THE KEPLER PROBLEM AND GEODESIC FLOWS IN SPACES OF CONSTANT CURVATURE

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Abstract. The main result of this paper is a theorem on the trajectory equivalence of phase flows on isoenergetic surfaces with a positive energy level in the Kepler problem and perturbed Kepler problem. The following two facts are crucial for proving it: firstly, an isomorphism of the phase flow on an isoenergetic surface in the Kepler problem and the geodesic flow in a constant curvature space. The isomorphism is studied in detail. In particular, all the integrals of the Kepler problem are obtained proceeding from the group-theory considerations. The second fact is a generalization of the theorem on structural stability of Anosov flows onto non-compact manifolds.

It is well-known that velocity hodographs of a mass point moving under the influence of Newton gravitation of a fixed centre are circles and straight lines (or their segments). It is readily seen that in a planar case the requirement of constancy of the power of the point 0 (in the elementary geometrical sense) with respect to the circles marks out hodographs of isoenergetic orbits. In turn, the geodesics of standard constant curvature metrics on a plane are also circles and straight lines (or their segments), the power of the point 0 with respect to the circles being constant. This is especially evident in the case of Lobachevskian space (in Poincaré's interpretation): with respect to circles orthogonal to the absolute (i.e. geodesics), the power of the point 0, being equal to the squared tangent, equals the squared radius of the absolute, i.e. a constant.

Such a coincidence makes us feel there is a certain connection between both the objects. In fact it does exist and was investigated by Jürgen Moser [7] in the case of negative energies*; the formulation of the corresponding result for a general case is given in [3] (details are given in [1]). The author hopes that the manner of reporting this connection in Section 1 of the present paper is the best one to fit the core of the problem.

The connection mentioned above consists in the fact that the change of time scale dt/r transforms the phase flow on an isoenergetic surface with energy level h into the geodesic flow on a manifold of constant curvature $-2h$ (though the latter flow acts not in a tangent bundle, but in a cotangent one). The manifolds in question are isometric to a sphere punctured at one point when $h < 0$, to Euclidian space punctured at one point when $h = 0$ and to Lobachevskian space punctured at one point when $h > 0$.

Being completed, these spaces become homogeneous. This makes it possible to build a complete system of first integrals (angular momentum and Laplace vector)

* And earlier by Györgyi [11] in the same case.

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proceeding from the Noether theorem. According to Smale, the first integral corresponding to the Lie group of symmetries takes values in the Lie co-algebra of the group. In connection with this we had to include here the calculation of the Lie algebras for the groups of motions of constant curvature spaces.

Geometrical objects used in the paper make it possible to give a geometrical interpretation to the eccentric anomaly and its analogue in a hyperbolic case.

Besides the problem of an attracting centre, the paper also deals with that of a repelling centre.

The geodesic flow in Lobachevskian space, being an Anosov flow, though a non-compact one, is structurally stable [4]. This, by virtue of all that has been said above, involves (see Theorem 2, Section 2) trajectory equivalence of phase flows on regularized isoenergetic surfaces ($h > 0$) of the Kepler problem and perturbed Kepler problem in a certain class of perturbations. A typical example of an admissible perturbation is that caused, e.g. by an oblateness of a planet which is the gravitation centre. A local version of a simple generalization to Theorem 2 seems to allow the study of the neighbourhood of second species periodic solution in a restricted problem of three bodies.

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1. The Kepler Problem and Geodesic Flows

Consider the Hamiltonian

$$H_0(Y, X) = \frac{|Y|^2}{2} - \frac{\mu}{|X|}, \qquad X,\ Y \in \mathbb{R}^n$$

with a non-zero μ. When $\mu > 0$ this is a Hamiltonian of a mechanical problem with a single attracting centre, where gravitation changes according to the inverse square law; when $\mu < 0$ this is a Hamiltonian of a problem with a repelling centre, repulsion changing according to the same law.

The isoenergetic surface

$$H_0 = \frac{|Y|^2}{2} - \frac{\mu}{|X|} = h \tag{1}$$

is projected, in accordance with various values of μ and h, into the following regions of impulse space

$$\mathcal{M}^{+}_{-2h} = \{Y \in \mathbb{R}^n : |Y|^2 > 2h\} \quad \text{if} \quad \mu > 0$$
$$\mathcal{M}^{-}_{-2h} = \{Y \in \mathbb{R}^n : |Y|^2 < 2h\} \quad \text{if} \quad \mu < 0, h > 0$$
$$\mathcal{M}^{-}_{-2h} = \varnothing \qquad\qquad \text{if} \quad \mu < 0, h \le 0$$

Further we shall sometimes use notations $\varkappa = -2h$, $\sigma = \pm$ without making special reservations.

We turn the regions $\mathcal{M}^{\sigma}_{-2h}$ into configuration space, changing the roles of co-ordinates and impulses by means of the canonical transformation i

$$P = Y,$$
$$Q = -X. \tag{2}$$

(P is coordinate, Q is impulse).

On the isoenergetic surface $H_0 = h$ we make a common change of time scale $\mathrm{d}t' = (1/|Q|)\mathrm{d}t$ using Poincaré's trick. Consider the new Hamiltonian

$$F' = |Q|(H_0(P, Q) - h) = |Q|\left(\frac{|P|^2}{2} - \frac{\mu}{|Q|} - h\right) = |Q|\left(\frac{|P|^2 - 2h}{2}\right) - \mu.$$

Its peculiarity is that on the surface $H_0 = h$ in which we are interested, the corresponding vector field differs from the initial one by the multiplier $|Q|$. Rejecting the inessential constant in F', we obtain the Hamiltonian

$$F'' = F' + \mu = |Q|\left(\frac{|P|^2 - 2h}{2}\right),$$

called parametric [10], i.e. the first-power homogeneous with respect to impulses Q. The common procedure of transforming a parametric Hamiltonian into a 'convex' one with respect to impulses, to which Legendre transformation can be applied, consists of passing to

$$F''' = \tfrac{1}{2}(F'')^2$$

(see ibid.). The passage on an isoenergetic surface $F'' = \mu$ or, on a $H_0 = h$, which is the same, from the Hamiltonian vector field with a Hamiltonian F'' to the field with a Hamiltonian F''' is accompanied by multiplying the field $(\partial F''/\partial Q, -(\partial F''/\partial P))$ by μ, i.e. with a new change of time scale $\mathrm{d}\tau = (1/\mu)\,\mathrm{d}t'$. Applying Legendre transformation to the Hamiltonian

$$F''' = \tfrac{1}{2}|Q|^2\left(\frac{|P|^2 - 2h}{2}\right)^2$$

we obtain the Lagrangian

$$L = \tfrac{1}{2}|\Xi|^2\left(\frac{2}{|P|^2 - 2h}\right)^2.$$

Vectors Q and Ξ are connected by the relation

$$\Xi = \left(\frac{|P|^2 - 2h}{2}\right)^2 Q. \tag{3}$$

Let us introduce on $\mathscr{M}^{\sigma}_{-2h}$ metric $g_{ij} = [4\delta_{ij}/(|P|^2 - 2h)^2]$ which is a metric of constant curvature. Now we may rewrite the Lagrangian L in the form

$$L(P, \Xi) = \tfrac{1}{2}g(\Xi, \Xi).$$

The corresponding Lagrangian vector field is called a spray associated with the metric

g [6]. The flow on a tangent bundle determined by it is a geodesic flow. Legendre transformation transfers our isoenergetic surface $F''' = \frac{1}{2}\mu^2$ (in other words, $F'' = \mu$ or $H_0 = h$) onto the fibre bundle $S_{|\mu|}(\mathscr{M}^\sigma_{-2h})$ of vectors tangent to $\mathscr{M}^\sigma_{-2h}$, whose length in metric g is $|\mu|$.

