PERIODIC ELLIPTIC MOTIONS IN A PLANAR RESTRICTED $(N+1)$-BODY PROBLEM

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(Received 5 November, 1976)

Abstract. Let $N \geqq 2$ mass points (primaries) move on a collinear solution of relative equilibrium of the N-body problem; i.e. suitably fixed on a uniformly rotating straight line. Consider the motion of a massless particle in the gravitational field of these primaries with arbitrarily given masses. An existence proof for periodic solutions (i.e. closed trajectories in a rotating coordinate system) will be given, in which the particle performs nearly keplerian elliptic motions about (and close to) any one of the primaries.

1. Introduction

We consider the planar N-body problem for $N \geqq 2$ mass points with given masses $m_k > 0$ and inertial position vectors q_k, $(k = 1, \ldots, N)$. We identify the plane of motion with the ordinary complex plane. Let the real numbers $e_1, \ldots, e_N$ denote the positions of a collinear central configuration belonging to the given masses; i.e. $(e_1, \ldots, e_N)$ is a solution of the system of equations

$$\sum_{n=1}^{N}{}^{*} m_n F(e_n - e_k) = -e_k \text{ (real)}, \; (k = 1, \ldots, N); \; F(z) = z\,|z|^{-3}. \tag{1}$$

(A star on a summation sign means that an undefined term; i.e. here the one for $n = k$, is to be deleted.) One can show (Moulton (1910), whose derivation however is not quite rigorous; or Smale (1970)) that the number of solutions of (1) is positive and finite for any choice of the masses; in particular, this number is $N!$, if $m_1, \ldots, m_N$ have different values. We take from the keplerian motions $q = q(\mathrm{t})$ with $\ddot{q} = -F(q)$ the circular one to form a collinear homographic solution

$$q_k = e_k q(t), \; (k = 1, \ldots, N), \; q(t) = e^{it} \tag{2}$$

of the N-body problem

$$\ddot{q}_k = \sum_{n=1}^{N}{}^{*} m_n F(q_n - q_k), \; (k = 1, \ldots, N), \; (\dot{} = \mathrm{d}/\mathrm{d}t). \tag{3}$$

Let p denote the inertial position vector of a particle of mass zero, which moves in the gravitational field of the N primaries, without disturbing their motion given by (2). The equation of motion for this $(N+1)$th body is

$$\ddot{p} = \sum_{k=1}^{N} m_k F(q_k - p), \quad (N \geq 2) \tag{4}$$

and we call (4) the 'circular collinear restricted n-body problem', $n = N + 1$.

If p is near q_N, say, its motion can be approximated by a Kepler-problem, and the possibility of periodic motions might be expected. If one is not content with nearly circular motions, but wants motions which are close to elliptic orbits with prescribed eccentricity, then the standard symmetry of the solution with respect to the rotating straight line carrying the N primaries, is the characteristic property to look for. This is familiar from the restricted 3-body problem, which is the special case of (4) with $N = 2$.

The goal of this paper is to prove the actual existence of such 'relative periodic' solutions of *elliptic* type of (4), which close in a rotating coordinate system after many revolutions of p about q_N and have periods close to $2\pi m$, where m is any prescribed natural number. We remark that the masses of the primaries are given in advance and that, for instance, m_N might be much smaller than the masses of the other bodies acting on p. Thus, even when $N = 2$, the existence of our solutions cannot be argued simply by invoking the implicit-function theorem with a 'sufficiently small' mass parameter. Our result includes (for $N = 2$) the long-periodic solutions of Conley (1963); and the ones derived in Arenstorf (1966), but now with a nicer proof.

The existence of nearly *circular* motions of p about q_N has been established by Perron (1937), though not for (4) but even for the corresponding $(N+1)$-body problem where p has a positive mass. His method (which uses the period, or equivalently, the radius of the orbit as a small parameter) is not capable of giving elliptic solutions also (whose period cannot be small).

