ON THE STABILITY OF CIRCULAR 'ASTEROID' ORBITS IN AN N-PLANETARY SYSTEM

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(Received 21 August, 1975)

Abstract. The restricted problem of the motion of a point of negligible mass ('asteroid') in an N-planetary system is considered. It is assumed that all the planets move about the central body ('Sun') along circular orbits in the same plane and the mean motions of the asteroid and the planets are incommensurable. The asteroid orbit evolution is described as a first approximation by secular equations with the perturbing function averaged by the mean longitudes of the asteroid and the planets. For small values of the asteroid orbit eccentricity an expression for the secular part of the perturbing function has been obtained. This expression holds for the arbitrary values of the asteroid orbit semi-axis which are different from those of the planet orbit radii. The stability of the asteroid circular orbits in a linear approximation with respect to the eccentricity is studied. The critical inclinations for a Solar system model are calculated.

1. Problem Formulation

Let us consider the motion of a point of zero mass attracted by a system of $(N+1)$ material points. Assume that N points of masses m_j $(j=1, 2, \ldots, N)$ move uniformly along circular coplanar orbits about the central point of mass $m_0 \gg m_{j\neq 0}$. Such a scheme can simulate as a first approximation the motion of an asteroid in a planetary system, hence below the point m_0 will be referred to as the Sun, m_j – as planets and the point of zero mass – as an asteroid.

Assume that the mean motions of the planets and the asteroid are incommensurable. Then the evolution of the asteroid orbit elements in a first approximation with respect to the perturbing masses of the planets is described by secular equations with the perturbing function averaged independently over mean longitudes of the planets λ_j and the asteroid λ. Such a formulation for the restricted nonresonance problem was described in Krasinsky (1973).

The secular part of the perturbing function can be written as

$$W = \frac{1}{2\pi} \int_0^{2\pi} V \, \mathrm{d}\lambda, \tag{1}$$

where

$$V = \frac{f}{2\pi} \sum_{j=1}^{N} m_j \int_0^{2\pi} \frac{\mathrm{d}\lambda_j}{\Delta_j}, \qquad \Delta_j^2 = r^2 + a_j^2 - 2ra_j \cos H_j, \tag{2}$$

Celestial Mechanics **13** (1976) 313–324.

f is the gravitational constant; $\mathbf{r}$, $\mathbf{r}_j$ are the heliocentric radii-vectors of the asteroid and the planets, respectively; $r=|\mathbf{r}|$, $a_j=|\mathbf{r}_j|=\text{const}$; H_j is the angle between the vectors $\mathbf{r}$ and $\mathbf{r}_j$. Lagrange equations with the function W

$$\frac{\mathrm{d}a}{\mathrm{d}t}=0, \qquad \frac{\mathrm{d}e}{\mathrm{d}t}=-\frac{\sqrt{1-e^2}}{na^2e}\frac{\partial W}{\partial\omega},$$

$$\frac{\mathrm{d}i}{\mathrm{d}t}=\frac{\operatorname{ctg} i}{na^2\sqrt{1-e^2}}\frac{\partial W}{\partial\omega}, \qquad \frac{\mathrm{d}\Omega}{\mathrm{d}t}=\frac{\operatorname{cosec} i}{na^2\sqrt{1-e^2}}\frac{\partial W}{\partial i}, \tag{3}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t}=\frac{\sqrt{1-e^2}}{na^2e}\frac{\partial W}{\partial e}-\frac{\operatorname{ctg} i}{na^2\sqrt{1-e^2}}\frac{\partial W}{\partial i}$$

have the three first integrals in involution

$$W=\text{const}, \qquad a=\text{const}, \qquad (1-e^2)\cos^2 i=\text{const};$$

($a, n, e, i, \Omega, \omega$ are the semi-major axis of the asteroid orbit, the mean motion, the eccentricity, the inclination to the planet motion plane, the ascending node longitude and the argument of perihelion latitude, respectively).

Therefore the problem under consideration is integrable as well as the circular twice-averaged restricted three-body one ($N=1$) (Moiseev, 1945).

For the case $N=1$ in Lidov (1961, 1962) Hill's approximation of the problem ($r\ll a_1$) was analysed; in Kozai (1962), Aksenov (1967) various expressions for the function W were obtained and the secular equations for the inner case of the problem ($r<a_1$) were studied; in Lidov and Ziglin (1974) the case of uniformly close-by orbits ($r\approx a_1$, $e\approx 0$, $\sin i\approx 0$) was considered.

2. Perturbing Function Transformation

Let us discuss the calculation of the function V and W. Usually in analytical investigations the expansion of the function $1/\Delta_j$ in a series over powers of ratios r/a_j when $r<a_j$ or a_j/r when $r>a_j$ is used. However these expansions, each separately, turn out to be inapplicable if during the motion r can be both smaller and greater than a_j. In Boda (1931), Petrovskaya (1970, 1972) for the restricted elliptical three-body problem the presentation of the function Δ_1 as

$$\Delta_1=(r+r_1)(1-\varepsilon_1)^{1/2} \tag{4}$$

was used, where r_1 is the distance from the planet to the attraction center and the parameter

$$\varepsilon_1=\frac{2rr_1(1+\cos H_1)}{(r+r_1)^2} \tag{5}$$

remains less than unit for all the values of r, H_1 apart from the point $r=r_1$, $H_1=0$. The introduction of such a parameter allows to obtain a united analytical expression for the function $1/\Delta_1$ in the form of a series over powers ε_1 converging everywhere except the points of the asteroid–planet collision.