The manifolds $\mathscr{M}^\sigma_\varkappa$ are geodesically incomplete: the geodesic can for finite time run to infinity. In particular, for this reason $\mathscr{M}^\sigma_\varkappa$, though a space of constant curvature, does not admit a transitive group of motions. In order to remove the singularity in the infinity, Levi-Civita (see, e.g. [9]) suggested subjecting the space of impulses to inversion and then completing it with zero.

Consider the manifolds

$$\hat{\mathscr{M}}^+_\varkappa = \left\{p \in \mathbb{R}^n : |p|^2 < \frac{1}{-\varkappa}\right\} \quad \text{if} \quad \varkappa < 0,$$

$$\hat{\mathscr{M}}^+_\varkappa = \mathbb{R}^n \quad \text{if} \quad \varkappa \geq 0,$$

$$\hat{\mathscr{M}}^-_\varkappa = \left\{p \in \mathbb{R}^n : |p|^2 > \frac{1}{-\varkappa}\right\} \quad \text{if} \quad \varkappa < 0$$

provided with metrics $\hat{g}_{ij} = [4\delta_{ij}/(1 + \varkappa|p|^2)^2]$ (when $\varkappa < 0$ this metric is called Poincaré's).

PROPOSITION. *Mapping of the inversion*

$$I^\sigma_\varkappa : \mathscr{M}^\sigma_\varkappa \backslash \{0\} \to \hat{\mathscr{M}}^\sigma_\varkappa \backslash \{0\}, \qquad P \mapsto p = \frac{P}{|P|^2}$$

realizes isometry between its source (with the metric g) and target (with the metric $\hat{g}$).

Proof. Let $\Xi \in T_P \mathscr{M}^\sigma_\varkappa$. Using an obvious formula for the differential of the inversion

$$(I^\sigma_\varkappa)_* : \Xi \mapsto \frac{\Xi}{|P|^2} - 2\frac{P}{|P|^4}\langle P, \Xi \rangle \tag{4}$$

one can easily see that the proposition is true:

$$\hat{g}_{I(P)}(I_*\Xi, I_*\Xi) = \hat{g}_p(I_*\Xi, I_*\Xi) = \frac{4}{(1 + \varkappa|p|^2)^2}\langle I_*\Xi, I_*\Xi \rangle =$$

$$= \frac{4}{\left(1 + \varkappa\dfrac{1}{|P|^2}\right)^2} \cdot \frac{|\Xi|^2}{|P|^4} = \frac{4|\Xi|^2}{(|P|^2 + \varkappa)^2} = g_P(\Xi, \Xi).$$

Consider a commutative diagram

$$\begin{array}{ccc} T^*(\mathscr{M}^\sigma_\varkappa \backslash \{0\}) & \xrightarrow{g^\flat} & T(\mathscr{M}^\sigma_\varkappa \backslash \{0\}) \\ \downarrow (I^*)^{-1} & & \downarrow I_* \\ T^*(\hat{\mathscr{M}}^\sigma_\varkappa \backslash \{0\}) & \xrightarrow[g^\flat]{} & T(\hat{\mathscr{M}}^\sigma_\varkappa \backslash \{0\}). \end{array} \tag{5}$$

The Legendre transformation mentioned above is actually a mapping of identification of $T^*\mathscr{M}_\varkappa^\sigma$ with $T\mathscr{M}_\varkappa^\sigma$, determined by the metric g, ($g^\flat$ in the diagram). This and the isometricity of I implies that the composition $I_*\circ g^\flat$ transfers the phase flow of the Hamiltonian F''' on the surface $F''' = \frac{1}{2}\mu^2$ into the geodesic flow acting on $S_{|\mu|}(\hat{\mathscr{M}}_\varkappa^\sigma)$. In Section 2 we shall need to pass from the Hamiltonian vector fields on $T^*\mathscr{M}_\varkappa^+$ to the vector fields on $T(\hat{\mathscr{M}}_\varkappa^+\backslash\{0\})$. To do this, we shall use the second branch of the diagram exploiting the canonicity of $(I^*)^{-1}$. Formulas for I^* and $\hat{g}^\flat$ that would be necessary in Section 2 are almost self-evident:

$$I^*: (p,q) \mapsto (P, Q) = \left(\frac{p}{|p|^2}, |p|^2 q - 2\langle p,q\rangle p\right); \tag{6}$$

$$\hat{g}^\flat: (p,q) \mapsto \left(p, \frac{(1+\varkappa|p|^2)^2}{4} q\right). \tag{7}$$

(Mapping $i^{-1}\circ I^*$ together with the change of time scale $(1/|X|)\mathrm{d}t$ is called Levi-Civita regularization [9].)

Note that, having applied an inversion, we have regularized, i.e. completed, only manifolds $\mathscr{M}_\varkappa^+$ with $\varkappa < 0$ (this will be exploited in Section 2) and simultaneously introduced new singularities into the rest. For this reason we consider another model of constant curvature spaces*; namely, consider in $\mathbb{R}^{n+1} = \{\zeta = (\zeta_1, \zeta_0): \zeta_1 \in \mathbb{R}^n, \zeta_0 \in \mathbb{R}\}$ the family of second-order surfaces

$$\langle R_\varkappa^{\mathrm{I}}\zeta, \zeta\rangle = \varkappa\zeta_1^2 + \zeta_0^2 = 1.$$

When $\varkappa > 0$, this is an ellipsoid; we shall denote it by $\mathscr{N}_\varkappa^+$. When $\varkappa < 0$, this is a hyperboloid of two sheets; the upper sheet will be denoted by $\mathscr{N}_\varkappa^-$, the lower one – by $\mathscr{N}_\varkappa^+$. When $\varkappa = 0$, this is a pair of parallel hyperplanes; the upper one will be excluded from considerations while the lower one will be denoted by $\mathscr{N}_0^+$. Figure 1 gives you an idea of the way these surfaces are situated when $n = 1$.

Consider two metrics on each of the surfaces $\mathscr{N}_\varkappa^\sigma$. The first one

$$g_{\mathrm{I}} = |\varkappa(\mathrm{d}\zeta_1)^2 + (\mathrm{d}\zeta_0)^2|$$

is, as a matter of fact, induced by the same quadratic form which determines the surface itself. (When $\varkappa = 0$, degeneration takes place: g_{I} identically equals zero.) The second metric which, when $\varkappa \neq 0$, is defined as

$$g_{\mathrm{II}} = (\mathrm{d}\zeta_1)^2 + \frac{1}{\varkappa}(\mathrm{d}\zeta_0)^2$$

and when $\varkappa = 0$ as

$$g_{\mathrm{II}} = (\mathrm{d}\zeta_1)^2$$

* Added in proof: Recently this model in the same context was used also by Belbruno [12].

