THE GRAVITY-PERTURBED LAMBERT PROBLEM: A KS VARIATION OF PARAMETERS APPROACH

R. C. ENGELS and J. L. JUNKINS
Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061, U.S.A.

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Abstract. A new formulation is presented for the perturbed Lambert problem. The formulation employs the variation-of-parameters method in the KS transformed state space to determine perturbations of a Keplerian Lambert solution. The approach is universal (in that its validity is not restricted to a particular energy domain). For the case of the second zonal harmonic (oblateness) perturbation, first order perturbations are carried out entirely analytically; non-iterative corrections are determined through solution of a pair of algebraic equations. For more general perturbations, numerical quadratures are required.

1. Introduction

The classical two-point-boundary-value-problem of celestial mechanics (Keplerian motion, for given initial and final position coordinates and for given time of flight, determine the initial velocity coordinates) is widely known as the 'Lambert's Problem'. This problem and its variations play a fundamental role in many navigation and guidance procedures. Recent papers by Sun (1979) and Battin (1970) have re-solved Winter's (1917) classical integral equation

$$\Delta t = \frac{1}{\sqrt{2\mu}} \int_{n/2}^{m/2} \left(\frac{1}{r} - \frac{1}{2a}\right)^{-1/2} \mathrm{d}r \tag{1}$$

in terms of hypergeometric functions. Sun's developments established the equivalence between solutions of (1) and solutions of the nonautonomous linear differential equation

$$\xi(1-\xi)\frac{\mathrm{d}^2\tau}{\mathrm{d}\xi^2} + (\tfrac{5}{2} - 3\xi)\frac{\mathrm{d}\tau}{\mathrm{d}\xi} - \tfrac{3}{4}\tau = \tfrac{1}{2}\sigma^3(1-\sigma^2)(1-\sigma^2\xi)^{-3/2} \tag{2}$$

where a = semi major axis; r = distance from occupied focus; μ = gravitational mass constant; $c = |\mathbf{r}_2 - \mathbf{r}_1|$ = chord length from $\mathbf{r}_1$ to $\mathbf{r}_2$; $m = r_1 + r_2 + c$; $n = r_1 + r_2 - c$; $\xi = m/4a$; $\sigma^2 = n/m$; $\tau = 4\Delta t(\mu/m^3)^{-1/2}$. Sun's theoretical developments and parametric studies brings the Keplerian Lambert problem to a rather complete state.

Recent papers by Jezewski (1976) and Andrus (1977) have addressed two new issues

(i) Formulation of two-point-boundary-value-problems in the state space resulting from the KS transformation, as developed in Stiefel and Scheifele (1971).

(ii) Perturbation of the Keplerian Lambert solution to account for oblateness of the central body.

The present paper extends the work of Jezewski and Andrus, and makes use of a KS-variation-of-parameters formulation developed by Bond (1976). An especially attractive, non-iterative perturbation of the Keplerian Lambert solution is developed, to first order in J_2.

2. Kustaanheimo–Stiefel (KS) Transformation

Let us consider the classical non-dimensional equations of motion of the perturbed two-body problem in rectangular coordinates, as given by

$$\frac{d^2\mathbf{r}}{dt^2}+\frac{1}{r^3}\mathbf{r}=-\frac{\partial V}{\partial \mathbf{r}}+\mathbf{P}\,, \tag{3}$$

where $\mathbf{r}$ represents the position vector, r the radial distance, V the perturbing potential and $\mathbf{P}$ the remaining perturbing force.

Introducing the fictitious time s, defined by

$$dt = r\,ds\,, \tag{4}$$

and the KS transformation

$$\mathbf{r}=L(\mathbf{u})\mathbf{u}\,, \tag{5}$$

where

$$L(\mathbf{u})=\begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{bmatrix} \tag{6}$$

we obtain the universal set of differential equations

$$\ddot{\mathbf{u}}+\alpha_T\mathbf{u}=\mathbf{Q}\,, \tag{7}$$

with

$$\alpha_T=\frac{1}{r}(\tfrac{1}{2}-\dot{\mathbf{u}}^T\dot{\mathbf{u}})-\frac{V}{2}\,, \tag{8}$$

$$\mathbf{Q}=-\tfrac{1}{4}\frac{\partial}{\partial \mathbf{u}}(rV)+\frac{r}{2}L^T\mathbf{P}\,, \tag{9}$$

$$r=\mathbf{u}^T\mathbf{u}\,, \tag{10}$$

$$(\dot{\ })\equiv\frac{d}{ds}(\ )\,. \tag{11}$$

The quantity α_T represents half of the negative non-dimensionalized total energy.

3. The Universal Keplerian Lambert Problem

With $\mathbf{P}=0$ and $V=0$, Equation (7) becomes

$$\ddot{\mathbf{u}}+\alpha_T\mathbf{u}=\mathbf{0}\,, \tag{12}$$

where

$$\alpha_T=\frac{1}{r}\left(\tfrac{1}{2}-\dot{\mathbf{u}}^T\dot{\mathbf{u}}\right). \tag{13}$$

The universal solution of Equation (12) is given by

$$\mathbf{u}=\mathbf{u}(0)c_0+\dot{\mathbf{u}}(0)sc_1\,, \tag{14}$$

in which the *Stumpff functions* (Stiefel and Scheifele, 1971; Bond, 1974) are abbreviated as

$$c_0\equiv c_0(\alpha_T s^2)\,,\qquad c_1\equiv c_1(\alpha_T s^2)\,, \tag{15}$$

and $\mathbf{u}(0)$, $\dot{\mathbf{u}}(0)$ are initial conditions.

Furthermore, using Equations (10) and (4) together with Equation (14) we can write

$$r=r(0)\tilde{c}_0+\dot{r}(0)s\tilde{c}_1+s^2\tilde{c}_2\,, \tag{16}$$

$$t=t(0)+r(0)s\tilde{c}_1+\dot{r}(0)s^2\tilde{c}_2+s^3\tilde{c}_3\,, \tag{17}$$

with

$$\dot{r}(0)=2\mathbf{u}^T(0)\dot{\mathbf{u}}(0)\,, \tag{18}$$

$$\tilde{c}_n\equiv c_n(4\alpha_T s^2),\qquad n=0,1,\ldots\,. \tag{19}$$

Because $\dot{\mathbf{u}}(0)$ is not known, we must eliminate $\dot{r}(0)$ from Equations (16) and (17). Using Equation (14) we have

$$\mathbf{u}^T(0)\mathbf{u}c_0=r(0)c_0^2+\mathbf{u}^T(0)\dot{\mathbf{u}}(0)s\tilde{c}_1\,, \tag{20}$$