2. Transformation of the Equations of Motion

We introduce a scaled relative position vector x in a rotating coordinate system by

$$x = \beta(p - q_N)e^{-it}, \quad \beta = m_N^{-1/3} > 0.$$

Since we chose not to order the e_k of (1) in any particular way, q_N here can indicate any one of the collinear primaries. Then, by (2), (3) and (4)

$$\ddot{x} + 2i\dot{x} - x = -F(x) + \beta \sum_{k=1}^{N-1} m_k[F(e_k - e_N - \beta^{-1}x) - F(e_k - e_N)]. \tag{5}$$

Using (1) this is equivalent to the Hamiltonian system

$$\dot{x}_j = G_{y_j}, \dot{y}_j = -G_{x_j}, (j = 1, 2); \; x = x_1 + ix_2, \; y = y_1 + iy_2; \tag{6}$$

$$G = \tfrac{1}{2}|y|^2 + x_2 y_1 - x_1 y_2 - |x|^{-1} - U = G(x, y), \tag{6a}$$

$$U = \beta e_N x_1 + \beta^2 \sum_{k=1}^{N-1} m_k(|e_N - e_k + \beta^{-1}x|^{-1} - |e_N - e_k|^{-1}). \tag{6b}$$

The constant terms in (6b) have been added so that $U = U(x)$ vanishes at $x = 0$, and then in second order, since also grad $U = 0$ at $x = 0$, by (1). Hence

$$|U(x)| \leqq c_1 |x|^2 \quad \text{for} \quad |x| \leqq c_0, \tag{7}$$

where c_0 and c_1 are suitable constants depending only on the given central configuration.

To treat elliptic motion of p about q_N; i.e. of x, introduce Delaunay variables u_1, u_2 (mod 2π) and v_1, v_2 instead of x and y by the (time-independent) canonical transformation

$$x = v_1[v_1(\cos\tau - \varepsilon) + iv_2 \sin\tau]e^{iu_2},\ y = r^{-1}[-v_1 \sin\tau + + iv_2 \cos\tau]e^{iu_2}, \tag{8a}$$

$$\varepsilon = (1 - v_1^{-2}v_2^2)^{1/2} > 0,\ r = v_1^2(1 - \varepsilon\cos\tau),\ 0 < |v_2| < v_1,\quad \tau - \varepsilon\sin\tau = u_1, \tag{8b}$$

where the last (Kepler's) equation is to be solved for $\tau = \tau(u_1, \varepsilon)$ and the solution to be substituted into the other equations. (8) is generated by

$$y_j = W_{x_j},\ u_j = W_{v_j};\ W = v_2 \operatorname{arc} x + \int_{r_1}^{|x|} [(r - r_1)(r_2 - r)]^{1/2} \frac{dr}{v_1 r},\quad r_{1,2} = v_1^2(1 \mp \varepsilon);$$

and implies, in particular,

$$|x| = r,\ |y|^2 = \frac{2}{r} - v_1^{-2},\ x_2 y_1 - x_1 y_2 = \operatorname{Im} x\bar{y} = -v_2.$$

Thus (8) transforms (6) into

$$\dot{u}_j = H_{v_j},\ \dot{v}_j = -H_{u_j},\ (j = 1, 2);\ H = -\tfrac{1}{2}v_1^{-2} - v_2 - U, \tag{9}$$

where U is given in (6b), but with $x = x(u, v)$ substituted from (8).

Since (9), being autonomous, conserves energy, we can eliminate time t; for instance, by introducing the (fast) angle u_1 as new independent variable along the solutions to be considered. On the latter $|x|$ is small; i.e. v_1 is small and thus $\dot{u}_1 > 0$ by (9). We rename

$$u_1 = s,\ u_2 = u,\ v_2 = v;\ ' = d/ds. \tag{10}$$

Then

$$u' = K_v,\ v' = -K_u;\ v_1 = -K,\ t = \int_{s_0}^{s} H_{v_1}^{-1}\, ds \tag{11a}$$

is equivalent to (9), where K is to be determined by solving for $-v_1$

$$H(u_1, u_2, v_1, v_2) = h \Leftrightarrow v_1 = -K(u_1, u_2, v_2; h) \tag{11b}$$

with a large negative constant h, and substituting from (10).