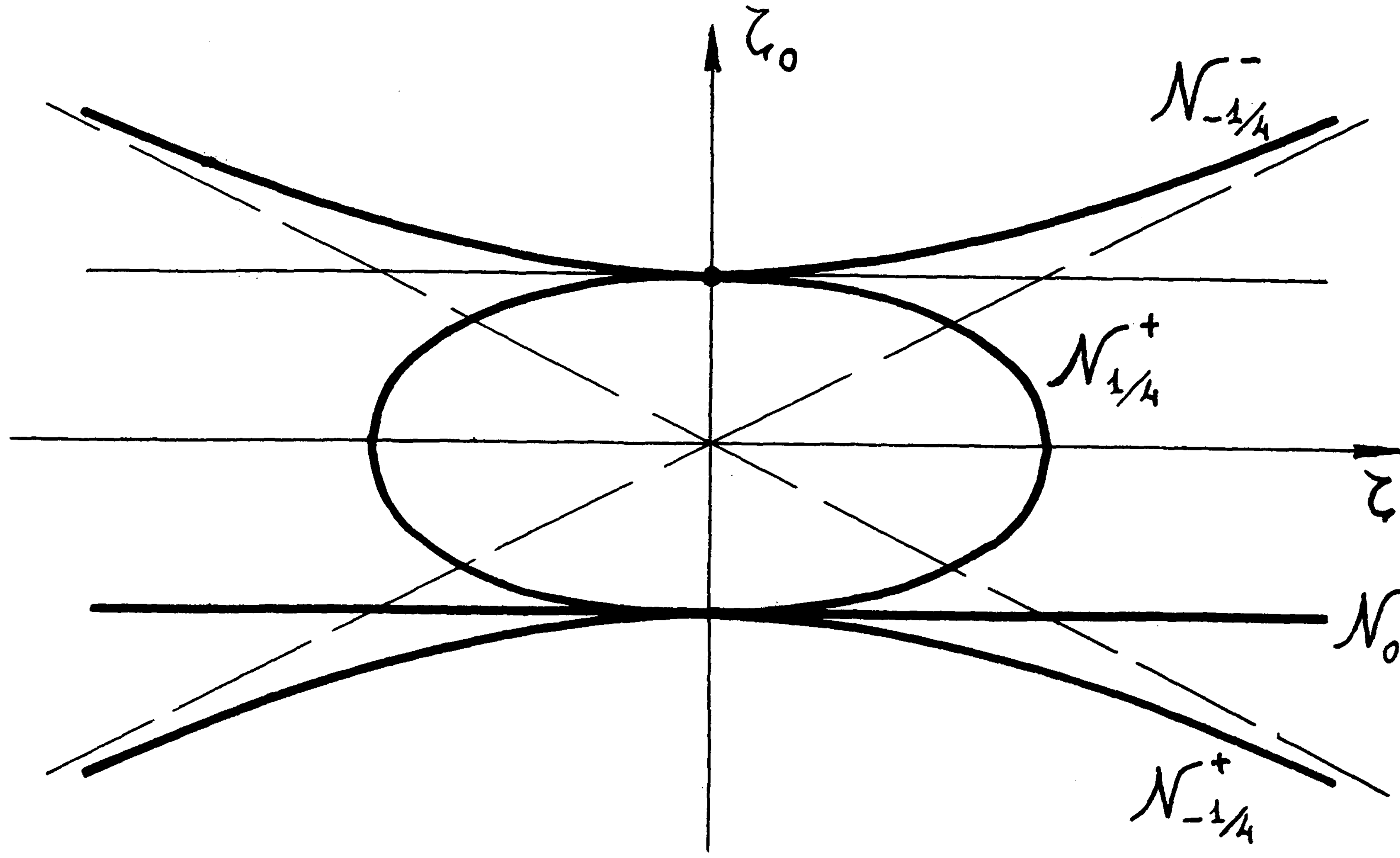

Fig. 1.

is positively defined on each surface. Evidently, the following relation takes place on the surface $\mathcal{N}^\sigma_\varkappa$:

$$g_{\rm I} = |\varkappa| g_{\rm II}.$$

The metric $g_{\rm II}$ is, of course, also induced by a quadratic form $R^{\rm II}_\varkappa$ in $\mathbb{R}^{n+1}$. Consider the mapping

$$\textstyle\sum^\sigma_\varkappa: \mathbb{R}^n \supset \hat{\mathcal{M}}^\sigma_\varkappa \to \mathcal{N}^\sigma_\varkappa \setminus \{(0, 1)\} \subset \mathbb{R}^{n+1}$$

inverse to the stereographic projection from the north pole (0, 1) to the space $\zeta_0 = 0$. Mappings $\sum^{-1}, \sum$ are described with the formulas

$$(\textstyle\sum^\sigma_\varkappa)^{-1}\begin{pmatrix}\zeta_1\\ \zeta_0\end{pmatrix} = p, \qquad \textstyle\sum^\sigma_\varkappa(p) = \begin{pmatrix}\zeta_1\\ \zeta_0\end{pmatrix},$$

$$p = \frac{\zeta_1}{1-\zeta_0}, \qquad \begin{pmatrix}\zeta_1\\ \zeta_0\end{pmatrix} = \frac{2}{1+\varkappa|p|^2}\begin{pmatrix} p \\ \dfrac{\varkappa|p|^2-1}{2}\end{pmatrix}, \tag{8}$$

PROPOSITION. *Mapping $\sum^\sigma_\varkappa$ realizes isometry between $\hat{\mathcal{M}}^\sigma_\varkappa$ and $\mathcal{N}^\sigma_\varkappa \setminus \{(0, 1)\}$ considered respectively with metrices $\hat{g}$ and $g_{\rm II}$.*

Proof. Differentiating (8), we obtain for $\xi \in T_p\hat{\mathcal{M}}^\sigma_\varkappa$

$$(\Sigma_x^\sigma)_*\,\xi = \frac{4}{(1+\varkappa|p|^2)^2}\begin{pmatrix}\left(\dfrac{1+\varkappa|p|^2}{2}\right)\xi - \varkappa p\langle p,\xi\rangle \\ \\ \varkappa\langle p,\xi\rangle\end{pmatrix} \tag{9}$$

and, consequently, for $\varkappa \neq 0$

$$(g_{\mathrm{II}})_{\varepsilon(p)}(\Sigma_*\,\xi, \Sigma_*\xi) = [(\Sigma_*\xi)_1]^2 + \frac{1}{\varkappa}[(\Sigma_*\xi)_0]^2 =$$

$$= \left[\frac{4}{(1+\varkappa|p|^2)^2}\right]^2\left(\frac{(1+\varkappa|p|^2)^2}{4}|\xi|^2 + \varkappa^2|p|^2\langle p,\xi\rangle^2 -\right.$$

$$\left. - \varkappa\langle p,\xi\rangle^2(1+\varkappa|p|^2) + \frac{1}{\varkappa}\varkappa^2\langle p,\xi\rangle^2\right) = \frac{4|\xi|^2}{(1+\varkappa|p|^2)^2} = \hat{g}_p(\xi,\xi).$$

For $\varkappa = 0$ the proposition is obvious.

