda}_N}\right]^{1/2}, \tag{76}$$

or

$$T = \frac{4}{n/s_0}\left[\frac{\Delta\phi}{s_0}\right]^{1/2} \frac{1}{\left[\sum \frac{1}{m P_{lmpq}^2} \sin m\beta_{lm}\right]^{1/2}}. \tag{77}$$

Dividing Equation (74) by Equation (77) the yearly ΔV requirement can be obtained (if n is expressed in radians per year). For small deadbands, $\Delta\phi$ cancels out from the resulting equation, which reads as follows:

$$\frac{\delta V/V}{T} = \frac{n/s_0}{3}\left(\frac{V_c}{V}\right)^2 \sum \frac{1}{m P_{lmpq}^2} \sin m\beta_{lm}. \tag{78}$$

For 24-hour orbits J_{22} dominates, but J_{31} and J_{33} are significant; J_{42}, and J_{44} are negligibly small. See Figures 10 and 11.

The stable and unstable nodes shift with the introduction of other critical terms. The new positions can be found easiest (see [19]) if the partial of the potential function with respect to the node is set to zero. Expressing Equation (1) with $\psi = m(\lambda_N - \lambda_{lm}) - q\omega$

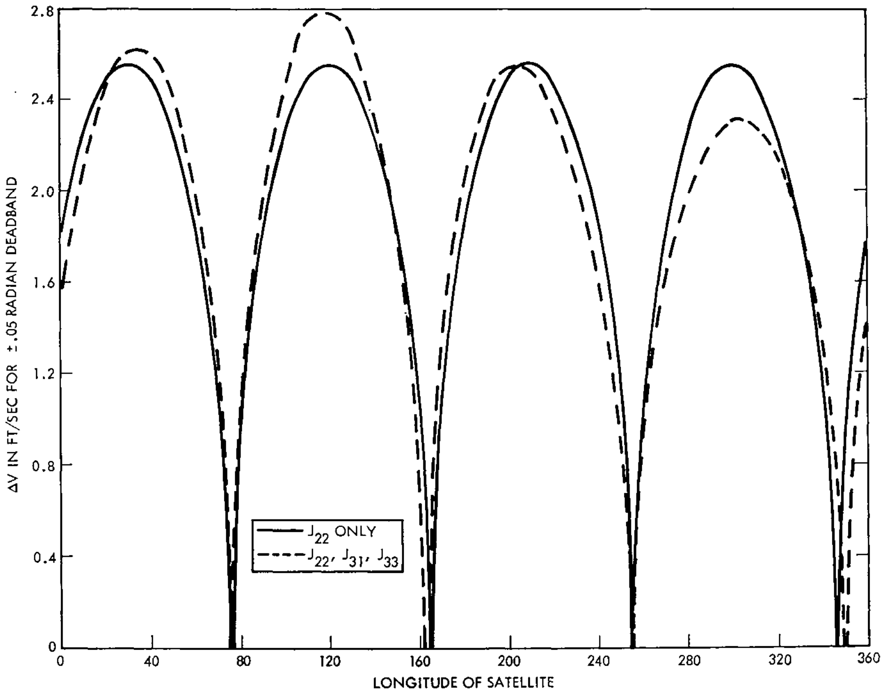


Fig. 10. Station-keeping requirement of a geostationary satellite.

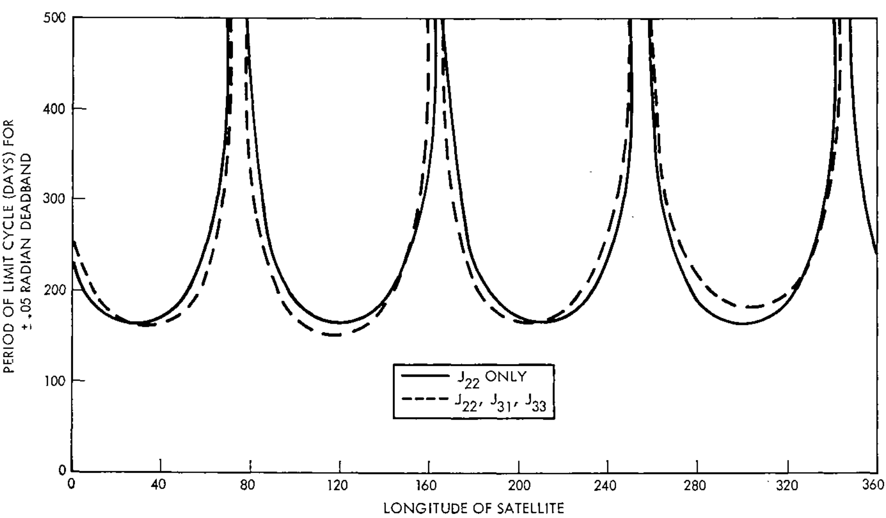


Fig. 11. Limit cycle periods for geostationary satellites.

we obtain

$$\frac{\partial V}{\partial \lambda_N} = \frac{\mu}{a} \sum m \left(\frac{a_e}{a} \right)^l J_{lm} F_{lmp}(i) G_{lpq}(e) \left\{ \begin{matrix} -\sin \\ \\ \cos \end{matrix} \right\}_{(l-m)\ \text{odd}}^{(l-m)\ \text{even}} [m(\lambda_N - \lambda_m) - q\omega] = 0,$$

where summation is on all critical terms. (79)

The values λ_0 which satisfy Equation (79) are the stable and unstable nodes under the influence of all critical terms considered. For eccentric, non-critically inclined orbits Equation (79) is time-dependent. Thus it yields a solution which is valid for only a specified time.