from which it follows that

$$\dot{r}(0)s\tilde{c}_1=2\mathbf{u}^T(0)\mathbf{u}c_0-2r(0)c_0^2\,, \tag{21}$$

and

$$\mathbf{u}^T(0)\dot{\mathbf{u}}(0)sc_1=\mathbf{u}^T(0)\mathbf{u}-r(0)c_0\,, \tag{22}$$

where we made use of the identity

$$s\tilde{c}_1=sc_1c_0\,. \tag{23}$$

Using Equations (21–22) together with Equations (16–17) we obtain

$$r=-r(0)+2\mathbf{u}^T(0)\mathbf{u}c_0+s^2\tilde{c}_2\,, \tag{24}$$

$$t=t(0)+\mathbf{u}^T(0)\mathbf{u}sc_1+s^3\tilde{c}_3\,, \tag{25}$$

where we used the identities

$$\tilde{c}_0 - 2c_0^2 = -1\,, \tag{26}$$

$$c_1^2 = 2\tilde{c}_2\,. \tag{27}$$

Equations (24–25) can now be solved for α_T and s_f given $r(0)$, $r(f)$, $\mathbf{u}(0)$,* $\mathbf{u}(t)$ and $t(f)-t(0)$, where s_f is the 'final' value of s.

Indeed, using Newton's method, we can iterate according to

$$\begin{Bmatrix}\alpha_T\\ s_f\end{Bmatrix}^{(k+1)} = \begin{Bmatrix}\alpha_T\\ s_f\end{Bmatrix}^{(k)} - \frac{1}{D^{(k)}}\begin{bmatrix}\mathscr{G}'_{s_f} & -\mathscr{F}'_{s_f}\\ -\mathscr{G}'_{\alpha_T} & \mathscr{F}'_{\alpha_T}\end{bmatrix}^{(k)}\begin{Bmatrix}\mathscr{F}\\ \mathscr{G}\end{Bmatrix}^{(k)} \tag{28}$$

with

$$D^{(k)} = (\mathscr{F}'_{\alpha_T}\mathscr{G}'_{s_f} - \mathscr{F}'_{s_f}\mathscr{G}'_{\alpha_T})^{(k)}\,, \tag{29}$$

$$\mathscr{F} \equiv r(f) + r(0) - s_f^2\tilde{c}_2(f) - 2\mathbf{u}^T(0)\mathbf{u}(f)c_0(f)\,, \tag{30}$$

$$\mathscr{G} \equiv t(f) - t(0) - s_f^3\tilde{c}_3(f) - \mathbf{u}^T(0)\mathbf{u}(f)s_f c_1(f)\,, \tag{31}$$

$$\mathscr{F}'_{s_f} \equiv s_f c_1(f)[2\alpha_T\mathbf{u}^T(0)\mathbf{u}(f) - c_0(f)]\,, \tag{32}$$

$$\mathscr{G}'_{s_f} \equiv -s_f^2\tilde{c}_2(f) - \mathbf{u}^T(0)\mathbf{u}(f)c_0(f)\,, \tag{33}$$

$$\mathscr{F}'_{\alpha_T} \equiv s_f^2\{\mathbf{u}^T(0)\mathbf{u}(f)c_1(f) - 2s_f^2[2\tilde{c}_4(f) - \tilde{c}_3(f)]\}\,, \tag{34}$$

$$\mathscr{G}'_{\alpha_T} \equiv -s_f^3\{2s_f^2[3\tilde{c}_5(f) - \tilde{c}_4(f)] + \tfrac{1}{2}\mathbf{u}^T(0)\mathbf{u}(f)[c_3(f) - c_2(f)]\}\,, \tag{35}$$

with

$$c_n(f) = c_n(\alpha_T s_f^2)\,.$$

In order to obtain the derivatives (32–35) we used

$$\frac{\partial c_0}{\partial s} = -\alpha_T s c_1\,,$$

$$\frac{\partial c_0}{\partial \alpha_T} = -\tfrac{1}{2}s^2 c_1,$$

$$\frac{\partial (sc_1)}{\partial s} = c_0\,,$$

$$\frac{\partial (sc_1)}{\partial \alpha_T} = \tfrac{1}{2}s^3(c_3 - c_2)\,,$$

$$\frac{\partial (s^2\tilde{c}_2)}{\partial s} = s\tilde{c}_1\,,$$

$$\frac{\partial (s^2\tilde{c}_2)}{\partial \alpha_T} = 2s^4(s\tilde{c}_4 - \tilde{c}_3)\,,$$

* For brevity, we denote initially and finally evaluated functions by $f(s=0) \equiv f(0)$ and $f(s=s_f) \equiv f(f)$.

$$\frac{\partial(s^3\tilde{c}_3)}{\partial s} = s^2\tilde{c}_2 ,$$

$$\frac{\partial(s^3\tilde{c}_3)}{\partial \alpha_T} = 2s^5(3\tilde{c}_5 - \tilde{c}_4) . \tag{36}$$

Numerical experiments with this algorithm indicate that, even with starting estimates off by 25%, it will converge in 3 to 6 iterations. It contains implicit singularities, of course, for 0° and 180° transfers (in which case the orbit plane is undefined).

Once α_T and s_f are determined we find the initial velocity $\dot{\mathbf{u}}(0)$ from Equation (14) with $s = s_f$ as

$$\dot{\mathbf{u}}(0) = \frac{1}{s_f c_1(f)}[\mathbf{u}(f) - \mathbf{u}(0)c_0(f)] . \tag{37}$$

The velocity in rectangular coordinates follows from

$$\dot{\mathbf{r}}(0) = B\dot{\mathbf{u}}(0) , \tag{38}$$

where

$$B = \frac{2}{r(0)}\begin{bmatrix} u_1(0) & -u_2(0) & -u_3(0) & u_4(0) \\ u_2(0) & u_1(0) & -u_4(0) & -u_3(0) \\ u_3(0) & u_4(0) & u_1(0) & u_2(0) \end{bmatrix} . \tag{39}$$

Also, note that the final position vector $\mathbf{u}(f)$ must be calculated from $\mathbf{r}(f)$ as indicated in Appendix A.