Remark. With respect to the metric g_{I} on $\mathcal{N}_\varkappa^\sigma$ diffeomorphism Σ is, in terms of [5], homothety with coefficient $\sqrt{|\varkappa|} = \sqrt{|2h|}$.

The manifolds $\mathcal{N}_\varkappa^\sigma$ are complete. The geodesic flows on the manifolds $S_{|\mu|}(\mathcal{N}_\varkappa^\sigma)$ may thus be regarded as regularized phase flows on the isoenergetic surfaces $H_0(Y, X) = h$.

If there exists a Lie group of symmetries (which, for the geodesic flow on $\mathcal{N}_\varkappa^\sigma$, is a transitive group of motions of $\mathcal{N}_\varkappa^\sigma$), the Noether theorem allows us to construct the first integrals. We shall use this theorem in Smale's interpretation [8], for which we shall need to calculate the Lie algebras of groups of motions and their action in $\mathcal{N}_\varkappa^\sigma$. The group of motions of the manifold $\mathcal{N}_\varkappa^\sigma$ with respect to the metric g_{II} may be described quite easily – it may be presented as a subgroup of $GL(n+1, \mathbb{R})$. The transformation $B \in GL(n+1, \mathbb{R})$ determines the motion of the manifold $\mathcal{N}_\varkappa^\sigma$, if, for one thing, it transfers $\mathcal{N}_\varkappa^\sigma$ into itself, i.e. preserves the form $R_\varkappa^{\mathrm{I}}$

$$B^* R_\varkappa^{\mathrm{I}} B = R_\varkappa^{\mathrm{I}} \tag{10}$$

and, when $\varkappa \leq 0$, does not exchange $\mathcal{N}_\varkappa^+$ and $\mathcal{N}_\varkappa^-$, and if, for another thing, it preserves the form $R_\varkappa^{\mathrm{II}}$

$$B^* R_\varkappa^{\mathrm{II}} B = R_\varkappa^{\mathrm{II}} \tag{11}$$

and in this way the metric g_{II}. The group described exhausts the whole group of motions of $\mathcal{N}_\varkappa^\sigma$, which is easily seen, e.g., from Theorem 6.7.9 of [5]. Requirements (10) and (11) are evident to be equivalent if $\varkappa \neq 0$. For $\varkappa = 0$ we may, in turn, obtain from (10) and (11) a more explicit description of the group of motions. Transformation B should merely be of the form

$$B = \begin{pmatrix} B' & b \\ 0 & 1 \end{pmatrix}, \tag{12}$$

where B' is an orthogonal matrix $n \times n$.

Now it is easy to find the Lie algebra of the group of motions. Let B_t be a one-parameter subgroup. Differentiating the relation $B_t^* R_\varkappa^{\mathrm{I}} B_t = R_\varkappa^{\mathrm{I}}$ (see (10)) we obtain, if $t = 0$, $A^* R_\varkappa^{\mathrm{I}} + R_\varkappa^{\mathrm{I}} A = 0$, where $A = (\mathrm{d}B_t/\mathrm{d}t)_{t=0}$. If A is presented as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$R_\varkappa^{\mathrm{I}} = \begin{pmatrix} \varkappa \,\mathrm{id} & 0 \\ 0 & 1 \end{pmatrix},$$

then from the relation obtained it follows that

$$\begin{pmatrix} \varkappa a^* & c^* \\ \varkappa b^* & d \end{pmatrix} + \begin{pmatrix} \varkappa a & \varkappa b \\ c & d \end{pmatrix} = 0,$$

which is equivalent to $\varkappa a^* = -\varkappa a$, $c = -\varkappa b^*$, $d = 0$.

In this fashion, the Lie algebra $\mathscr{I}_\varkappa$ of the group of motions, when $\varkappa \neq 0$, consists of the matrices

$$\begin{pmatrix} a & b \\ -\varkappa b^* & 0 \end{pmatrix},$$

where a is skew-symmetric. From representation (12) for the group of motions of $\mathscr{N}_0^+$ it is readily seen that such a representation for Lie algebra is also true if $\varkappa = 0$.

An action of the Lie algebra $\mathscr{I}_\varkappa$ on $\mathscr{N}_\varkappa^\sigma$ is a mapping

$$\alpha : \mathscr{I}_\varkappa \to \Gamma(T\mathscr{N}_\varkappa^\sigma)$$

which puts into correspondence to $A \in \mathscr{I}_\varkappa$ the vector field on $\mathscr{N}_\varkappa^\sigma$ generated by a one-parameter group of transformations corresponding to the element A of the Lie algebra. For any point $\zeta \in \mathscr{N}_\varkappa^\sigma$

$$\alpha_\zeta : \mathscr{I}_\varkappa \to T_\zeta \mathscr{N}_\varkappa^\sigma$$

means the mapping which puts into correspondence to the element A the value of the vector field $\alpha(A)$ at the point ζ. An explicit expression for α_ζ is

$$\alpha_\zeta(A) = \begin{pmatrix} a & b \\ -\varkappa b^* & 0 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_0 \end{pmatrix}.$$

According to [8], composition J of the mappings $g_{\mathrm{II}}^{\#} : T\mathscr{N}_\varkappa^\sigma \to T^*\mathscr{N}_\varkappa^\sigma$ and

$$J_1 : T^*\mathscr{N}_\varkappa^\sigma \to \mathscr{I}_\varkappa^*$$

the latter being dual to α_ζ on every fibre of $T_\zeta^*\mathcal{N}$, is the first integral of the geodesic flow. Mapping J may be set by giving the value $J\eta$ for each $\eta \in T_\zeta\mathcal{N}$ on the elements $A \in \mathcal{J}_\varkappa$. In our case

$$\langle J\eta, A\rangle = g_{\text{II}}(\eta, \alpha_\zeta(A)) = \langle \eta_1, a\zeta_1 + \zeta_0 b\rangle - \eta_0\langle \zeta_1, b\rangle = \\ = \langle \eta_1, a\zeta_1\rangle + \langle \zeta_0\eta_1 - \eta_0\zeta_1, b\rangle. \tag{13}$$

It is clear that the composition of the through mapping

$$U\colon (Y, X) \overset{i}{\rightarrow} (P, Q) \overset{g^b}{\rightarrow} (P, \Xi) \overset{I*}{\rightarrow} (p, \xi) \overset{\Sigma*}{\rightarrow} (\zeta, \eta) \tag{14}$$

and the first integral J of the geodesic flow on $\mathcal{N}_\varkappa^\sigma$ is the first integral of the Kepler problem. Formulas setting the mapping U may be obtained from (14) by means of fairly simple calculations, using (2), (3), (4) and (9)

$$\zeta_1 = \frac{2}{|Y|^2 - 2h} Y, \qquad \zeta_0 = -\frac{2}{|Y|^2 - 2h}\frac{|Y|^2 + 2h}{2},$$

$$\eta_1 = \langle Y, X\rangle - \frac{|Y|^2 - 2h}{2} X, \qquad \eta_0 = -2h\langle Y, X\rangle.$$