3. Periodicity Conditions for Elliptic Motions

We consider the equations of motion in the form (6). As in the restricted three-body problem, the symmetry conditions

$$x(t) = \overline{x(t)}, y(t) = -\overline{y(t)} \quad \text{at} \quad t = 0 \quad \text{and} \quad t = t_1 > 0 \tag{12}$$

imply that the solution $z(t) = (x(t), y(t))$ is a periodic function of t with period $T = 2t_1$; (the bar means complex conjugation). For, $\overline{(x(T-t), -y(T-t))}$ is a solution of (6) also, which agrees with $z(t)$ at $t = t_1$, by (12), hence for all t, which implies $z(T) = z(0)$, again by (12), and thus the statement. By (8) the conditions (12) translate into

$$u_1 \equiv u_2 \equiv 0 \bmod \pi \quad \text{at} \quad t = 0 \quad \text{and} \quad t = t_1 \tag{13}$$

for solutions of (9), since Kepler's equation in (8b) yields $\tau = u_1$, if $u_1 = k\pi$ (k integer).

If one replaces U in (6) and in (9) by zero, one obtains the Kepler-problem (for p about q_N) in a rotating coordinate system; i.e.

$$\dot{u}_1 = v_1^{-3}, \dot{u}_2 = -1, \dot{v}_1 = 0, \dot{v}_2 = 0. \tag{14}$$

This has the particular solutions $u_j = u_j^*(t)$, $v_j = v_j^*(t)$ given by

$$u_1^* = a^{-3/2}\, t, u_2^* = -t, v_1^* = a^{1/2} > 0, v_2^* = b^*; \tag{15a}$$

with

$$a = a^* = |m^*/k^*|^{2/3}, b^* = (m^*/k^*)^{1/3}(1 - \varepsilon^{*2})^{1/2}, 0 < \varepsilon^* < 1, \tag{15b}$$

where $m^* > 0$ and $k^* \neq 0$ are relatively prime integers and $\operatorname{sng} b^* = \operatorname{sng} k^*$. The constants a^*, b^* have been chosen so that $x = x^*(t; \varepsilon^*, k^*, m^*)$ given by

$$x^* = a^*(\cos\tau - \varepsilon^* \pm i\sqrt{1 - \varepsilon^{*2}} \sin\tau)e^{-it}, t = \frac{m^*}{|k^*|}(\tau - \varepsilon^* \sin\tau) \tag{16a}$$

according to (8), ($\pm = \operatorname{sgn} k^*$) describes a rotating elliptic motion of semi-major axis $a = a^*$ and eccentricity ε^*, closing after $k^* - m^*$ revolutions about $x = 0$ as focus, with period $T^* = 2\pi m^*$ (in t).

In order that we can use this motion as an approximation to analogous solutions of (6) and (9), we require that $x = x^*$ satisfies (7) for all t. Therefore we choose $|k^*|$ sufficiently large compared to m^*. Then

$$T^* = 2\pi m^*, h^* = \frac{-1}{2a^*} - b^* \ll 0; \tag{16b}$$

i.e. the period can be chosen arbitrarily large, while the 'energy' $H = h^*$ must be chosen sufficiently large negative. Now, the functions $u_j^*(t)$ in (15a) satisfy the periodicity conditions (13) with $t_1 = \frac{1}{2} T^*$. Therefore we require that the analogous solution of (9), to be derived in the following, satisfies the specific periodicity conditions

$$u_1 = u_2 = 0 \quad \text{at} \quad t = 0; \, u_1 = \pi \left|k^*\right|, u_2 = -\pi m^* \quad \text{at} \quad t = t_1. \tag{17}$$

Here t_1 and the initial values of v_1 and v_2 are to be determined so as to satisfy the second half of these equations. Finally, let

$$u(s;h), v(s;h) \quad \text{with} \quad u(0;h) = 0, v(0;h) = b^* \tag{18}$$

denote the solution of (11) with the indicated initial values, where b^* is from (15b). Since (11) and (9) are equivalent, (17) after fixing v_2 initially at b^* becomes equivalent to

$$u(\pi \left|k^*\right|; h) = -\pi m^*, \, t_1 = \int_0^{\pi|k^*|} H_{v_1}^{-1} \, \mathrm{d}s, \tag{19}$$

using the notation from (10). The first equation here is to be solved for h near h^*; the second then yields the half-period.