It can be readily seen that Δ_1 along with (4) allows the following presentation

$$\Delta_1 = (r^2 + r_1^2)^{1/2}(1 - \delta_1)^{1/2}, \tag{6}$$

where the parameter

$$\delta_1 = \frac{2rr_1 \cos H_1}{r^2 + r_1^2} \tag{7}$$

has the same properties as ε_1 ($\delta_1 = \pm 1$ at $r = r_1$, $H_1 = 0, \pi$).

Though the introduction of δ_1 preserves the irrational factor $(r^2 + r_1^2)^{1/2}$ in (6), from our point of view using this parameter simplifies the process of obtaining the secular part of the perturbing function.

In the case being considered $r_j = a_j = \text{const}$ ($j = 1, 2, \ldots, N$). Using the presentation

$$\Delta_j = \sqrt{a_j^2 + r^2}\left(1 - \frac{2a_j r \cos H_j}{a_j^2 + r^2}\right)^{1/2} \tag{8}$$

and integrating over λ_j in (2) we can find

$$V = \frac{2f}{\pi} \sum_{j=1}^{N} \frac{m_j}{\sqrt{a_j^2 + r^2}} \frac{K(\varkappa_j)}{\sqrt{1 + \xi_j}}, \tag{9}$$

where

$$\varkappa_j = \sqrt{\frac{2\xi_j}{1 + \xi_j}}, \qquad \xi_j = \frac{2a_j r \cos \varphi}{a_j^2 + r^2},$$

φ is the latitude of the asteroid with respect to the planet motion plane; $K(\varkappa_j)$ is the complete elliptical integral of the first kind with the modulus $\varkappa_j$.

Note that (9) can be obtained by the simple transformation of the expression for the force function of a system of Gauss circular rings. Such a system was used by Duboshin (1945) for describing the mutual perturbations of Saturn's satellites.

Since $\xi_j \leqslant 1$ ($j = 1, 2, \ldots, N$) the function V can be written as a sum of N power series with respect to ξ_j uniformly converging everywhere except the circumferences $r = a_j$, $\varphi = 0$. Using the expansions of the functions $K(\varkappa_j)$ and $(1 + \xi_j)^{-1/2}$ in a series over the powers ξ_j we find

$$V = f \sum_{j=1}^{N} \frac{m_j}{\sqrt{a_j^2 + r^2}} F(\tfrac{1}{4}, \tfrac{3}{4}; 1; \xi_j^2), \tag{10}$$

where $F(\frac{1}{4}, \frac{3}{4}; 1; \xi_j^2) = \sum_{k=0}^{\infty} B_k \xi_j^{2k}$ is a hypergeometric function;

$$B_k = \frac{(4k)!}{2^{6k}(k!)^2(2k)!}, \qquad B_0 = 1,\ B_1 = \tfrac{3}{16},\ B_2 = \tfrac{105}{1024}, \ldots. \tag{11}$$

As k increases the coefficients B_k slowly decrease and with $k \to \infty$, $B_k \approx 1/k\pi\sqrt{2}$. The function W can be expressed

$$W = \frac{f}{2\pi} \sum_{j=1}^{N} m_j \sum_{k=0}^{\infty} B_k (2a_j)^{2k} \int_0^{2\pi} \frac{(r \cos \varphi)^{2k}}{(a_j^2 + r^2)^{2k+1/2}} \, d\lambda, \tag{12}$$

where

$$r = \frac{a(1 - e^2)}{1 + e \cos (U - \omega)},$$

$$\cos^2 \varphi = 1 - \sin^2 i \sin^2 U,$$

$$d\lambda = \frac{r^2}{a^2 \sqrt{1 - e^2}} \, dU,$$

U is the argument of the asteroid latitude.

The subintegral expression in (12) is sufficiently complicated. We have succeded in calculating the function W analytically only in several cases by means of the series expansion over powers e and $\sin i$. Besides, the closed expression for W can be obtained in the extreme cases when the condition

$$\frac{2a_j r}{a_j^2 + r^2} \ll 1$$

is satisfied for all j. All these cases were discussed in Vashkovjak (1975). The present paper deals only with the analysis of near-circular orbits of the asteroid.