4. KS Variation of Parameters

A variation of parameters formulation for $\mathbf{P} = \mathbf{0}$ is given by Bond (1974), leading to the following solution form

$$\mathbf{u} = -\boldsymbol{\alpha} c_0 + \boldsymbol{\beta} s c_1 , \tag{40}$$

$$\dot{\mathbf{u}} = \boldsymbol{\alpha}\alpha_T s c_1 + \boldsymbol{\beta} c_0 , \tag{41}$$

$$t = \tau + a s\tilde{c}_1 + b s^2\tilde{c}_2 + s^3\tilde{c}_3 , \tag{42}$$

with

$$\alpha_T = \frac{1}{r(0)}[\tfrac{1}{2} - \dot{\mathbf{u}}^T(0)\dot{\mathbf{u}}(0)] - \frac{V(0)}{2} , \tag{43}$$

and the differential equations for the slowly varying elements

$$\dot{\boldsymbol{\alpha}} = \mathbf{Q} s c_1 , \tag{44}$$

$$\dot{\boldsymbol{\beta}} = \mathbf{Q} c_0 , \tag{45}$$

$$\dot{a} = -G s\tilde{c}_1 , \tag{46}$$

$$\dot{b} = G\tilde{c}_0 , \tag{47}$$

$$\dot{\tau} = Gs^2\tilde{c}_2 , \tag{48}$$

$$\mathbf{Q} = -\frac{1}{4}\frac{\partial}{\partial \mathbf{u}}(rV) , \tag{49}$$

$$G = -rV + 2\mathbf{u}^T\mathbf{Q} , \tag{50}$$

also, note that

$$r = a\tilde{c}_0 + bs\tilde{c}_1 + s^2\tilde{c}_2 , \tag{51}$$

$$a = \boldsymbol{\alpha}^T\boldsymbol{\alpha} , \tag{52}$$

$$b = -2\boldsymbol{\alpha}^T\boldsymbol{\beta} . \tag{53}$$

This set of equations is used below as the basis of a new solution for the perturbed Lambert problem.

5. The Perturbed Universal Lambert Problem

We wish to follow an approach similar to that of the unperturbed motion. Therefore, it is helpful to eliminate $\dot{\mathbf{u}}(0)$ from Equations (42) and (51). We start by multiplying Equation (40) by $\boldsymbol{\alpha}^T$ and using Equations (52–53), to obtain

$$\boldsymbol{\alpha}^T\mathbf{u} = -ac_0 - \tfrac{1}{2}bsc_1 . \tag{54}$$

Using this result we can transform Equations (42) and (51) into

$$r = -a - 2\boldsymbol{\alpha}^T\mathbf{u}c_0 + s^2\tilde{c}_2 , \tag{55}$$

$$t = \tau - \boldsymbol{\alpha}^T\mathbf{u}sc_1 + s^3\tilde{c}_3 . \tag{56}$$

Let us write these equations for $s = s_f$

$$r(f) = -a(f) - 2\boldsymbol{\alpha}^T(f)\mathbf{u}(f)c_0(f) + s_f^2\tilde{c}_2(f) , \tag{57}$$

$$t(f) = \tau(f) - \boldsymbol{\alpha}^T(f)\mathbf{u}(f)s_fc_1(f) + s_f^3\tilde{c}_3(f) , \tag{58}$$

and assume that the potential V is a function of $\mathbf{u}$ only and is proportional to a small parameter ε

$$V = \varepsilon V_1(\mathbf{u}) . \tag{59}$$

Because of Equation (59) we can write

$$\mathbf{u}(s) = \mathbf{u}_0(s) + \varepsilon\mathbf{u}_1(s) + 0(\varepsilon^2) , \tag{60}$$

with

$$\mathbf{u}_0(s) = \mathbf{u}(0)c_0(\alpha_T s^2) + \dot{\mathbf{u}}(0)sc_1(\alpha_T s^2) . \tag{61}$$

Next, let us consider an unperturbed Lambert problem with $\mathbf{r}(0)$, $\mathbf{r}(f)$ and $t(f) - t(0)$ given, and solve for s_{f0} and α_{T0}. Then, from Equations (37), (41), and (36) it follows that

$$\dot{\mathbf{u}}_0(0) = \frac{1}{s_{f0} c_1(\alpha_{T0} s_{f0}^2)} [\mathbf{u}_0(s_{f0}) - \mathbf{u}(0) c_0(\alpha_{T0} s_{f0}^2)] . \tag{62}$$

Note that $\mathbf{u}_0(0) = \mathbf{u}(0)$ and $\mathbf{u}_0(s_{f0})$ are constructed from $\mathbf{r}(0)$ and $\mathbf{r}(f)$ as shown in Equations (A12–A13) of Appendix A.

Because of Equations (59) we can also write

$$s_f = s_{f0} + \varepsilon s_{f1} , \tag{63}$$

$$\alpha_T = \alpha_{T0} + \varepsilon \alpha_{T1} , \tag{64}$$

$$\mathbf{u}(f) = \mathbf{u}_0(s_{f0}) + \varepsilon \mathbf{u}_1 , \tag{65}$$

$$a(f) = a_0(s_{f0}) + \varepsilon a_1 , \tag{66}$$

$$\boldsymbol{\alpha}(f) = \boldsymbol{\alpha}_0(s_{f0}) + \varepsilon \boldsymbol{\alpha}_1 , \tag{67}$$

$$\boldsymbol{\beta}(f) = \boldsymbol{\beta}_0(s_{f0}) + \varepsilon \boldsymbol{\beta}_1 , \tag{68}$$

$$\tau(f) = \tilde{\tau}_0(s_{f0}) + \tilde{\varepsilon}\tau_1 . \tag{69}$$

Using Equations (63–69) in Equations (57–58), and discarding terms in ε^2, we obtain

$$\begin{aligned} r(f) = {} & -a_0(s_{f0}) + s^2\tilde{c}_2|_0 - 2\boldsymbol{\alpha}_0^T(s_{f0})\mathbf{u}_0(s_{f0})c_0|_0 + \\ & + \varepsilon\Bigg\{ -a_1 + s_{f1}\frac{\partial(s^2\tilde{c}_2)}{\partial s}\bigg|_0 + \alpha_{T1}\frac{\partial(s^2\tilde{c}_2)}{\partial\alpha_T}\bigg|_0 - \\ & -2\Big[(\boldsymbol{\alpha}_0^T(s_{f0})\mathbf{u}_1 + \boldsymbol{\alpha}_1^T\mathbf{u}_0(s_{f0})) \times \\ & \times c_0|_0 + \boldsymbol{\alpha}_0^T(s_{f0})\mathbf{u}_0(s_{f0})\Big(s_{f1}\frac{\partial c_0}{\partial s}\bigg|_0 + \alpha_{T1}\frac{\partial c_0}{\partial \alpha_T}\bigg|_0\Big)\Big]\Bigg\} , \end{aligned} \tag{70}$$

$$\begin{aligned} t(f) = {} & \tau_0(s_{f0}) + s^3\tilde{c}_3|_0 - \boldsymbol{\alpha}_0^T(s_{f0})\mathbf{u}_0(s_{f0})sc_1|_0 + \\ & + \varepsilon\Bigg\{ \tilde{\tau}_1 + s_{f1}\frac{\partial(s^3\tilde{c}_3)}{\partial s}\bigg|_0 + \alpha_{T1}\frac{\partial(s^3\tilde{c}_3)}{\partial\alpha_T}\bigg|_0 - \\ & -\Big[(\boldsymbol{\alpha}_0^T(s_{f0})\mathbf{u}_1 + \boldsymbol{\alpha}_1^T u_0(s_{f0}))sc_1|_0 + \\ & + \boldsymbol{\alpha}_0^T(s_{f0})\mathbf{u}_0(s_{f0})\Big(s_{f1}\frac{\partial(sc_1)}{\partial s}\bigg|_0 + \alpha_{T1}\frac{\partial(sc_1)}{\partial \alpha_T}\bigg|_0\Big)\Big]\Bigg\} , \end{aligned} \tag{71}$$

where, using Equations (44)–(50).