4. Method for Solving the Periodicity Condition

We want to show that the first equation in (19) can be satisfied by proper choice of h. By (11b) this is equivalent to determining the initial value of v_1, since the initial values of u_1, u_2 and v_2 are already given. The apparent loss of a parameter when fixing v_2 initially at b^* is unessential, since ε^* in (15b) can be considered as a free parameter.

We observe that the equation to be solved for h in (19) does not depend on any parameter (besides h), so that the customary method of solution by the ordinary implicit-function theorem (which requires knowledge of a solution for a special value of that parameter already and non-vanishing of a determinant at that solution) is not now applicable. Instead, we shall guarantee the existence of a solution by applying the following special case of a general theorem first stated and proved in Arenstorf (1968).

THEOREM: *Let $f(h)$ and $g(h)$ be real-valued differentiable function of the real variable h in $\left|h\text{-}h^*\right| < \rho$, and $f'(h^*) \neq 0$ for some real h^*, (where the prime now denotes the derivative with respect to h). Let g be 'close' to f; i.e.*

$$\left|g(h^*) - f(h^*)\right| < \gamma\rho \left|f'(h^*)\right|, \, (0 < \gamma < \tfrac{1}{2}, \rho > 0) \tag{20a}$$

and

$$\left|g'(h) - f'(h^*)\right| < \gamma \left|f'(h^*)\right| \quad \text{for} \quad \left|h - h^*\right| < \rho. \tag{20b}$$

Then there exists a solution $h = h_\infty$ of $g(h) = f(h^*)$ with $\left|h_\infty - h^*\right| < \rho$.

Proof. Define recursively for $n = 1, 2, \ldots.$

$$h_1 = h^*, h_{n+1} = S(h_n) \quad \text{with} \quad S(h) = h - [g(h) - f(h^*)]/f'(h^*). \tag{21}$$

Then, by the mean-value theorem and (20b)

$$|S(x) - S(y)| < \gamma |x - y| \quad \text{for} \quad |x - h^*| < \rho, |y - h^*| < \rho;$$

i.e. S is a local contraction. Hence by (21), as seen by induction over n,

$$|h_n - h^*| < \rho, |h_{n+1} - h_n| < \rho\gamma^n, \qquad (n \geqq 1)$$

using (20a) for the start. This yields

$$|h_{n+k} - h_n| < \sum_{j=n}^{n+k-1} |h_{j+1} - h_j| < 2\rho\gamma^n, \qquad (k, n \geqq 1).$$

Thus $\lim h_k = h_\infty$ exists (in the stated domain) and $h_\infty = S(h_\infty)$ by continuity of S. This implies the statement of the theorem. We observe that this existence proof is constructive.

In order to apply this theorem with $g(h) = u(\pi |k^*|; h)$ to the first equation in (19) we have to find another function $f(h)$ and suitable values h^*, γ and ρ so that g is close to f in the above sense. Therefore we return to the equations of motion (11), which together with (18) determine the function $g(h)$. Motivated by the approximations leading to (16), we replace H in (11b) and K in (11) by

$$H^0 = -\tfrac{1}{2} v_1^{-2} - v_2 \quad \text{and} \quad K^0 = -[-2(v + h)]^{-1/2} = K^0(v; h), \tag{22}$$

where h is a large negative parameter, and we introduce the solutions of the Hamiltonian system

$$u' = K_v^0, v' = -K_u^0; (u, v) = (\xi, \eta) \quad \text{at} \quad s = 0; \tag{23a}$$

i.e.

$$u = \xi - s[-2(\eta + h)]^{-3/2} \equiv \phi(s; \xi, \eta; h), v = \eta. \tag{23b}$$