3. Near – Circular Orbits

Assume that the asteroid orbit eccentricity is sufficiently small. Let us expand the integrand of (12) in a series over the powers e and truncate the terms of the order e^4 and up. All the required calculations carried out, the following expression for the function W can be obtained

$$W = \sum_{j=1}^{N} \sum_{k=0}^{\infty} M_k^{(j)} \{ R_k(i) + e^2 [S_k^{(j)}(i) + T_k^{(j)}(i) \cos 2\omega] \}. \tag{13}$$

Here

$$R_k(i) = F(-k, \tfrac{1}{2}; 1; \sin^2 i), \tag{14}$$

$$\begin{aligned} S_k^{(j)}(i) = {} & \tfrac{1}{4}\alpha_j (4k + 1)[(4k + 3)\alpha_j - 3] F(-k, \tfrac{1}{2}; 1; \sin^2 i) + \\ & + \frac{k}{2} [3 - 2\alpha_j (4k + 1) - \sin^2 i] F(1 - k, \tfrac{1}{2}; 1; \sin^2 i) + \\ & + \frac{k}{2} \sin^2 i [\alpha_j (4k + 1) - \tfrac{1}{2}] F(1 - k, \tfrac{3}{2}; 2; \sin^2 i) + \\ & + k(k - 1) \Big[F(2 - k, \tfrac{1}{2}; 1; \sin^2 i) - \\ & - \sin^2 i \left(1 - \frac{\sin^2 i}{2}\right) F(2 - k, \tfrac{3}{2}; 2; \sin^2 i) \Big], \end{aligned} \tag{15}$$

$$T_k^{(j)}(i) = \tfrac{1}{4}\alpha_j(4k+1)[(4k+3)\alpha_j - 3][F(-k, \tfrac{1}{2}; 1; \sin^2 i) - F(-k, \tfrac{3}{2}; 2; \sin^2 i)] + \frac{k}{2}[3 - 2\alpha_j(4k+1) + \sin^2 i]F(1-k, \tfrac{1}{2}; 1; \sin^2 i) + \frac{k}{2}[2\alpha_j(4k+1) - 3]\left(1 - \frac{\sin^2 i}{2}\right)F(1-k, \tfrac{3}{2}; 2; \sin^2 i) + k(k-1)\left[F(2-k, \tfrac{1}{2}; 1; \sin^2 i) - \left(1 - \frac{\sin^2 i}{2}\right)^2 F(2-k, \tfrac{3}{2}; 2; \sin^2 i)\right], \tag{16}$$

$$M_k^{(j)} = \frac{fm_j B_k}{\sqrt{a_j^2 + a^2}}\eta_j^{2k}, \qquad \eta_j = \frac{2a_j a}{a_j^2 + a^2} \leqslant 1, \qquad \alpha_j = \frac{a^2}{a_j^2 + a^2} \leqslant 1. \tag{17}$$

Set of the secular Equation (3) at small e has the form

$$\frac{da}{dt} = 0,$$

$$\frac{di}{dt} = -\frac{2\operatorname{ctg} i}{na^2} e^2 \sin 2\omega\, A(i),$$

$$\frac{d\Omega}{dt} = -\frac{\cos i}{na^2}\sum_{j=1}^{N}\sum_{k=0}^{\infty} M_k^{(j)} k F(1-k, \tfrac{3}{2}; 2; \sin^2 i), \tag{18}$$

$$\frac{de}{dt} = \frac{2e\sin 2\omega}{na^2} A(i),$$

$$\frac{d\omega}{dt} = \frac{2}{na^2}[B(i) + A(i)\cos 2\omega],$$

where

$$A(i) = \sum_{j=1}^{N}\sum_{k=0}^{\infty} M_k^{(j)} T_k^{(j)}(i),$$
$$B(i) = \sum_{j=1}^{N}\sum_{k=0}^{\infty} M_k^{(j)}\left[S_k^{(j)}(i) + \frac{k}{2}\cos^2 i\, F(1-k, \tfrac{3}{2}; 2; \sin^2 i)\right]. \tag{19}$$

Set (18) evidently describes the real evolution of the asteroid orbit elements only in the case when the eccentricity remains small throughout the considered time interval. Hence it is interesting to find out under what conditions the particular solution of the set $e=0$ turns out to be stable with respect to eccentricity.

When $e=0$, $i=i_0=\text{const}$,

$$\Omega = \Omega_0 - \frac{\cos i_0}{na^2}(t - t_0) \sum_{j=1}^{N} \sum_{k=0}^{\infty} M_k^{(j)} k F(1 - k, \tfrac{3}{2}; 2; \sin^2 i_0),$$

where Ω_0 is the value of Ω at the initial moment of time t_0 and the dependence $\omega(t)$ can be found from the last expression of (18).

An a linear approximation with respect to the eccentricity for the elements $l = e\cos\omega$, $h = e\sin\omega$ the set of linear differential equations with constant coefficients is obtained

$$\frac{dl}{dt} = \frac{2}{na^2}[A(i_0) - B(i_0)]h, \qquad \frac{dh}{dt} = \frac{2}{na^2}[A(i_0) + B(i_0)]l. \tag{20}$$

If the condition

$$A^2(i_0) - B^2(i_0) < 0 \tag{21}$$

is satisfied the trivial solution of linear set (20) is stable and the dependences $l(t)$ and $h(t)$ are defined by the following formulas

$$l = l_0 \cos\tau + h_0 \frac{A - B}{\sqrt{B^2 - A^2}} \sin\tau,$$

$$h = h_0 \cos\tau + l_0 \frac{A + B}{\sqrt{B^2 - A^2}} \sin\tau, \tag{22}$$

where

$$\tau = \frac{2}{na^2}\sqrt{B^2 - A^2}\,(t - t_0),$$

l_0, h_0 are the initial values of l and h for $t=t_0$.