9. Summary and Conclusions

(1) Resonance was treated through the motion of the longitude of the ascending node of the mean satellite ('the stroboscopic mean node').

(2) The differential equation of the motion of the 'stroboscopic mean node' in general form is not amenable to analytic solution.

(3) In order to characterize the resonance regimes, a resonance parameter R was introduced. This parameter is defined as the ratio of circulation period to orbit period. A satellite in a circular orbit has only one circulation period, corresponding to the time taken by the mean node to drift from one stable node to another. Satellites in eccentric orbits have a number of different circulation periods and corresponding resonance parameters. For low-altitude satellites the resonance parameters are at least in the low tens and for high-altitude satellites they can be less than ten if the orbit period is not commensurate. As commensurability is approached they can become arbitrarily high.

(4) If the resonance parameters are less than 10, the differential equation of the stroboscopic mean node can be ignored and Lagrange's Planetary Equations can be analytically integrated. The result is the classical solution.

(5) If one or several of the resonance parameters are in the low tens, a small divisor appears in the classical solution. The result is an oscillation impressed on the secularly precessing orbit, with period equal to the circulation period.

(6) For resonance parameters in the high tens or greater, the differential equation of the stroboscopic mean node must be solved. The procedure is first to neglect a mean node rate term. Next the summation sign must be removed from the differential equation, which can be done for critically inclined orbits and for tesseral terms with the same order m. In the case of circular orbits, one term clearly dominates and the others can be ignored.

(7) The solution for the stroboscopic mean node in the rapid circulation regime can be approximated by one additional term with twice the frequency of the circulation. In the slow circulation regime the solution is in the form of an elliptic integral. The slow circulation period is not defined by the initial drift velocity alone, but by the field strength of the critical tesseral as well.

(8) In the libration regime, the pendulum-type differential equation can be solved by elliptic integrals under the same restrictions as were imposed in the case of slow circulation.

(9) The solution in the libration regime agrees with the solutions published in [1–10], the solution in the circulation regime agrees with that of [10] and with results obtained by numerical integration.

(10) The full differential equation yields a small drag or thrust type of force that

is noticeable only in the case of the 'stalled' pendulum; i.e., on the borderline between libration and circulation.

(11) For station-keeping applications, analytic solutions are available for any number of tesserals.

References

[1] Morando, M. B.: 1963, 'Orbites de Résonance des Satellites de 24H', *Bull. Astron.* **24**, 47–67.
[2] Alan, R. R.: 1963, 'On the Motion of Nearly Synchronous Satellites', Royal Aircraft Establishment TR No. 64078.
[3] Blitzer, L.: 1962, 'Effect of Ellipticity of the Equator on 24-Hour Nearly Circular Satellite Orbits', *J. Geophys. Res.* **67**, 329–335.
[4] Blitzer, L.: 1963, 'The Perturbed Motion of 24-Hour Satellites Due to Equatorial Ellipticity', *J. Geophys. Res.* **68**, 950–952.
[5] Blitzer, L.: 1966, 'Synchronous and Resonant Satellite Orbits Associated with Equatorial Ellipticity', *ARS J.* **32**, 1016–1019.
[6] Blitzer, L.: 1965, 'Equilibrium Positions and Stability of 24-Hour Satellite Orbits', *J. Geophys. Res.* **70**, 3987–3992.
[7] Blitzer, L.: 1966, 'Satellite Resonances and Librations Associated with Tesseral Harmonics of the Geopotential', *J. Geophys. Res.* **71**, 3557–3992.
[8] Garfinkel, B.: 1965, 'The Disturbing Function for an Artificial Satellite', *Astron. J.* **70**, 688–704.
[9] Garfinkel, B.: 1965, 'Tesseral Harmonic Perturbations of an Artificial Satellite', *Astron. J.* **70**, 784–786.
[10] Garfinkel, B.: 1966, 'Formal Solution in the Problem of Small Divisors', *Astron. J.* **71**, 657–669.
[11] Yionoulis, S. M.: 1965, 'A Study of the Resonance Effects Due to the Earth's Potential Function', *J. Geophys. Res.* **70**, 5991–5996; (1966) **71**, 1289–1291.
[12] Francis, M. P., Gedeon, G. S., and Douglas, B. C.: 1966, 'Perturbations of Repeating Ground-track Satellites by Tesseral Harmonics in the Gravitational Potential,' *AIAA J.* **4**, 1281–1286.
[13] Gedeon, G. S., Douglas, B. C., and Palmiter, M. T.: 1967, 'Resonance Effects on Eccentric Satellite Orbits', *J. Astronaut. Sci.* **14**, 147–157.
[14] Gedeon, G. S. and Dial, O. L.: 1967, 'Along Track Oscillations of a Satellite Due to Tesseral Harmonics', *AIAA J.* **5**, 593–595.
[15] Kaula, W. M.: 1966, *Theory of Satellite Geodesy*, Blaisdell.
[16] Anderle, R. and Smith, S.: 1968, 'Observation of Twenty-Seventh and Twenty-Eight Order Gravity Coefficients Based on Doppler Observations', *J. Astronaut. Sci.* **15**, 1–4.
[17] Douglas, B. C. and Wagner, C. A.: 1968, 'A Perturbation Analysis of Existing Resonant Satellites', NASA X-643-68-338.
[18] Brown, E. W. and Shook, C. A.: 1964, *Planetary Theory*, Dover Publications, New York, p. 219.
[19] Blitzer, L.: 1966, 'Equilibrium Satellite Orbits and Resonances', TRW Systems, IOC 3412.10-2.