$$a_0(s_{f0}) = r(0) , \tag{72}$$

$$\tau_0(s_{f0}) = t(0) , \tag{73}$$

$$\boldsymbol{\alpha}_0(s_{f0}) = -\mathbf{u}(0) , \tag{74}$$

$$\boldsymbol{\beta}_0(s_{f0}) = \dot{\mathbf{u}}(0) , \tag{75}$$

$$\boldsymbol{\alpha}_1 = \int_0^{s_{f0}} \mathbf{Q}_1 s c_1(\alpha_{T0}s^2)\,\mathrm{d}s\,, \tag{76}$$

$$\boldsymbol{\beta}_1 = \int_0^{s_{f0}} \mathbf{Q}_1 c_0(\alpha_{T0}s^2)\,\mathrm{d}s\,, \tag{77}$$

$$a_1 = -\int_0^{s_{f0}} G_1 s\tilde{c}_1(\alpha_{T0}s^2)\,\mathrm{d}s\,, \tag{78}$$

$$\tau_1 = \int_0^{s_{f0}} G_1 s^2\tilde{c}_2(\alpha_{T0}s^2)\,\mathrm{d}s\,, \tag{79}$$

$$\mathbf{Q}_1 = -\frac{1}{4}\left[\frac{\partial}{\partial \mathbf{u}}(rV_1)\right]\Bigg|_{\mathbf{u}=\mathbf{u}_0}, \tag{80}$$

$$G_1 = [-rV_1 + 2\mathbf{u}^T\mathbf{Q}_1]|_{\mathbf{u}=\mathbf{u}_0}\,, \tag{81}$$

$$\mathbf{u}_0 = \mathbf{u}(0)c_0(\alpha_{T0}s^2) + \dot{\mathbf{u}}_0(0)sc_1(\alpha_{T0}s^2)\,, \tag{82}$$

$$\dot{\mathbf{u}}_0(0) = \frac{1}{s_{f0}c_1(\alpha_{T0}s_{f0}^2)}[\mathbf{u}_0(s_{f0}) - \mathbf{u}(0)c_0(\alpha_{T0}s_{f0}^2)]\,, \tag{83}$$

in which $\mathbf{u}_0(s_{f0})$ is known.

Furthermore, because s_{f0} and α_{T0} are solutions of the unperturbed Lambert problem, it follows from Equations (24–25) that

$$r(f) = -a_0(s_{f0}) + s^2\tilde{c}_2|_0 - 2\boldsymbol{\alpha}_0^T(s_{f0})\mathbf{u}_0(s_{f0})c_0|_0\,, \tag{84}$$

$$t(f) = \tau_0(s_{f0}) + s^3\tilde{c}_3|_0 - \boldsymbol{\alpha}_0^T(s_{f0})\mathbf{u}_0(s_{f0})sc_1|_0\,, \tag{85}$$

so that, Equations (70–71) reduce to

$$\begin{aligned}&\left[\frac{\partial(s^2\tilde{c}_2)}{\partial s}\bigg|_0 + 2\mathbf{u}^T(0)\mathbf{u}_0(s_{f0})\frac{\partial c_0}{\partial s}\bigg|_0\right]s_{f1} + \\ &\qquad + \left[\frac{\partial(s^2\tilde{c}_2)}{\partial \alpha_T}\bigg|_0 + 2\mathbf{u}^T(0)\mathbf{u}_0(s_{f0})\frac{\partial c_0}{\partial \alpha_T}\bigg|_0\right]\alpha_{T1} = \\ &\qquad = a_1 + 2[\boldsymbol{\alpha}_1^T\mathbf{u}_0(s_{f0}) - \mathbf{u}^T(0)\mathbf{u}_1]c_0|_0\,,\end{aligned} \tag{86}$$

$$\begin{aligned}&\left[\frac{\partial(s^3\tilde{c}_3)}{\partial s}\bigg|_0 + \mathbf{u}^T(0)\mathbf{u}_0(s_{f0})\frac{\partial(sc_1)}{\partial s}\bigg|_0\right]s_{f1} + \\ &\qquad + \left[\frac{\partial(s^3\tilde{c}_3)}{\partial \alpha_T}\bigg|_0 + \mathbf{u}^T(0)\mathbf{u}_0(s_{f0})\frac{\partial(sc_1)}{\partial \alpha_T}\bigg|_0\right]\alpha_{T1} = \\ &\qquad = -\tau_1 + [\boldsymbol{\alpha}_1^T\mathbf{u}_0(s_f) - \mathbf{u}^T(0)\mathbf{u}_1]sc_1|_0\,.\end{aligned} \tag{87}$$

Equations (86–87) represent a set of two simultaneous *algebraic* equations in the unknowns s_{f1} and α_{T1}. Therefore, *no iteration is necessary* to solve for s_{f1} and α_{T1}. The actual initial velocity $\dot{\mathbf{u}}(0)$ is now obtained from Equation (40),

$$\dot{\mathbf{u}}(0)=\frac{1}{s_f c_1(\alpha_T s_f^2)}[\mathbf{u}(f)+\boldsymbol{\alpha}(f)c_0(\alpha_T s_f^2)]-\varepsilon\,\boldsymbol{\beta}_1\,. \tag{88}$$

6. Computational Summary

In this paper, the authors derived an algorithm to solve the perturbed universal Lambert problem. The procedure works as follows:

(1) Given: $\mathbf{r}(0)$, $\mathbf{r}(f)$, $t(f)-t(0)$.

(2) Construct $\mathbf{u}(0)$ from Equations (A2–A3) and $\mathbf{u}_0(s_{f0})$ from Equations (A12–A13).

(3) Solve the unperturbed universal Lambert problem using Equations (28–37) with $\mathbf{u}(f)=\mathbf{u}_0(s_{f0})$, yielding α_{T0}, s_{f0} and $\dot{\mathbf{u}}_0(0)$. This involves an iterative process.

(4) $\dot{\mathbf{u}}_0(0)$ in Equation (82) is now known and $\mathbf{u}_0$ can be evaluated as a function of s.

(5) It is now possible to determine $\boldsymbol{\alpha}_1$ and $\boldsymbol{\beta}_1$ from Equations (76–77). These integrals can be computed in closed form as shown in Appendix B or by any suitable numerical quadrature scheme.