The equation

$$A^2(i^*) - B^2(i^*) = 0 \tag{23}$$

is used for determining the so-called critical inclinations (Krasinsky, 1972). When $i_0=i^*$, $d\omega/dt=0$. If $A+B=0$, $\omega=0, \pi$ and if $A-B=0$, $\omega=\pm\pi/2$.

4. Calculation of the Critical Inclinations

Let us first consider the case $N=1$. It is known (Lidov, 1961, 1962) that with $a \to 0$ the critical value $\sin^2 i^* \to \frac{2}{5}$. With $a \to \infty$ $\sin^2 i^* \to \frac{4}{5}$. These limiting values can be obtained by solving Equation (23). Confining ourselves in formulas (19) to the items for which $k=0, 1$ we obtain

$$A(i^*) = C\left(1 - \frac{x}{4}\right)\sin^2 i^*, \qquad B(i^*) = C(\tfrac{4}{5} - \sin^2 i^*), \tag{24}$$

where

$$C = \tfrac{15}{16} \frac{f m_1 a_1^2 a^2}{(a_1^2 + a^2)^{5/2}},$$

$$x = \alpha_1(11 - 7\alpha_1), \qquad \alpha_1 = \frac{a^2}{a_1^2 + a^2}.$$

In this case the solution of (23) takes the form

$$\sin^2 i^* = \tfrac{4}{5} \frac{1 - \sqrt{1 - z}}{z}, \tag{25}$$

where

$$z = \frac{x}{2}\left(1 - \frac{x}{8}\right).$$

With

$$a \to 0, \alpha_1 \to 0, z \to 0, \sin^2 i^* \to \tfrac{2}{5}.$$

With

$$a \to \infty, \alpha_1 \to 1, z \to 1, \sin^2 i^* \to \tfrac{4}{5}.$$

There is a qualitative difference between these cases. If with $a \to 0$ the circular orbits are stable in a linear approximation with respect to e at $\sin^2 i_0 < \frac{2}{5}$ and unstable at $\sin^2 i_0 > \frac{2}{5}$ then with $a \to \infty$ linear stability is preserved at any value of $\sin^2 i_0 \neq \frac{4}{5}$. If in formulas (19) the items with $k=2$ are taken into account then for determining the critical inclinations we shall have the following asymptotic formulas

$$\sin^2 i^* = \tfrac{2}{5} - \tfrac{29}{100}\left(\frac{a}{a_1}\right)^2 \qquad (a \ll a_1) \tag{26}$$

$$\left.\begin{aligned} \sin^2 i_1^* &= \tfrac{4}{5} - \tfrac{6}{25}\left(\frac{a_1}{a}\right)^2 \\ \sin^2 i_2^* &= \tfrac{4}{5} - \tfrac{9}{50}\left(\frac{a_1}{a}\right)^2 \end{aligned}\right\} \qquad (a \gg a_1). \tag{27}$$

In the case $a \gg a_1$ linear stability takes place if $\sin^2 i_0 < \sin^2 i_1^*$ or $\sin^2 i_0 > \sin^2 i_2^*$. If $\sin^2 i_1^* < \sin^2 i_0 < \sin^2 i_2^*$ then instability appears. Note that the taking into consideration of the additional items in formulas (19) with $k=2$ is qualitatively equivalent to that of the fourth zonal harmonic (as well as of the second one) in the problem of the evolution of near-circular satellite orbits in the non-central gravitational field of a planet (Kugaenko and Elyasberg, 1968).

For $a < a_1$ the critical inclination values were calculated by Kozai (1962) and for arbitrary $a \neq a_1$ by Krasinsky (1972, 1973, 1974). It should be noted that with $a \to a_1$ the results obtained by Kozai and Krasinsky differ. It is likely to be explained by the difficulties in the calculation of the critical inclinations in the region $a \approx a_1$. In Krasinsky (1972) and later in Lidov and Ziglin (1974) the asymptotic formula was obtained which defines the critical inclinations for the inner case of the problem with $a \to a_1$

$$\cos i^* = 1 - \tfrac{1}{4}\left(1 - \frac{a}{a_1}\right). \tag{28}$$

Numerical solution of (23) which has been obtained for $N=1$ gives the critical inclination dependence on a, coinciding with the results of Krasinsky (1973, 1974). It is interesting to note that the dependence of i^* on a in the region $0 \leqslant a \leqslant a_1$ can be approximated with sufficient accuracy by the empirical formula

$$\sin^2 i^* = \tfrac{2}{5} \cos \frac{\pi a}{2a_1}. \tag{29}$$

The absolute approximation error does not exceed 0.01 in the region $0.1a_1 < a < < 0.9a_1$ and 0.001 in the region $a < 0.1a_1$ and $a > 0.9a_1$. With $a \to a_1$ from formula (29) it follows that

$$\cos i^* = 1 - \frac{\pi}{10}\left(1 - \frac{a}{a_1}\right). \tag{30}$$

At great values of a formulas (27) can be used for the calculation of the critical inclinations. At $a \approx a_1$ in formulas (19) a great number of terms should be taken into account.