Substituting $U(Y, X)$ for η in (13) we get, after some transformations,

$$\langle \eta_1, a\zeta_1\rangle + \langle \zeta_0\eta_1 - \eta_0\zeta_1, b\rangle = \frac{2}{|Y|^2 - 2h}\langle Y, X\rangle\langle Y, a(Y)\rangle - \langle X, a(Y)\rangle + \\ + \langle |Y|^2 X - \langle Y, X\rangle Y - \frac{|Y|^2 - 2h}{2} X, b\rangle.$$

The fact that a is skew-symmetric implies $\langle Y, a(Y)\rangle = 0$; from the expression of energy integral (1) follows

$$\frac{|Y|^2 - 2h}{2} = \frac{\mu}{|X|}.$$

Therefore

$$\langle J \circ U(Y, X), A\rangle = -\langle X, a(Y)\rangle + \langle |Y|^2 X - \langle Y, X\rangle Y - \mu\frac{X}{|X|}, b\rangle.$$

This expression has an especially simple form in a three-dimensional case

$$-\langle X, a(Y)\rangle + \langle [Y, [X, Y]] - \mu X_0, b\rangle.$$

It is well-known (see, e.g. [2]) that for a three-dimensional skew-symmetric matrix a there exists a vector ω such that for all Y the equality $a(Y) = [\omega, Y]$ takes place. In accordance with this $-\langle X, a(Y)\rangle = \langle \omega, [X, Y]\rangle$.

The pair of vectors $c, f \in \mathbb{R}^3$ may be identified with an element of the Lie co-algebra $\mathscr{I}^*_\varkappa$ defining the action (c,f) on 4×4-matrices

$$A = \begin{pmatrix} a & b \\ -\varkappa b^* & 0 \end{pmatrix} \in \mathscr{I}_\varkappa$$

(a is skew-symmetric) in the following manner

$$\langle (c,f), A \rangle = \langle c, \omega \rangle + \langle f, b \rangle,$$

where vector ω is constructed in a standard way proceeding from matrix a.

As we have just seen above, Smale's interpretation of the Noether theorem implies that the pair consisting of the angular momentum vector $[X, Y]$ and the Laplace vector $[Y, [X, Y]] - \mu X_0$ and identified with an element of $\mathscr{I}^*_\varkappa$ is the first integral of the Kepler problem.

It should be noted that the pair of vectors $([X, Y], [Y, [X, Y]] - \mu X_0)$ with different (Y, X) belongs, in general, to different Lie co-algebras $\mathscr{I}^*_\varkappa$. Belonging to co-algebra $\mathscr{I}^*_{-2h}$ is determined, of course, by the value of the energy constant

$$h = \frac{|Y|^2}{2} - \frac{\mu}{|X|}.$$

Using manifolds $\mathscr{N}^\sigma_\varkappa$ with the metric g_{I} allows us to give a simple geometric interpretation to the eccentric anomalies E and H. Namely, the following proposition is true: PROPOSITION. *Geodesic parameter l on the manifold $\mathscr{N}^{+}_{-2h}$ with the metric g_{I} in case of $h < 0$ coincides with the eccentric anomaly E, and in case $h > 0$ with its analogue H.*

Proof. On the surface $S_{|\mu|}$ $(\mathscr{M}^\sigma_\varkappa)$ let us choose an arbitrary point s moved by the geodesic flow φ^τ. The speed magnitude of motion of its projection $\pi(s)$ in $\mathscr{M}^\sigma_\varkappa$ is

$$\left| \frac{\mathrm{d}\pi(\varphi^\tau(s))}{\mathrm{d}\tau} \right|_g = |\mu| .$$

The speed magnitude of motion of the image of $\pi(s)$ under isometry I and homothety $\sum$ is

$$\left| \frac{\mathrm{d}[(\sum \circ I \circ \pi)(\varphi^\tau(s))]}{\mathrm{d}\tau} \right|_{g_I} = \sqrt{|2h|}\, |\mu| .$$

Hence, keeping in mind that the time τ is related to physical time t as

$$\mathrm{d}\tau = \frac{1}{\mu |Q|} \mathrm{d}t = \frac{1}{\mu r} \mathrm{d}t$$

we obtain the relation between the time t and geodesic parameter l on $\mathscr{N}^\sigma_\varkappa$ in the metric g_{I}

$$\mathrm{d}l = \operatorname{sign} \mu \frac{\sqrt{|2h|}}{r} \mathrm{d}t . \tag{15}$$

From the Kepler equations relating time with eccentric anomaly E in the elliptic case and its analogue H in the hyperbolic one

$$E - e\sin E = \frac{\sqrt{|\mu|}}{|a|^{3/2}}(t-T), \qquad e\,\mathrm{sh}\,H - H = \frac{\sqrt{|\mu|}}{|a|^{3/2}}(t-T),$$

and also

$$\frac{r}{|a|} = 1 - \cos E, \qquad \frac{r}{|a|} = e\,\mathrm{ch}\,H - 1, \qquad a = -\frac{\mu}{2h}$$

it follows that

$$(1 - e\cos E)\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\sqrt{|\mu|}}{|a|^{3/2}}, \qquad (e\,\mathrm{ch}\,H - 1)\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\sqrt{|\mu|}}{|a|^{3/2}}$$

and, therefore,

$$\frac{r}{|a|}\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\sqrt{|\mu|}}{|a|^{3/2}}, \qquad \frac{r}{|a|}\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\sqrt{|\mu|}}{|a|^{3/2}}.$$

Thus

$$\mathrm{d}E = \frac{\sqrt{|2h|}}{r}\,\mathrm{d}t, \qquad \mathrm{d}H = \frac{\sqrt{|2h|}}{r}\,\mathrm{d}t. \tag{16}$$

Comparing (15) and (16) we see that the proposition is true.

2. Structural Stability of Hyperbolic Keplerian Motions

In Section 1 we have proved that the phase flow of the Kepler problem on an isoenergetic surface with $h > 0$ is trajectorily equivalent to the geodesic flow in Lobachevskian space. The same turns out to be true in case of a perturbed Kepler problem with a Hamiltonian

$$H(Y, X) = \frac{|Y|^2}{2} - \frac{1}{|X|} + K(Y, X),$$

where perturbation K decreases at infinity rapidly enough. The crucial fact here is that the geodesic flow in Lobachevskian space is an Anosov flow and, therefore, it is structurally stable. A suitable version of the theorem on structural stability is proved in [4]. Before formulating it, we shall provide the phase space $S = S_1\ (\hat{\mathscr{M}}_\varkappa^+)$ of the geodesic flow with Riemannian structure.

Let us introduce the metric G on $T\hat{\mathscr{M}}_\varkappa^+$, assuming vertical and horizontal subspaces [5] in $TT\hat{\mathscr{M}}_\varkappa^+$ orthogonal and inducing into them the metric from metric $\hat{g}$, respectively, by inclusion and projection. The metric on S – a natural metric of the submanifold of the $T\hat{\mathscr{M}}_\varkappa^+$ will be denoted by G also.

By V we denote restriction to S of the spray associated with metric $\hat{g}$, i.e. the vector field which determines the geodesic flow.