(6) Evaluate q_1 from Equation (A17) and determine $\mathbf{u}_1$ from Equation (A21).

(7) Determine a_1 and τ_1 from Equations (79–80) and s_{f1} and α_{T1} from Equations (86–87).

(8) Find $\dot{\mathbf{u}}(0)$ from Equation (88).

Note that there is no need for iteration to obtain the corrections α_{T1} and s_{f1} which account for the effect of perturbations; these are found by simply solving a set of two simultaneous *algebraic* equations. It is also clear that the solution is universal, i.e. it applies to all possible energies (α_T arbitrary), excluding the 0° and 180° transfers (which are, as usual, singular due to the nonuniqueness of the orbit plane).

7. Numerical Results

The algorithm as discussed in the previous section has been programmed on FORTRAN H on the Virginia Tech IBM 370/158 computer.

We consider here three test cases all with the same initial position vector

$$x(0)=6\,478\text{ km}$$

$$y(0)=0\text{ km}$$

$$z(0)=0\text{ km}$$

and time of flight

$$t(f)-t(0)=1800.0009\,\text{g sec}$$

and we adopt the physical constants

$$R = 6\,378.135 \text{ km}$$

$$\mu = 398\,600.8 \text{ km}^3/\text{s}^2$$

$$J_2 = 0.001\,082\,615\,7\,.$$

The final position vectors were chosen in such a way as to produce a positive negative and zero value for the total energy constant α_T, corresponding to elliptic, parabolic, and hyperbolic transfer orbits, respectively.

Case 1: $x(f) = 10\,970.928$ km

$y(f) = 1\,435.480$ km

$z(f) = 4\,304.951$ km

Total Energy $= -32.063\,76$ kNkm.

Case 2: $x(f) = 12\,534.300$ km

$y(f) = 4\,654.640$ km

$z(f) = 12\,518.690$ km

Total Energy $= 0$ kNkm.

Case 3: $x(f) = 24\,689.469$ km

$y(f) = 4\,986.430$ km

$z(f) = 3\,324.004$ km

Total Energy $= 29.436\,30$ kNkm.

The following converged solutions results for the initial velocity vector were obtained

Case 1: $\dfrac{dx(0)}{dt} = 7.000\,00 \text{ km s}^{-1}$

$\dfrac{dy(0)}{dt} = 1.000\,00 \text{ km s}^{-1}$

$\dfrac{dz(0)}{dt} = 3.000\,00 \text{ km s}^{-1}$

Case 2: $\dfrac{dx(0)}{dt} = 7.000\,00 \text{ km s}^{-1}$

$\dfrac{dy(0)}{dt} = 3.000\,00 \text{ km s}^{-1}$

$\dfrac{dz(0)}{dt} = 8.070\,161 \text{ km s}^{-1}$

Case 3: $\dfrac{dx(0)}{dt} = 13.000\,00 \text{ km s}^{-1}$

$$\frac{dy(0)}{dt} = 3.000\,00 \text{ km s}^{-1}$$

$$\frac{dz(0)}{dt} = 2.000\,00 \text{ km s}^{-1} .$$

The above initial conditions were also used as initial conditions for conventional Cowell integrations of the equations of motion (3), to provide a basis for discussion of errors.

It should be noted that the program is capable of producing variable accuracy in the initial velocity vector. However, the object is to generate just enough precise digits in the initial velocity vector so as to guarantee an acceptable final position vector. In the present examples, the initial velocities are correct in the fifth decimal. This produces final impact position vector components errors of less than 10 m. These errors are well within the tolerance that should be enforced for J_2-approximation of the gravity field.

8. Concluding Remarks

The above results provide significant extensions of the central two-point boundary value problem of celestial mechanics. Advantage has been taken of certain properties of the KS transformation and associated variation-of-parameter formulations to obtain a reasonably compact and efficient perturbation of the Keplerian Lambert problem. Given a preliminary solution of the Keplerian Lambert problem, the present developments provide algebraic, non-iterative corrections to account for the J_2 perturbation. These corrections' precision has been found entirely satisfactory (10 m or less terminal errors) for ballistic trajectories. The analytical results are general, however, and apply to all species of elliptic, parabolic, and hyperbolic orbits, except for the cases involving a non-unique orbit plane (180° transfers, 0° transfers, and rectilinear transfers); for all of these cases, appropriate modifications are necessary to obtain a unique solution. Comparisons with other approaches are made by Junkins *et al.* (1971).

Appendix A

The initial position vector $\mathbf{u}(0)$ can be constructed from Equation (5)

$$\mathbf{r}(0) = L(\mathbf{u}(0))\mathbf{u}(0) , \qquad \text{(A1)}$$

which leads to the following solution

$$u_1^2(0) = \tfrac{1}{2}(r(0) + x(0)) ,$$

$$\left.\begin{aligned} u_2(0) &= \frac{y(0)u_1(0)}{r(0)+x(0)} \\ u_3(0) &= \frac{z(0)u_1(0)}{r(0)+x(0)}, \\ u_4(0) &= 0 \quad \text{(chosen)} \end{aligned}\right\} \quad \text{if} \quad x(0) \geq 0 \tag{A2}$$

or

$$\left.\begin{aligned} u_2^2(0) &= \tfrac{1}{2}(r(0)-x(0)), \\ u_1(0) &= \frac{y(0)u_2(0)}{r(0)-x(0)} \\ u_3(0) &= 0 \quad \text{(chosen)}, \\ u_4(0) &= \frac{z(0)u_2(0)}{r(0)-x(0)}. \end{aligned}\right\} \quad \text{if} \quad x(0) < 0$$

Equations (A2–A3) completely determine $\mathbf{u}(0)$.

Next, let us derive some useful relationships. Given two vectors $\mathbf{v}(v_1, v_2, v_3, v_4)$ and $\mathbf{w}(w_1, w_2, w_3, w_4)$ we define the following bilinear relationship

$$l(\mathbf{v}, \mathbf{w}) \equiv v_4 w_1 - v_3 w_2 + v_2 w_3 - v_1 w_4 . \tag{A4}$$

From Equation (A4) it follows immediately that

$$l(\mathbf{u}(0), \mathbf{u}(0)) = 0 . \tag{A5}$$

It can also be shown (Stiefel and Scheiffle, 1971) that

$$l(\mathbf{u}(0), \dot{\mathbf{u}}(0)) = 0 . \tag{A6}$$

From Equation (14) it follows that

$$\begin{aligned} l(\mathbf{u}(0), \mathbf{u}) &= l(\mathbf{u}(0), \mathbf{u}(0)c_0 + \dot{\mathbf{u}}(0)sc_1) = \\ &= l(\mathbf{u}(0), \mathbf{u}(0))c_0 + l(\mathbf{u}(0), \dot{\mathbf{u}}(0))sc_1 , \end{aligned} \tag{A7}$$

so that, because of Equations (A5–A6) we obtain

$$l(\mathbf{u}(0), \mathbf{u}) = 0, \quad \text{for all} \quad s. \tag{A8}$$

In particular, for $s = s_f$, Equation (A8) becomes

$$l(\mathbf{u}(0), \mathbf{u}(f)) = 0 . \tag{A9}$$

Note that Equation (A9) holds rigorously for Keplerian motion, but must be relaxed correctly in the presence of perturbations.