For later application we remark that the transformation $(\xi,\eta) \to (u,v)$ given by (23b) is canonical. Now we define

$$f(h) = \phi(\pi |k^*|; 0, b^*; h), g(h) = u(\pi |k^*|; h) \tag{24}$$

having chosen the initial values for (23) as in (18). Then, by (23b), and with h^* from (16b) and (15b) we have $f(h^*) = -\pi m^*$. Thus, the periodicity condition in (19) can be rewritten with (24) as

$$g(h) = f(h^*). \tag{25}$$

To guarantee the existence of a solution h of this equation, it remains to verify that the functions f and g of (24) satisfy the assumptions (20a, b) of the existence theorem with suitable γ and ρ. To begin with we have by (24), (23b), (16b) and (15b)

$$f'(h) = -3\pi |k^*| [-2(b^* + h)]^{-5/2}, f'(h^*) = -3\pi a m^* \neq 0, \tag{26}$$

and

$$|f'(h) - f'(h^*)| < 15\pi |k^*| \rho [a^{-1} - 2\rho]^{-7/2} < 180\pi m^* a^2 \rho, \tag{27}$$

if

$$|h - h^*| < \rho \quad \text{and} \quad 0 < a\rho < \tfrac{1}{4};\ a = |m^*/k^*|^{2/3}. \tag{28}$$

In order to estimate $g(h) - f(h)$ and its derivative by h we need more precise information about the solutions of (11a, b), which we shall now derive.

5. Estimates for Solutions of the Restricted n-Body Problem

We define

$$R = K(s, \phi, \eta; h) - K^0(\eta; h) = R(s, \xi, \eta; h) \tag{29}$$

with K from (11b) and K^0 from (22), having substituted $\phi = \phi(s;\xi,\eta;h)$ from (23b) for u and η for v. The canonical transformation given in (23b) transforms the canonical equations of the restricted (N + 1)-body problem in (11a) to

$$\xi' = R_\eta, \eta' = -R_\xi; (\xi, \eta) = (u, v) \quad \text{at} \quad s = 0. \tag{30}$$

By (24) and (23b) we are interested in the difference

$$Z = u(s; h) - \phi(s; 0, b^*; h) = \xi - s[-2(\eta + h)]^{-3/2} + s[-2(b^* + h)]^{-3/2}, \tag{31a}$$

where

$$\xi = \xi(s; h), \eta = \eta(s; h) \quad \text{with} \quad \xi(0; h) = 0, \eta(0; h) = b^* \tag{31b}$$

denotes the solution of (30) with the indicated initial values, as required in (18). In particular, we need $Z = Z(s;h)$ and its partial derivative $Z_h(s; h)$ (by h) at $s = \pi |k^*|$, for verification of (20a, b). By (31)

$$Z_h(s; h) = \xi_h - 3s[-2(\eta + h)]^{-5/2}(\eta_h + 1) + 3s[-2(b^* + h)]^{-5/2}. \tag{32}$$

Differentiating in (30) by h, we obtain, with (31b)

$$\begin{aligned} \xi_h' &= \xi_h R_{\eta\xi} + \eta_h R_{\eta\eta} + R_{\eta h};\ \xi_h(0; h) = 0, \\ \eta_h' &= -\xi_h R_{\xi\xi} - \eta_h R_{\xi\eta} - R_{\xi h};\ \eta_h(0; h) = 0, \end{aligned} \tag{33}$$

where $' = d/ds$, and a variable as index denotes the corresponding partial derivative.

Thus we have to integrate (30) and (33) over a long interval, $0 \leq s \leq \pi |k^*|$, since $|k^*|$ must be sufficiently large to keep the perturbations small, as we already remarked for (16b).

We need estimates for the partial derivatives of R occurring in (30) and (33). Since only the magnitude, measured in powers of the small parameter $\delta = a^{1/2} > 0$ matters, we will use the familiar 0-symbol:

$$f = 0(\delta^r) \quad \text{means} \quad |f| \leqq c\delta^r \quad \text{with} \quad \delta = |m^*/k^*|^{1/3} < 1$$

for any function f, where c is a constant which may depend on m^* but not on k^*, and may change from one application to another. We now restrict the variables occuring in R to be real and to satisfy

$$0 \leqq s \leqq \pi |k^*|, \; |\eta - b^*| < \delta^2, \; |h - h^*| < \rho; \; 4\delta(\rho + \delta^2) < 1 \tag{34}$$

thus assuring (28), and we observe that ξ like $u = u_2$ is an angular variable. Then, with (16b) and (22)

$$\delta\left(1 - \frac{\delta}{2}\right) < -K^0(\eta; h) = [-2(h + \eta)]^{-1/2} < \delta\left(1 + \frac{\delta}{2}\right). \tag{35}$$