Thus for a true determination of i^* with $a = 1.03a_1$ it turned out necessary to take into consideration the items with the numbers $k \leqslant 5500$ in the sums by 'k'.

To get an idea of the critical inclinations in the N-planetary problem we shall consider the case $N=2$ that allows to reveal the main qualitative peculiarities.

Let first $a_1 \ll a_2$. For the values of a meeting the condition $a_1 \ll a \ll a_2$ the critical inclinations can be obtained by using formula (25) in which

$$x = \frac{\displaystyle\sum_{j=1}^{2} \frac{m_j a_j^2 \alpha_j (11 - 7\alpha_j)}{(a_j^2 + a^2)^{5/2}}}{\displaystyle\sum_{j=1}^{2} \frac{m_j a_j^2}{(a_j^2 + a^2)^{5/2}}}. \tag{31}$$

Taking into account of the ratios a_1/a, a/a_2 being small and introducing the dimensionless parameter σ characterizing the relation of disturbing accelerations from the first and the second body by the formula

$$\sigma = \frac{m_1 a_1^2 a_2^3}{m_2 a^5},$$

we obtain

$$x = \frac{4\sigma}{\sigma + 1}.$$

In so doing the critical inclinations are defined by the formulas

$$\sin^2 i_1^* = \frac{4}{5}\frac{\sigma+1}{\sigma+2}, \qquad \sin^2 i_2^* = \frac{4}{5}\frac{\sigma+1}{\sigma}. \tag{32}$$

If $\sin^2 i_0 < \sin^2 i_1^*$ or $\sin^2 i_0 > \sin^2 i_2^*$ then the solution of linear system (20) is stable. If $\sin^2 i_1^* < \sin^2 i_0 < \sin^2 i_2^*$ then instability takes place. Note that Formulas (32) are similar to those defining the critical inclinations in a single particular case of the twice-averaged Hill's problem with the central planet oblateness when its equatorial plane coincides with the outer body orbit plane taken into account (Vashkovjak, 1974). The calculations carried out for $m_2 = m_1$, $a_2 = 50a_1$ showed (Vashkovjak, 1975) that within the range $6a_1 < a < 9a_1$ solution (32) practically coincides with the numerical solution of Equation (23).

In the same paper in the plane of the parameters a, $\sin^2 i_0$ the regions of instability of the asteroid circular orbits for different values of m_1, m_2, a_1, a_2 were plotted. Here we give just the calculation results for the two typical cases $a_2 = 2a_1$, $m_2 = 50m_1$ (Figure 1), $m_2 = 0.2m_1$ (Figure 2).