Next, let us derive the vector $\mathbf{u}(f)$ which is needed to solve for $\dot{\mathbf{u}}(0)$ in Equation (35). Writing Equation (5) for $s = s_f$ yields

$$\mathbf{r}(f) = L(\mathbf{u}(f))\mathbf{u}(f) . \tag{A10}$$

Because the fourth component of this relation is an identity, namely $l(\mathbf{u}(f), \mathbf{u}(f)) = 0$ we only have three equations for four unknowns. An extra condition is necessary. This condition is given by Equation (A9). The set of simultaneous equations to determine the four components of $\mathbf{u}(f)$ can be written in matrix form as follows

$$\begin{Bmatrix} x(f) \\ y(f) \\ z(f) \\ 0 \end{Bmatrix} = \begin{bmatrix} u_1(f) & -u_2(f) & -u_3(f) & u_4(f) \\ u_2(f) & u_1(f) & -u_4(f) & -u_3(f) \\ u_3(f) & u_4(f) & u_1(f) & u_2(f) \\ u_4(0) & -u_3(0) & u_2(0) & -u_1(0) \end{bmatrix} \begin{Bmatrix} u_1(f) \\ u_2(f) \\ u_3(f) \\ u_4(f) \end{Bmatrix}. \tag{A11}$$

A convenient way to solve Equations (A11) is given in Jezewski (1976).

For $x(f) \geqslant 0$

$$\begin{aligned} u_1^2(f) &= \frac{r(f)+x(f)}{2(1+P^2)}, \\ u_4(f) &= Pu_1(f), \\ u_2(f) &= [y(f)u_1(f)+z(f)u_4(f)]/[r(f)+x(f)], \\ u_3(f) &= [z(f)u_1(f)-y(f)u_4(f)]/[r(f)+x(f)], \\ P &= \frac{u_4(0)[r(f)+x(f)]+u_2(0)z(f)-u_3(0)y(f)}{u_1(0)[r(f)+x(f)]+u_2(0)y(f)+u_3(0)z(f)}, \end{aligned} \tag{A12}$$

or, for $x(f) < 0$

$$\begin{aligned} u_3^2(f) &= \frac{r(f)-x(f)}{2(1+Q^2)}, \\ u_2(f) &= Qu_3(f), \\ u_1(f) &= [z(f)u_3(f)+y(f)u_2(f)]/[r(f)-x(f)], \\ u_4(f) &= [z(f)u_2(f)-y(f)u_3(f)]/[r(f)-x(f)], \\ Q &= \frac{u_2(0)[r(f)-x(f)]+u_1(0)y(f)+u_4(0)z(f)}{u_3(0)[r(f)-x(f)]+u_1(0)z(f)-u_4(0)y(f)}. \end{aligned} \tag{A13}$$

These equations can be used to evaluate $\mathbf{u}(f)$ in the case no perturbation is present. In particular the vector $\mathbf{u}_0(s_{f0})$ in Equation (62) can be evaluated from Equations (A12–A13).

Finally, let us derive the vector $\mathbf{u}_1$. Again, we write Equation (5) as

$$\mathbf{r}(f) = L(\mathbf{u}(f))\mathbf{u}(f), \tag{A14}$$

and again it is seen that an extra boundary condition is needed.

Let us write Equation (40) for $s = s_f$ and form bilinear terms with $\mathbf{u}(0)$,

$$l(\mathbf{u}(0), \mathbf{u}(f)) = -l(\mathbf{u}(0), \boldsymbol{\alpha}(f))c_0(\alpha_T s_f^2) + l(\mathbf{u}(0), \boldsymbol{\beta}(f))s_f c_1(\alpha_T s_f^2). \tag{A15}$$

From Equations (67–68) it then follows that

$$l(\mathbf{u}(0), \boldsymbol{\alpha}(f)) = -l(\mathbf{u}(0), \mathbf{u}(0)) + \varepsilon l(\mathbf{u}(0), \boldsymbol{\alpha}_1)\,, \tag{A16}$$

$$l(\mathbf{u}(0), \boldsymbol{\beta}(f)) = l(\mathbf{u}(0), \dot{\mathbf{u}}(0)) + \varepsilon l(\mathbf{u}(0), \boldsymbol{\beta}_1)\,, \tag{A17}$$

and, because of Equations (A5–A6), we can write Equation (A15) as

$$\varepsilon q_1 \equiv l(\mathbf{u}(0), \mathbf{u}(f)) = \varepsilon l(\mathbf{u}(0), -\boldsymbol{\alpha}_1 c_0(\alpha_{T0} s_{f0}^2) + \boldsymbol{\beta}_1 s_{f0} c_1(\alpha_{T0} s_{f0}^2))\,, \tag{A18}$$

which constitutes the new missing boundary condition. Therefore, we can write

$$\begin{Bmatrix} x(f) \\ y(f) \\ z(f) \\ \varepsilon q_1 \end{Bmatrix} = \begin{bmatrix} u_1(f) & -u_2(f) & -u_3(f) & u_4(f) \\ u_2(f) & u_1(f) & -u_4(f) & -u_3(f) \\ u_3(f) & u_4(f) & u_1(f) & u_2(f) \\ u_4(0) & -u_3(0) & u_2(0) & -u_1(0) \end{bmatrix} \begin{Bmatrix} u_1(f) \\ u_2(f) \\ u_3(f) \\ u_4(f) \end{Bmatrix}, \tag{A19}$$

or using Equation (65) together with Equation (A11) for $\mathbf{u}(f) = \mathbf{u}_0(s_{f0})$,

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ q_1 \end{Bmatrix} = \begin{bmatrix} u_{01}(s_{f0}) & -u_{02}(s_{f0}) & -u_{03}(s_{f0}) & u_{04}(s_{f0}) \\ u_{02}(s_{f0}) & u_{01}(s_{f0}) & -u_{04}(s_{f0}) & -u_{03}(s_{f0}) \\ u_{03}(s_{f0}) & u_{04}(s_{f0}) & u_{01}(s_{f0}) & u_{02}(s_{f0}) \\ u_4(0) & -u_3(0) & u_2(0) & -u_1(0) \end{bmatrix} \begin{Bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \end{Bmatrix}, \tag{A20}$$

which can be inverted to obtain the vector $\mathbf{u}_1$.