THEOREM 1. *Let $\mathscr{X}^{1(2)}(S)$ be the space of C^2-smooth vector fields W on S, bounded in the metric G and also having bounded covariant differentials ∇W and $\nabla^2 W$. Let*

$$\|W\|_1 = \sup_{s\in S} \{|W(s)|_G, |(\nabla W)(s)|_G\}$$

be the norm in $\mathscr{X}^{1(2)}$. Then the vector field V is structurally stable in the class $\mathscr{X}^{1(2)}$, i.e. for any $W \in \mathscr{X}^{1(2)}$, C^1-close to V, there exists a homeomorphism $u: S \to S$ C^0-close to identity and transferring the trajectories of vector field V into the trajectories of W.

In Section 1 we have seen that in order to regularize a phase flow of the Hamiltonian H_0 on an isoenergetic surface $H_0 = h > 0$ it suffices to pass to the coordinates p, q and to the Hamiltonian

$$F_0: (p,q) \overset{I^*}{\longmapsto} (P,Q) \overset{i^{-1}}{\longmapsto} (Y,X) \overset{\tilde{H}_0}{\dashrightarrow} \mathbb{R},$$

where $\tilde{H}_0 = \frac{1}{2}(|X|(H_0 - h) + 1)^2$. From (6) it follows that $|X|^2 = |p|^4|q|^2$ and, therefore,

$$F_0(p,q) = \frac{1}{2}|q|^2\left(\frac{1-2h|p|^2}{2}\right)^2 \tag{17}$$

To regularize a perturbed phase flow the same procedure should be undertaken, substituting H for H_0, which will result in

$$F(p,q) = \frac{1}{2}|q|^2\left(\frac{1-2h|p|^2}{2}\right)^2 \times \\ \times\left(1 + \frac{2}{1-2h|p|^2}|p|^2 K\left(\frac{p}{|p|^2}, 2\langle p,q\rangle p - |p|^2 q\right)\right)^2. \tag{18}$$

Hence it is clear that a perturbed phase flow can be regularized if the function

$$k(p,q) = |p|^2 K\left(\frac{p}{|p|^2}, 2\langle p,q\rangle p - |p|^2 q\right)$$

extends up to a smooth one onto a neighbourhood of a completed hypersurface $F(p,q) = \frac{1}{2}$. If, say, K has a form

$$K(Y,X) = \frac{\Phi(X)}{|X|}, \tag{19}$$

where Φ is a function smooth in the whole $\mathbb{R}^n$, then $k(p,q)$, being equal to

$$k(p,q) = |p|^2\frac{\Phi(2\langle p,q\rangle - |p|^2 q)}{|p|^2|q|} = \frac{\Phi(2\langle p,q\rangle p - |p|^2 q)}{|q|}$$

can be extended up to the smooth one, if Φ is sufficiently small, since, as can be easily demonstrated, the value $|q|$ is, in this case, separated from zero on the surface $F = \frac{1}{2}$.

THEOREM 2. *Let $h > 0$ and let K have the form* (19). *If the function $\Phi \in C^3(\mathbb{R}^n)$ for a certain ε which, in general, depends on h satisfies the inequalities*

$$|\Phi| < \varepsilon, \left|\frac{\partial \Phi}{\partial X}\right| < \varepsilon \min\left\{\frac{1}{|X|}, 1\right\}, \left|\frac{\partial^2 \Phi}{\partial X^2}\right| < \varepsilon \min\left\{\frac{1}{|X|^2}, 1\right\}, \tag{20}$$

$$\left|\frac{\partial^3 \Phi}{\partial X^3}\right| < \text{const} \min\left\{\frac{1}{|X|^3}, 1\right\}$$

then phase flows on regularized isoenergetic surfaces which correspond to $H_0 = h$ and $H = h$ are trajectorily equivalent.

The trajectories connected by conjugative homeomorphism have the same impulses Y at $\pm\infty$.

Proof. In $T^*\hat{\mathscr{M}}^+_{-2h}\backslash\{\text{zero section}\}$ consider two Hamiltonian vector fields: an unperturbed one, with a Hamiltonian F_0, and a perturbed one, with a Hamiltonian F. Each of them interests us for the most part on the level surface of its Hamiltonian with the value $\frac{1}{2}$, i.e. respectively on $F_0 = \frac{1}{2}$. Our aim is to find the mapping of the unperturbed surface onto the perturbed one, which transfers trajectories of one Hamiltonian field into those of the other one. Here it seems pertinent to make use of the theorem on structural stability of a geodesic flow (Theorem 1), and to this effect we map onto S both surfaces with vector fields defined on them. First let us map $\{F = \frac{1}{2}\}$ onto $\{F_0 = \frac{1}{2}\}$ by means of

$$\Pi : (p, q) \mapsto \left(p, \left(1 + \frac{2}{1 - 2h|p|^2} k(p, q)\right) q\right).$$

(The form of this mapping suggests itself when comparing (17) and (18).) Further let us map onto S by means of (7) the surface $F_0 = \frac{1}{2}$ with two vector fields already defined on it. Then, because of diagram (5) being commutative, from the unperturbed Hamiltonian vector field we obtain on S the field V determining the geodesic flow. Let us denote by W the field on S obtained from the perturbed Hamiltonian field. The difference $W - V$ will be denoted by Δ. In this way,

$$W_{(p,\xi)} = \hat{g}^\flat_* \Pi_* \begin{pmatrix} \dfrac{\partial F}{\partial q} \\ -\dfrac{\partial F}{\partial p} \end{pmatrix}_{(p,q)}$$

$$\text{and} \quad \Delta_{(p,\xi)} = \hat{g}^\flat_* \left(\Pi_* \begin{pmatrix} \dfrac{\partial F}{\partial q} \\ -\dfrac{\partial F}{\partial p} \end{pmatrix}_{(p,q)} - \begin{pmatrix} \dfrac{\partial F_0}{\partial q} \\ -\dfrac{\partial F_0}{\partial p} \end{pmatrix}_{\Pi(p,q)} \right) \tag{21}$$

(p is a coordinate and q is an impulse). The fact that V and W are actually defined not only on the surface S but also in its neighbourhood can significantly simplify verification of Theorem 1 conditions if the following lemma is used.

LEMMA 1. *Let ∇ be a covariant differential in the manifold $T\hat{\mathscr{M}}_{\varkappa}^{+}$ defined by the metric G, ∇' – a covariant differential in S defined by the metric $G|_S$; then for any smooth vector field W defined in the neighbourhood of S and tangential to S*

$$\|\nabla' W\|_G \le \|\nabla W\|_G, \; \|\nabla'^2 W\|_G \le \|\nabla^2 W\|_G + \|\nabla W\|_G + \|W\|_G.$$

The proof is sketched in [4].