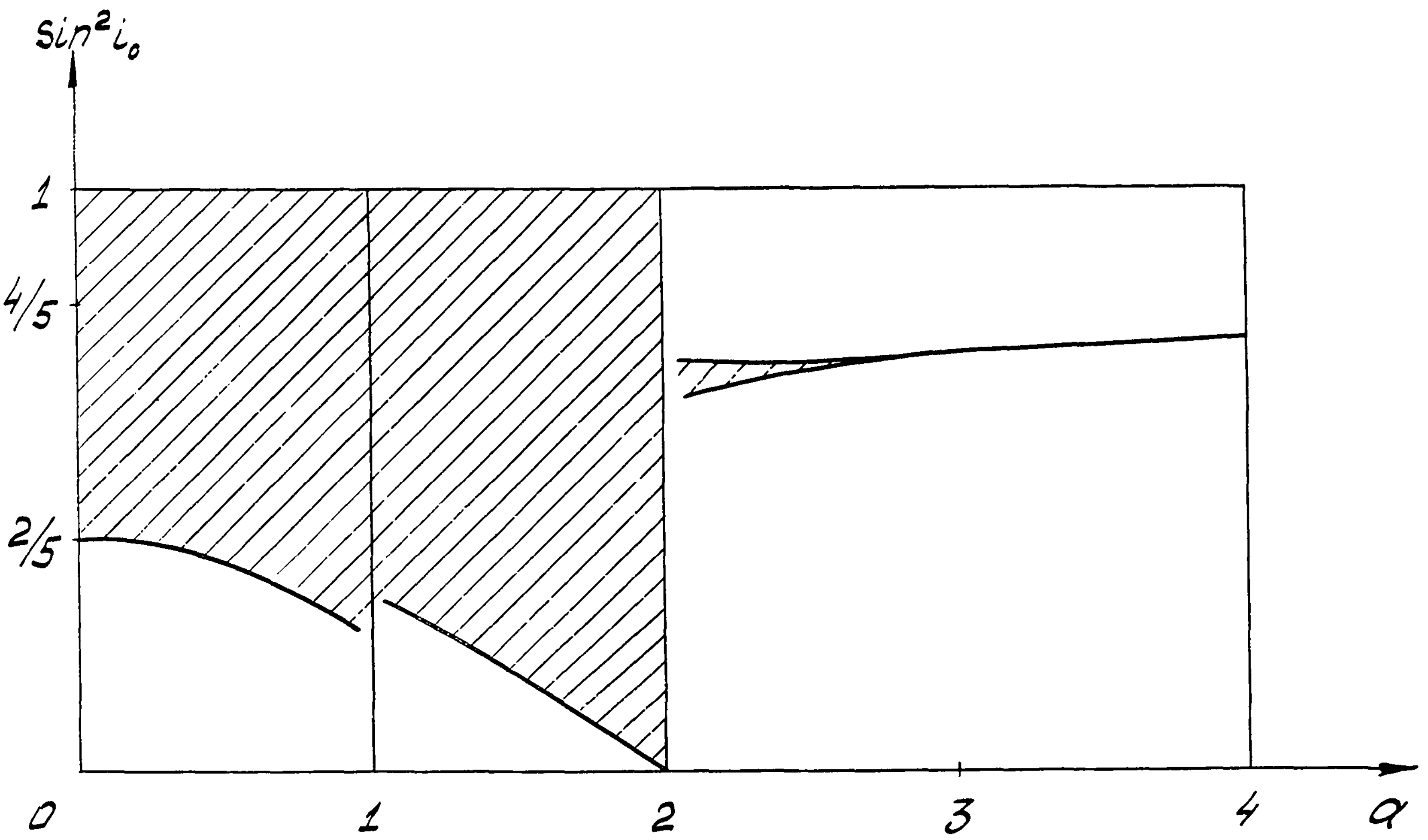

Fig. 1. $N=2$, $m_2=50$, $m_1=1$, $a_2=2$, $a_1=1$.

In the first case the boundaries of the shaded instability regions, except the $a \approx a_1$ region, are located so as they would be in the absence of the mass m_1. For the second case the availability of the linear stability region ($a > a_1$, $\sin^2 i_0 > \frac{4}{5}$) is typical. With $a \to a_1$ and $a \to a_2$ some calculation difficulties in equation (23) arise associated with a very slow convergence (19).

In addition to the examples described above, calculations of the critical inclinations in the nine-planetary system simulating the solar one have been carried out. The following values of a_j and m_j have been taken.

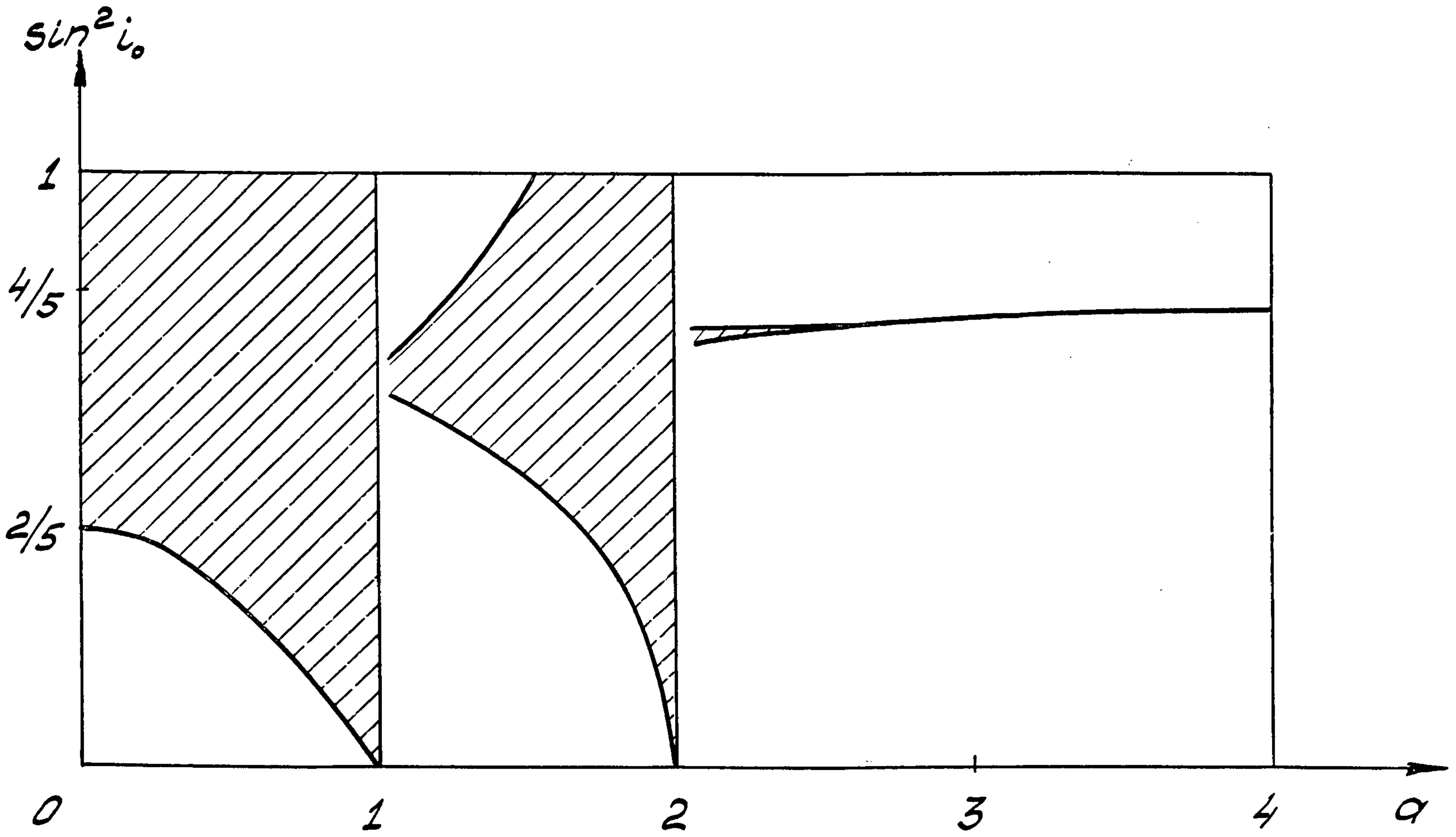

Fig. 2. $N=2$, $m_2=0.2$, $m_1=1$, $a_2=2$, $a_1=1$.