A convenient way to solve Equations (A20) is

$$\begin{aligned} u_{11} &= \frac{2q_1(u_{01}^2 + u_{04}^2)u_{04}}{P}\,, \\ u_{14} &= \frac{-u_{01}}{u_{04}} u_{11}\,, \\ u_{12} &= \frac{1}{2(u_{01}^2 + u_{04}^2)}[y u_{11} + z u_{14}]\,, \\ u_{13} &= \frac{1}{2(u_{01}^2 + u_{04}^2)}[z u_{11} - y u_{14}]\,, \\ P &= 2(u_{01}^2 + u_{04}^2)(u_4(0)u_{04} + u_1(0)u_{01}) + \\ &\quad + y(u_2(0)u_{01} - u_3(0)u_{04}) + z(u_2(0)u_{04} + u_3(0)u_{01})\,, \end{aligned} \tag{A21}$$

where

$$x \equiv x(f), \quad \text{etc.}, \quad u_{0i} \equiv u_{0i}(s_{f0}) \qquad (i = 1, 2, 3, 4)\,, \tag{A22}$$

using the fact that

$$\mathbf{u}_0^T(s_{f0})\mathbf{u}_1 = 0\,.$$

Appendix B

The solution of the perturbed universal Lambert problem necessitates the evaluation of integrals of Equations (76–80). In the case of the J_2 oblateness perturbation, it is possible to resolve these integrals in closed form. The perturbing potential V_1 is given by

$$V_1 = \frac{1}{2}\left(\frac{3z^2 - r^2}{r^5}\right) \tag{B1}$$

and

$$\mathbf{Q}_1 = -\tfrac{1}{2}[K\mathbf{u}]|_{\mathbf{u}=\mathbf{u}_0} \tag{B2}$$

with

$$K = \frac{1}{r^5}[3rzM - (6z^2 - r^2)\mathbf{1}] \tag{B3}$$

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{B4}$$

It can be shown that

$$G_1 = [rV_1]|_{\mathbf{u}=\mathbf{u}_0} \quad \text{or} \quad G_1 = [\mathbf{u}^T\mathbf{Q}_1]|_{\mathbf{u}=\mathbf{u}_0} . \tag{B5}$$

Using the following substitution

$$w = \frac{sc_1}{c_0}, \qquad sc_1 \equiv sc_1(\alpha_T s^2), \qquad c_0 \equiv c_0(\alpha_T s^2), \tag{B6}$$

it is easy to show that

$$c_0^2 = \frac{1}{1+\alpha_T w^2}, \qquad s^2 c_1^2 = \frac{w^2}{1+\alpha_T w^2},$$

$$sc_1 c_0 = \frac{w}{1+\alpha_T w^2}, \qquad \tilde{c}_0 = \frac{1-\alpha_T w^2}{1+\alpha_T w^2},$$

$$\mathrm{d}s = \frac{\mathrm{d}w}{1+\alpha_T w^2} \tag{B7}$$

and

$$r_0 = \frac{\xi}{1+\alpha_T w^2}, \qquad z_0 = \frac{B_1 + B_2 w + B_3 w^2}{1+\alpha_T w^2}, \tag{B8}$$

$$\xi = A_1 + A_2 w + A_3 w^2 , \tag{B9}$$

$$\begin{aligned} &A_1 = r(0), \qquad A_2 = \dot{r}(0), \qquad A_3 = \dot{\mathbf{u}}^T(0)\dot{\mathbf{u}}(0), \\ &B_1 = \mathbf{u}^T(0)M\mathbf{u}(0), \qquad B_2 = 2\mathbf{u}^T(0)M\dot{\mathbf{u}}(0), \qquad B_3 = \dot{\mathbf{u}}^T(0)M\dot{\mathbf{u}}(0). \end{aligned} \tag{B10}$$

Next, let us write integrals of Equations (76–80) as follows

$$\boldsymbol{\alpha}_1 = -\tfrac{1}{2}(T_1\mathbf{u}(0) + T_2\dot{\mathbf{u}}(0)), \tag{B11}$$

$$\boldsymbol{\beta}_1 = -\tfrac{1}{2}(T_3\mathbf{u}(0) + T_1\dot{\mathbf{u}}(0)), \tag{B12}$$

$$a_1 = -\tfrac{1}{2}T_4, \tag{B13}$$

$$\tau_1 = \tfrac{1}{2}T_5, \tag{B14}$$

with

$$T_1 = \int_0^{s_{f0}} K(\mathbf{u}_0)sc_1c_0\,\mathrm{d}s,$$

$$T_2 = \int_0^{s_{f0}} K(\mathbf{u}_0)s^2c_1^2\,\mathrm{d}s,$$

$$T_3 = \int_0^{s_{f0}} K(\mathbf{u}_0)c_0^2\,\mathrm{d}s,$$

$$T_4 = \int_0^{s_{f0}} g_1(\mathbf{u}_0)sc_1c_0\,\mathrm{d}s,$$

$$T_5 = \frac{1}{2}\int_0^{s_{f0}} g_1(\mathbf{u}_0)s^2c_1^2\,\mathrm{d}s, \tag{B15}$$

$$g_1 = \frac{3z^2 - r^2}{r^4},$$

$$c_i \equiv c_i(\alpha_{T0}s^2), \qquad \tilde{c}_i \equiv \tilde{c}_i(\alpha_{T0}s^2). \tag{B16}$$

First, let us compute integrals T_1, T_2 and T_3. Using Equations (B8–B10) it can be shown that

$$T_1 = 3\Big(\sum_{i=0}^{6} c_iI_{i+1}\Big)M - \Big(\sum_{i=0}^{6} \mathrm{d}_iI_{i+1}\Big)\mathbf{1},$$

$$T_2 = 3\Big(\sum_{i=0}^{6} c_iI_{i+2}\Big)M - \Big(\sum_{i=0}^{6} \mathrm{d}_iI_{i+2}\Big)\mathbf{1},$$

$$T_3 = 3\Big(\sum_{i=0}^{6} c_iI_i\Big)M - \Big(\sum_{i=0}^{6} \mathrm{d}_iI_i\Big)\mathbf{1}, \tag{B17}$$

where

$$I_i = \int_0^{s_{f0}(w)} \frac{w^i}{\xi^5} \mathrm{d}w \quad (i = 0, 1, \ldots, 3), \tag{B18}$$

and

$$\begin{aligned}
&c_0 = b_0, \quad c_1 = b_1, \quad c_2 = b_2 + \alpha_{T0} b_0, \quad c_3 = b_3 + \alpha_{T0} b_1, \\
&c_4 = b_4 + \alpha_{T0} b_2, \quad c_5 = \alpha_{T0} b_3, \quad c_6 = \alpha_{T0} b_4, \\
&b_0 = A_1 B_1, \quad b_1 = A_1 B_2 + A_2 B_1, \quad b_2 = A_1 B_3 + A_2 B_2 + A_3 B_1, \\
&b_3 = A_2 B_3 + A_3 B_2, \quad b_4 = A_3 B_3, \\
&d_0 = e_0, \quad d_1 = e_1, \quad d_2 = e_2 + \alpha_{T0} l_0, \quad d_3 = e_3 + \alpha_{T0} l_1, \\
&d_4 = e_4 + \alpha_{T0} e_2, \quad d_5 = \alpha_{T0} e_3, \quad d_6 = \alpha_{T0} e_4, \\
&e_0 = 6B_1^2 - A_1^2, \quad e_1 = 12 B_1 B_2 - 2A_1 A_2, \\
&e_2 = 6B_2^2 - A_2^2 + 12 B_1 B_3 - 2A_1 A_3, \\
&e_3 = 12 B_2 B_3 - 2A_2 A_3, \quad e_4 = 6B_3^2 - A_3^2,
\end{aligned} \tag{B19}$$

where all constants are independent of s.