Estimates, entering the conditions of Theorem 1, for covariant differentials of W and Δ, internal with respect to S, due to lemma, may be replaced by corresponding estimates for covariant differentials, external with respect to S. These latter, together with the estimate of C^0-norm of the field Δ, are

$$\sup\{|\Delta_{(p,\xi)}|_G: (p,\xi)\in S\} < C_1\varepsilon,$$

$$\sup\{|(\nabla\Delta)_{(p,\xi)}(Z)|_G: (p,\xi)\in S; Z\in T_{(p,\xi)}T\hat{\mathscr{M}}_{\varkappa}^{+}; |Z|_G = 1\} < C_1\varepsilon,$$

$$\sup\{|(\nabla^2 W)_{(p,\xi)}(Z,U)|_G: (p,\xi)\in S; Z, U\in T_{(p,\xi)}T\hat{\mathscr{M}}_{\varkappa}^{+};\; |Z|_G = |U|_G = 1\} < C_2$$

(estimate of ∇W follows from estimate of $\nabla\Delta$ and from $V\in\mathscr{X}^{1(2)}$). Here

$$((\nabla\Delta)(Z))^m = \sum_j \frac{\partial\Delta^m}{\partial x^j} Z^j + \sum_{i,j} \Gamma_{ij}^m \Delta^i Z^j, \tag{22}$$

$$\begin{aligned}((\nabla^2 W)(Z,U))^m = {} & \sum_{i,j} \frac{\partial^2 W^m}{\partial x^i \partial x^j} Z^j U^i + \sum_{i,j,l} \frac{\partial \Gamma_{il}^m}{\partial x^j} W^l Z^j U^i + \\ & + \sum_{i,j,l} \Gamma_{il}^m \frac{\partial W^l}{\partial x^j} Z^j U^i + \sum_{i,j,s} \Gamma_{js}^m \frac{\partial W^s}{\partial x^i} Z^j U^i + \\ & + \sum_{i,j,s,l} \Gamma_{js}^m \Gamma_{il}^s W^l Z^j U^i - \sum_{i,j,s} \Gamma_{ji}^s \frac{\partial W^m}{\partial x^s} Z^j U^i - \\ & - \sum_{i,j,s,l} \Gamma_{ji}^s \Gamma_{sl}^m W^l Z^j U^i\end{aligned} \tag{23}$$

(by x^j we mean p_j when $j \le n$ and ξ_{j-n} when $j > n$).

Estimates relating to the metric tensor G and coefficients Γ_{ij}^m are given by a lemma whose proof is omitted.

Formulating the lemma and further on we use the notation $\varphi = \varphi(p) = (1 + \varkappa|p|^2)/2$.

LEMMA 2. *The metric tensor G of the manifold $T\hat{\mathscr{M}}_{\varkappa}^{+}$ at the points of S obeys the estimate*

$$\frac{C^-}{\varphi^2}|Z|^2 < G(Z,Z) < \frac{C^+}{\varphi^2}|Z|^2. \tag{24}$$

On S Christoffel's symbols of the Riemannian connection ∇ of the manifold $T\hat{\mathscr{M}}_{\varkappa}^{+}$ and their derivatives have asymptotics

$$\Gamma_{ij}^{m} = O\left(\frac{1}{\varphi}\right), \qquad \frac{\partial \Gamma_{ij}^{m}}{\partial x^{s}} = O\left(\frac{1}{\varphi^{2}}\right) \tag{25}$$

when $|p|$ *tends to* $1/\sqrt{-\varkappa}$.

In this fashion, it only remains to estimate the magnitude of coordinates of W and Δ and their derivatives with respect to coordinates p, ξ of the manifold $T\hat{\mathscr{M}}_{\varkappa}^{+}$ at the points of S. With this end in view, we estimate derivatives with respect to coordinates p, q and use the formulas for the change of coordinates for derivatives to estimate (p, ξ)-derivatives. Let us write out explicit formulas for (p, ξ)-coordinates of the fields W and Δ as functions of coordinates p, q.

For brevity we also denote

$$\alpha = \alpha(p,q) = 1 + \frac{2}{1+\varkappa|p|^{2}}\, k(p,q) = 1 + \frac{k}{\varphi}.$$

In these notations

$$F_0 = \tfrac{1}{2}|q|^2\varphi^2, \qquad F = \tfrac{1}{2}|q|^2\varphi^2\alpha^2,$$
$$\Pi:(p,q)\mapsto(p,\alpha q), \qquad \hat{g}^{\flat}:(p,q)\mapsto(p,\varphi^2 q).$$

And, as one can easily see,

$$\begin{pmatrix} \dfrac{\partial F_0}{\partial q} \\[2ex] -\dfrac{\partial F_0}{\partial p} \end{pmatrix}_{\Pi(p,q)} = \begin{pmatrix} \varphi^2\alpha q \\[2ex] -\varkappa|q|^2\alpha^2\varphi p \end{pmatrix},$$

$$\begin{pmatrix} \dfrac{\partial F}{\partial q} \\[2ex] -\dfrac{\partial F}{\partial p} \end{pmatrix}_{(p,q)} = \begin{pmatrix} \varphi^2\alpha^2 q + |q|^2\varphi^2\alpha\dfrac{\partial \alpha}{\partial q} \\[2ex] -\varkappa|q|^2\alpha^2\varphi p - |q|^2\varphi^2\alpha\dfrac{\partial\alpha}{\partial p} \end{pmatrix}, \tag{26}$$

$$\Pi_* = \begin{pmatrix} \mathrm{id} & 0 \\[2ex] q\otimes\dfrac{\partial\alpha}{\partial p} & \alpha\,\mathrm{id} + q\otimes\dfrac{\partial\alpha}{\partial q} \end{pmatrix}, \qquad \hat{g}^{\flat}_* = \begin{pmatrix} \mathrm{id} & 0 \\[2ex] 2\varkappa\varphi q\otimes p & \varphi^2\,\mathrm{id} \end{pmatrix}.$$

These formulas combined with (21) testify to the fact that (p, ξ)-coordinates of the vector fields W and Δ are polynomial (in a generalized sense) combinations of values

$$\varphi,\ \alpha,\ \frac{\partial\alpha}{\partial p},\ \frac{\partial\alpha}{\partial q},\ p,\ q. \tag{27}$$

The following lemma is useful for estimating them.

LEMMA 3. *For the function $k(p, q)$ and its derivatives on the surface $F = \frac{1}{2}$ the following asymptotics take place*

$$\frac{\partial^{i+j} k}{\partial p^i \partial q^j} = \varepsilon O(\varphi^{j+1}), \ (i + j \le 2); \quad \frac{\partial^{i+j} k}{\partial p^i \partial q^j} = O(\varphi^{j+1}), \quad (i + j = 3).$$

The proof of the lemma is based upon the estimates (20) and the inequality

$$\frac{1}{2} \frac{1}{\varphi} < |q| < 2 \frac{1}{\varphi}, \qquad (p, q) \in \{F = \tfrac{1}{2}\}$$

and is of no special interest.

The values (27) have the asymptotics

$$\varphi = O(\varphi), \ \alpha = O(1), \ \frac{\partial \alpha}{\partial p} = \varepsilon O\left(\frac{1}{\varphi}\right), \ \frac{\partial \alpha}{\partial q} = \varepsilon O(\varphi), \ p = O(1), \ q = O\left(\frac{1}{\varphi}\right). \quad (28)$$

It is not difficult to see (using, in particular, Lemma 3) that the following remark is true for the values (27).

Remark. Differentiating with respect to p decreases the order of the value with respect to φ by no more than 1, and differentiating with respect to q increases it by no less than 1.

This is equally true of (p, ξ)-coordinates of the fields W and Δ as polynomial combinations of the values (27).