j	1	2	3	4	5	6	7	8	9
a_j	0.387	0.723	1.0	1.52	5.20	9.58	19.1	30.2	39.8
m_j	0.055	0.805	1.0	0.106	314.0	94.0	14.4	17.0	0.110

The results of the numerical solving of Equation (23) for the case being considered are presented in Figure 3. Figure 4 gives the region $0 \leqslant a \leqslant a_5$ larger scaled. From these figures one can see that the structure of the shaded instability regions qualitatively coincides with that of the regions in the two-planetary system (Figures 1–2). In the

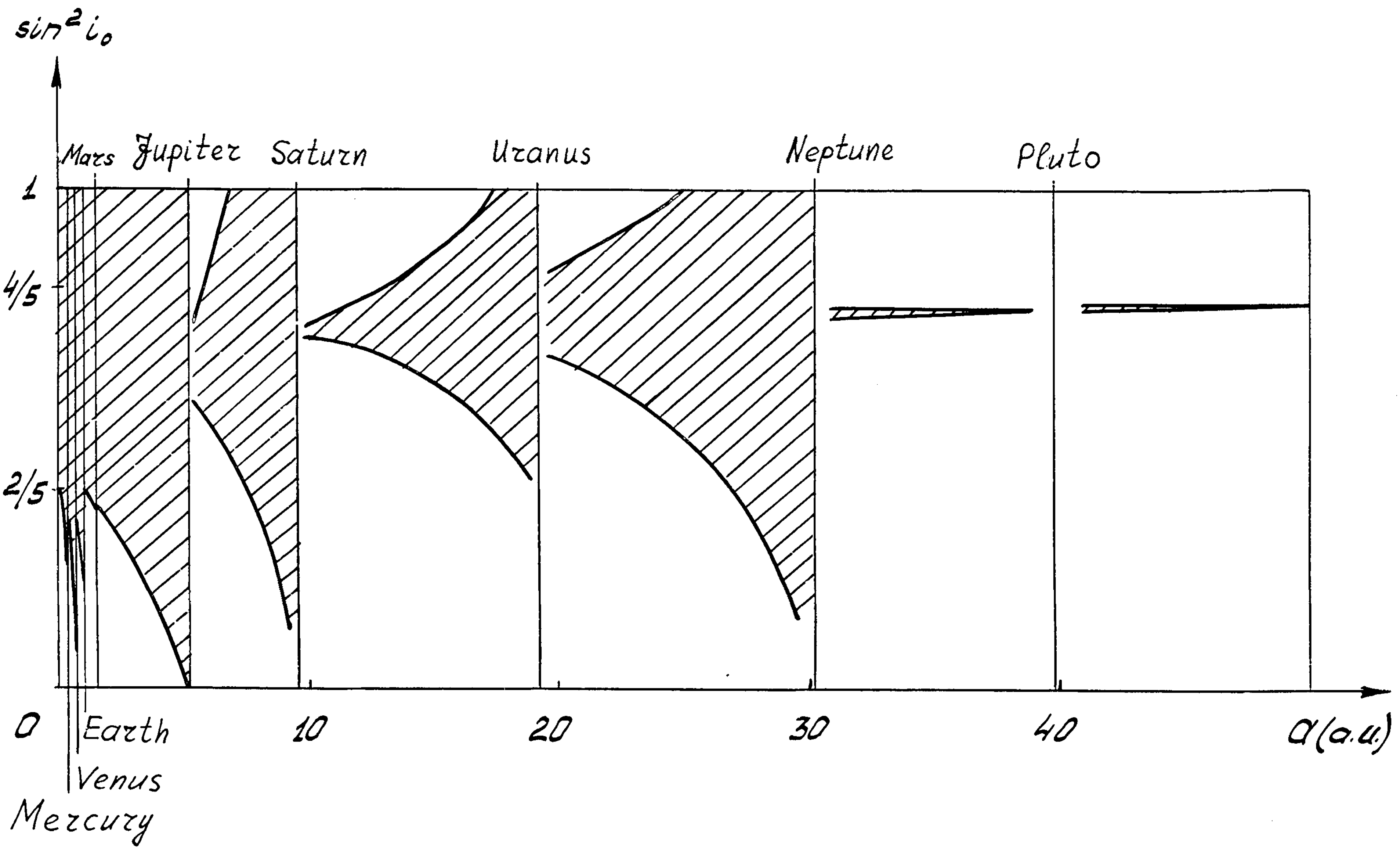

Fig. 3. $N=9$, $1 \leqslant j \leqslant 9$.