Integrals (B18) may be computed as indicated below.

The discriminant of the quadratic form ξ is

$$A_2^2 - 4A_1 A_3 = 4[\mathbf{u}^T(0)\dot{\mathbf{u}}(0)]^2 - [\mathbf{u}^T(0)\mathbf{u}(0)][\dot{\mathbf{u}}^T(0)\dot{\mathbf{u}}(0)], \tag{B20}$$

which is always smaller or equal to zero because of Schwartz Inequality. Therefore, in all cases we can write the following partial fractions expansion

$$\frac{w^i}{\xi^5} = \frac{1}{A_1^5} \sum_{j=1}^{5} \frac{K_j^{(i)}}{(w - w_1)^j} + \frac{\bar{K}_j^{(i)}}{(w - \bar{w}_1)^j}, \tag{B21}$$

with

$$w_1 = \frac{-A_2 + i(4A_1 A_3 - A_2^2)^{1/2}}{2A_1}, \qquad i = \sqrt{-1}, \tag{B22}$$

$$K_{5-j}^{(i)} = \frac{1}{j!} \frac{\mathrm{d}^j}{\mathrm{d}w^j} \left(\frac{w^i}{(w - w_1)^5} \right) \Bigg|_{w = w_1} \qquad \begin{cases} j = 0, 1, 2, 3, 4 \\ i = 0, 1, \ldots, 8 \end{cases} \tag{B23}$$

and $\bar{K}_j^{(i)}$ is the complex conjugate of $K_j^{(i)}$.

A useful recurrence relation is

$$\begin{aligned}
K_l^{(i+1)} = K_{l+1}^{(1)} + w_1 K_l^{(i)}, \qquad & i = 0 \; (l = 1, 2, 3, 4), \\
& i = 1 \; (l = 1, 2, 3, 4), \quad \text{etc.}
\end{aligned} \tag{B24}$$

with

$$K_5^{(0)} = \frac{1}{(w_1 - \bar{w}_1)^5}, \qquad K_4^{(0)} = \frac{-5K_5^{(0)}}{(w_1 - \bar{w}_1)},$$

$$K_3^{(0)} = \frac{-3K_4^{(0)}}{(w_1 - \bar{w}_1)}, \qquad K_2^{(0)} = -\frac{7}{3}\frac{K_3^{(0)}}{(w_1 - \bar{w}_1)},$$

$$K_1^{(0)} = \frac{-2K_2^{(0)}}{(w_1 - \bar{w}_1)}, \tag{B25}$$

and

$$K_5^{(i)} = w_1^i K_5^{(0)}, \qquad i = 1, 2, \ldots, 8. \tag{B26}$$

From Equation (B21) we have

$$I_i = \frac{1}{A_1^5} 2Re \sum_{j=1}^{5} \int_0^{s_{f0}(w)} \frac{K_j^{(i)}}{(w - w_1)^j} \mathrm{d}w, \tag{B27}$$

where Re means 'real part' and we assume that w takes on real values only. Finally,

$$I_i = \frac{1}{A_1^5} 2Re K_1^{(i)} \ln(w - w_1) - \sum_{j=2}^{5} \frac{K_j^{(i)}}{(j-1)(w - w_1)^{j-1}} \Bigg|_{w=0}^{w=(sc_1/c_0)} \tag{B28}$$

$$i = 0, 1, \ldots, 8$$

$$w \text{ real.}$$

The integrals T_4 and T_5 can be evaluated in a similar manner

$$T_4 = \sum_{i=0}^{4} f_i \bar{I}_{i+1},$$

$$T_5 = \tfrac{1}{2} \sum_{i=0}^{4} f_i \bar{I}_{i+2}, \tag{B29}$$

with

$$\bar{I}_i = \int_0^{s_{f0}(w)} \frac{w^i}{\xi^4} \mathrm{d}w, \qquad i = 0, 1, \ldots, 6, \tag{B30}$$

and

$$f_0 = 3B_1^2 - A_1^2, \qquad f_1 = 6B_1B_2 - 2A_1A_2,$$

$$f_2 = 3B_2^2 - A_2^2 + 6B_1B_3 - 2A_1A_3,$$

$$f_3 = 6B_2B_3 - 2A_2A_3, \qquad f_4 = 3B_3^2 - A_3^2, \tag{B31}$$

so that

$$\bar{I}_i = \frac{1}{A_1^4} 2Re\tilde{K}_1^{(i)} \ln(w - w_1) - \sum_{j=2}^{4} \frac{\tilde{K}_j^{(i)}}{(j-1)(w-w_1)^{j-1}} \Bigg|_{w=0}^{w=(sc_1/c_0)} \tag{B32}$$

$i = 0, 1, \ldots, 8$

w real

with

$$\tilde{K}_l^{(i+1)} = \tilde{K}_{l+1}^{(i)} + w_1 \tilde{K}_l^{(i)}, \tag{B33}$$

for

$$i = 0\ (l = 1, 2, 3), \qquad i = 1\ (l = 1, 2, 3), \ldots$$

and

$$\tilde{K}_4^{(0)} = \frac{1}{(w_1 - \bar{w}_1)^4}, \qquad \tilde{K}_3^{(0)} = -\frac{4\tilde{K}_4^{(0)}}{w_1 - \bar{w}_1},$$

$$\tilde{K}_2^{(0)} = -\frac{5}{2}\frac{\tilde{K}_3^{(0)}}{w_1 - \bar{w}_1}, \qquad \tilde{K}_1^{(0)} = -\frac{\tilde{K}_2^{(0)}}{w_1 - \bar{w}_1},$$

$$\tilde{K}_4^{(i)} = w_1^i K_4^{(0)}, \qquad i = 1, 2, \ldots, 6. \tag{B34}$$

It should also be noted that integrals (76–80) may also be evaluated by any suitable numerical method (e.g. the trapezoidal rule in connection with Romberg's principle).

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