A SHORT DERIVATION OF THE SPERLING-BURDET EQUATIONS

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Abstract. A short derivation is given of the regularized equtions of motion for the perturbed two-body problem. This method is then applied to the slightly modified time transformation $dt/ds = r/\omega$.

As is well known, the equations of motion for the perturbed two-body problem can be written in the form

$$\ddot{\mathbf{x}} + \frac{K^2\mathbf{x}}{r^3} = \bar{\mathbf{P}} \tag{1}$$

In the unperturbed case, the integrals of Equation (1) are
angular momentum:

$$\mathbf{c} = \mathbf{x} \times \dot{\mathbf{x}}, \qquad (2)^{\dagger}$$

energy:

$$\omega^2 = \frac{2K^2}{r} - v^2, \tag{3}$$

Laplace or eccentricity vector:

$$\mathbf{A} = \mathbf{x}\left(\frac{K^2}{r} - \omega^2\right) - r\dot{r}\dot{\mathbf{x}}. \tag{4}$$

In the perturbed case, respectively,

$$\frac{d\mathbf{c}}{dt} = \mathbf{x} \times \mathbf{P}, \tag{5}$$

$$\frac{d\omega^2}{dt} = -2\mathbf{P}\cdot\dot{\mathbf{x}}, \tag{6}$$

$$\frac{d\mathbf{A}}{dt} = 2\mathbf{x}\dot{\mathbf{P}}\cdot\dot{\mathbf{x}} - \dot{\mathbf{x}}\mathbf{P}\cdot\mathbf{x} - r\dot{r}\dot{\mathbf{P}}. \tag{7}$$

By using Equations (2) through (7) and the time transformation $dt/ds = r$, Equation (1) can be transformed into a (perturbed) harmonic oscillator. (See, for example, Burdet, 1968). A new short derivation is presented here.

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† Dots indicate differentiation with respect to physical time t, primes with respect to s.

Multiply the Laplace vector [Equation (4)] by r and introduce the fictitious time defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}s}, \tag{8}$$

to obtain

$$r\mathbf{A} = \mathbf{x}(K^2 - r\omega^2) - r'\mathbf{x}'. \tag{9}$$

Differentiate this equation once with respect to s and rearrange. This yields

$$r'(\mathbf{x}'' + \omega^2\mathbf{x} + \mathbf{A} - r^2\mathbf{P}) + \mathbf{x}'(r'' + \omega^2 r - K^2 - r\mathbf{P}\cdot\mathbf{x}) = 0. \tag{10}$$

It is now shown that both quantities in parentheses vanish. Because

$$\mathbf{x} \times (\mathbf{x}'' + \omega^2\mathbf{x} + \mathbf{A} - r^2\mathbf{P}) = \frac{\mathrm{d}}{\mathrm{d}s}(r\mathbf{C}) - r'\mathbf{C} - r^2\mathbf{x} \times \mathbf{P} = 0,$$

the vector

$$\mathbf{x}'' + \omega^2\mathbf{x} + \mathbf{A} - r^2\mathbf{P}$$

is collinear with $\mathbf{x}$; by Equation (10), this vector is collinear with $\mathbf{x}'$. Thus,

$$\mathbf{x}'' + \omega^2\mathbf{x} + \mathbf{A} - r^2\mathbf{P} = 0. \tag{11*}$$

From Equation (10), it follows immediately that

$$r'' + \omega^2 r - K^2 = r\mathbf{P}\cdot\mathbf{x}. \tag{12}$$

If, instead of Equation (8), the transformation

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{r}{\omega} \tag{13}$$

is used, Equations (6) and (7) become

$$\frac{\mathrm{d}\omega}{\mathrm{d}s} = -\frac{1}{\omega}\mathbf{P}\cdot\mathbf{x}' \tag{14}$$

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}s} = 2\mathbf{x}\mathbf{P}\cdot\mathbf{x}' - \mathbf{x}'\mathbf{P}\cdot\mathbf{x} - rr'\mathbf{P}, \tag{15}$$

respectively. Proceeding as before, it follows that

$$\mathbf{x}'' + \mathbf{x} + \frac{\mathbf{A}}{\omega^2} = \frac{1}{\omega^2}(r^2\mathbf{P} + \mathbf{P}\cdot\mathbf{x}'\mathbf{x}') \tag{16}$$

$$r'' + r - \frac{K^2}{\omega^2} = \frac{1}{\omega^2}(r\mathbf{P}\cdot\mathbf{x} + r'\mathbf{P}\cdot\mathbf{x}'). \tag{17}$$

* In the special case where $\dot{\mathbf{x}}$ and $\mathbf{x}$ are collinear, Equation (11) reduces to Equation (12), which is obtained by introducing Equation (8) into Equation (3) and differentiating.

The motion is completely determined by the set of Equations (13)–(16), which is of order 11. This system however, is not stable in Liapuniov's sense if the perturbing forces are set equal to zero because Equation (13) is non-linear. In this case, Equation (17) has to be added raising the order of the system to 13 while making r a new dependent variable.

Note that Equations (11) and (12) are uniform, but that Equations (16) and (17) are not; that is, Equations (16) and (17) fail when $\omega = 0$ (a parabola). Finally, the time transformation (13) leading to the special form of the oscillator Equations (16) and (17) was first given in Stiefel and Scheifele (1971), for use in K/S theory. It is shown in the above reference that this time transformation leads to better numerical behavior in conservative cases, because the oscillator frequency in (16) and (17) is constant.

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References

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