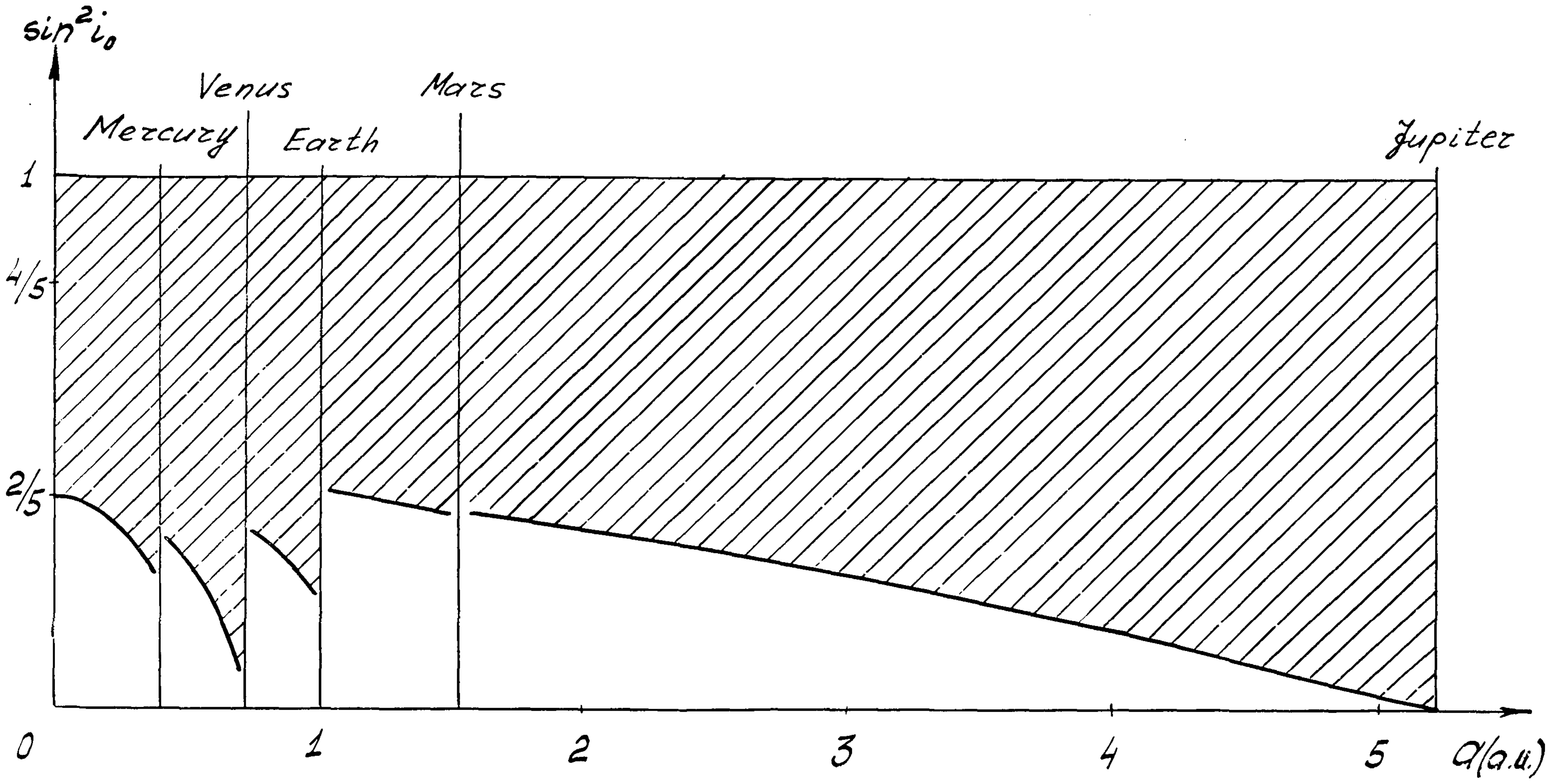

Fig. 4. $N=9$, $1 \leqslant j \leqslant 5$.

region $0 \leqslant a \leqslant a_1$ and $a_1 < a < a_2$ Venus has the main influence on the value of the critical inclination, and in the region $a_3 < a < a_4$ and $a_4 < a < a_5$ – Jupiter does, the influence of Mercury and Mars being inconsiderable. In addition the main Earth's effect, Venus and Jupiter also have a pronounced effect influence on the intermediate region $a_2 < a < a_3$. In the region $a > a_5$ the critical inclinations are determined by the joint effect of the major planets Jupiter, Saturn, Uranus and Neptune, the effect of Pluto being negligible.

Note that the critical values of $\sin^2 i^*$ at $a < a_5$ do not exceed $\approx \frac{2}{5}$. At $a > a_8$ the instability region is rather narrow. It is of interest to note that in the region $a_5 < a < a_8$ the criterion of linear stability of the asteroid circular orbits is valid for large inclinations to the planet motion plane, in particular for $i_0 = 90°$.

Acknowledgements

The author is thankful to M. L. Lidov for useful recommendations and O. S. Ryzhina for assistance in obtaining the numerical results.

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