From (26) and (28) it is easy to obtain

$$W = O(\varphi), \qquad \Delta = \varepsilon O(\varphi) \quad (29)$$

and, according to the remark

$$\frac{\partial \Delta}{\partial p} = \varepsilon O(1), \qquad \frac{\partial \Delta}{\partial q} = \varepsilon O(\varphi^2), \quad (30)$$

$$\frac{\partial W}{\partial p} = O(1), \qquad \frac{\partial W}{\partial p} = O(\varphi^2),$$

$$\frac{\partial^2 W}{\partial p^2} = O\left(\frac{1}{\varphi}\right), \qquad \frac{\partial^2 W}{\partial p \partial q} = O(\varphi), \qquad \frac{\partial^2 W}{\partial q^2} = O(\varphi^3). \quad (31)$$

Let us introduce the notations $f = \hat{g}^\flat \circ \Pi$, $e = f^{-1}$. The derivatives with respect to p, ξ in which we are interested are expressed by the formulas

$$\frac{\partial \Delta}{\partial (p, \xi)} = \frac{\partial \Delta}{\partial (p, q)} \cdot \frac{\partial e}{\partial (p, \xi)}, \ \frac{\partial W}{\partial (p, \xi)} = \frac{\partial W}{\partial (p, q)} \cdot \frac{\partial e}{\partial (p, \xi)}, \quad (32)$$

$$\frac{\partial^2 W}{\partial (p, \xi)^2} (\cdot, \cdot) = \left\{ \frac{\partial^2 W}{\partial (p, q)^2} \left(\frac{\partial e}{\partial (p, \xi)} \cdot, \frac{\partial e}{\partial (p, \xi)} \cdot \right) \right\} -$$
$$- \left[\frac{\partial W}{\partial (p, q)} \cdot \frac{\partial e}{\partial (p, \xi)} \right] \left\{ \frac{\partial^2 f}{\partial (p, q)^2} \left(\frac{\partial e}{\partial (p, \xi)} \cdot, \frac{\partial e}{\partial (p, \xi)} \cdot \right) \right\} \quad (33)$$

(the derivatives with respect to p, ξ are taken at the points of S).

It is not difficult to write out component-wise asymptotic for f_* and e_*:

$$f_* = \hat{g}^\flat_* \circ \Pi_* = \begin{pmatrix} \mathrm{id} & 0 \\ O(1) & O(\varphi^2) \end{pmatrix} \begin{pmatrix} \mathrm{id} & 0 \\ \varepsilon O\left(\frac{1}{\varphi^2}\right) & O(1) \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ O(1) & O(\varphi^2) \end{pmatrix} \tag{34}$$

$$e_* = \Pi_*^{-1} \circ \hat{g}^\sharp_* = \begin{pmatrix} \mathrm{id} & 0 \\ \varepsilon O\left(\frac{1}{\varphi^2}\right) & O(1) \end{pmatrix} \begin{pmatrix} \mathrm{id} & 0 \\ O\left(\frac{1}{\varphi^2}\right) & O\left(\frac{1}{\varphi^2}\right) \end{pmatrix} =$$

$$= \begin{pmatrix} \mathrm{id} & 0 \\ O\left(\frac{1}{\varphi^2}\right) & O\left(\frac{1}{\varphi^2}\right) \end{pmatrix}. \tag{35}$$

From the asymptotics (30) and (35) and the formula (32), asymptotics for $\partial \Delta / \partial(p, \xi)$ follow immediately:

$$\frac{\partial \Delta}{\partial(p, \xi)} = \frac{\partial \Delta}{\partial(p,q)} \cdot \frac{\partial e}{\partial(p, \xi)} = \begin{pmatrix} \varepsilon O(1) & \varepsilon O(\varphi^2) \\ \varepsilon O(1) & \varepsilon O(\varphi^2) \end{pmatrix} \begin{pmatrix} \mathrm{id} & 0 \\ O\left(\frac{1}{\varphi^2}\right) & O\left(\frac{1}{\varphi^2}\right) \end{pmatrix} = \varepsilon O(1) \tag{36}$$

and, in a similar manner,

$$\frac{\partial W}{\partial(p, \xi)} = \frac{\partial W}{\partial(p,q)} \cdot \frac{\partial e}{\partial(p, \xi)} \cdot = O(1) \tag{37}$$

From the component-wise asymptotics (34) and the remark we get estimates analogous to (31):

$$\frac{\partial^2 f}{\partial p^2} = O\left(\frac{1}{\varphi}\right), \frac{\partial^2 f}{\partial p \partial q} = O(\varphi), \frac{\partial^2 f}{\partial q^2} = O(\varphi^3) \tag{38}$$

(the estimates are weakened here and there). The similarity of (31) and (38) implies that of estimates for the expressions in curly brackets entering (33). Due to a special form of e, estimation of these expressions becomes a bit easier:

$$\left\| \frac{\partial^2 f}{\partial (p,q)^2} \left(\frac{\partial e}{\partial(p, \xi)} \cdot, \frac{\partial e}{\partial(p, \xi)} \cdot \right) \right\| \leq$$

$$\left\| \frac{\partial^2 f}{\partial p^2} \right\| + 2 \left\| \frac{\partial^2 f}{\partial p \partial q} \right\| \left(\left\| \frac{\partial e_q}{\partial p} \right\| + \left\| \frac{\partial e_q}{\partial \xi} \right\| \right) + \left\| \frac{\partial^2 f}{\partial q^2} \right\| \left(\left\| \frac{\partial e_q}{\partial p} \right\| + \left\| \frac{\partial e_q}{\partial \xi} \right\| \right)^2$$

which, with regard to (35) and (38), allows us to write out the asymptotics of the left hand side of this inequality

$$\frac{\partial^2 f}{\partial(p,q)^2}\left(\frac{\partial e}{\partial(p,\xi)}\cdot, \frac{\partial e}{\partial(p,\xi)}\cdot\right) = O\left(\frac{1}{\varphi}\right).$$

As a matter of fact, estimating of $\partial^2 W/\partial(p,\xi)^2$ is over:

$$\frac{\partial^2 W}{\partial(p,\xi)^2} = O\left(\frac{1}{\varphi}\right). \tag{39}$$

We still have to estimate $|(\nabla\Delta)(Z)|_G$ and $|(\nabla^2 W)(Z,U)|_G$. From the condition $|Z|_G = |U|_G = 1$ and the first inequality of (24) follows $|Z|$, $|U| < C\varphi$. Taking this into account, as well as (25), (29), (36), (37), and (39) we infer from (22) and (23) the asymptotics

$$(\nabla\Delta)(Z) = \varepsilon O(\varphi),$$
$$(\nabla^2 W)(Z,U) = O(\varphi).$$

Adding here (29) and using the second inequality of (24), we obtain

$$|\Delta|_G = \varepsilon O(1),$$
$$|(\nabla\Delta)(Z)|_G = \varepsilon O(1),$$
$$|(\nabla^2 W)(Z,U)|_G = O(1).$$

Thus, we have verified the fulfilment of Theorem 1 conditions. Homeomorphism $\Pi^{-1}\circ\hat{g}^\sharp\circ u\circ\hat{g}^\flat$: $\{F_0 = \frac{1}{2}\} \to \{F_0 = \frac{1}{2}\}$ transfers trajectories of the Hamiltonian field $(\partial F_0/\partial q, -(\partial F_0/\partial p)$ into those of the field $(\partial F/\partial q, -(\partial F/\partial p)$. Theorem 2 is proved.

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