By (11b) and (9), and using the notation of (10), we can determine

$$K = K(s, u, v; h) = -v_1 \quad \text{from} \quad v_1 = [-2(h + v + U)]^{-1/2} > 0 \tag{36}$$

with $U = U(x)$ of (6b) and $x = x(u_1, u_2, v_1, v_2)$ of (8), by iteration. Here

$$|U(x)| < 4c_1 v_1^4 \quad \text{for} \quad |x| < 2v_1^2 < c_0 \tag{37}$$

by (7); hence (36) with $v = \eta$, and (35) yield $v_1 < 2\delta$, and more precisely

$$K(s, u, \eta; h) = K^0(\eta; h)\,[1 - 2UK^{02}]^{-1/2} = K^0(\eta; h) + 0(\delta^7), \tag{38}$$

if δ is sufficiently small (which is to mean that δ is smaller than some constructable constant, which only for brevity we do not exhibit). Also, by the foregoing and (15b) and then by (8b), since $v_1 = -K$ and $v_2 = v = \eta$,

$$|v_1 - \delta| < \delta^2, \; v_2 v_1^{-1} = \pm (1 - \varepsilon^{*2})^{1/2} + 0(\delta), \; 0 < \varepsilon_1 < \varepsilon < \varepsilon_2 < 1, \tag{39}$$

where ε_1 and ε_2 depend only on ε^*; if δ is sufficiently small. Now, by (29) and (38)

$$R = -\int_0^U [-2(h + \eta + z)]^{-3/2}\, dz, \; U = U(x), \; x = x(s, \phi, -K, \eta) = \\ = x_1 + ix_2. \tag{40}$$

By (6b) and (37) U has a convergent power series expansion beginning with second order terms in x_1 and x_2, and by (8) and (39) x_1 and x_2 are real-analytic functions of

u_1, u_2, v_1, v_2 there. By (36) then K is real-analytic in its four variables, and with ϕ from (23b) the same holds for R, on the domain described by (34). Thus all partial derivatives of R are finite there, and can be calculated from (40). By (8) and (39)

$$\tau_\varepsilon = \frac{\sin\tau}{1-\varepsilon\cos\tau} = 0(1),\ \varepsilon_{v_1} = \varepsilon^{-1} v_1^{-3} v_2^2 = 0(\delta^{-1}),$$
$$\varepsilon_{v_2} = -\varepsilon^{-1} v_1^{-2} v_2 = 0(\delta^{-1}),$$

thus

$$x, x_{u_2}, x_\tau, x_\varepsilon = 0(\delta^2);\ x_{v_1}, x_{v_2} = 0(\delta). \tag{41}$$

Similarly one recognizes that another partial differentiation in (41) by v_1 or v_2 decreases the order (in δ) by one, while differentiation by u_2 does not change the order. Also, by (23b), (34) and (35)

$$\phi_\xi = 1,\ \phi_\eta = \phi_h = 0(\delta^2),\ \phi_{hh} = 0(\delta^4);\ K_h^0 = 0(\delta^3),\ K_{hh}^0 = 0(\delta^5). \tag{42}$$

In (40) we can replace K by $K^0 + R$ according to (29), thus obtaining an implicit equation for R (which again can be solved by iteration). Differentiating this equation by ξ,η or h we get, by (36) with (23b)

$$R_\xi = K^3 U' x_\xi,\ R_\zeta = -3\int_0^U [-2(h+\eta+z)]^{-5/2}\,\mathrm{d}z + K^3 U' x_\zeta,$$
$$(\zeta = \eta \quad \text{or} \quad h), \tag{43a}$$

$$x_\xi = x_{u_2} - x_{v_1} R_\xi,\ x_\zeta = x_{u_2}\phi_h - x_{v_1}(K_h^0 + R_\zeta) + x_{v_2}\eta_\zeta;\ U' = \operatorname{grad} U. \tag{43b}$$

Solving for the derivatives of R we get with (36), (37), (39), (41), (42) and since $U' = 0(|x|)$

$$R_\xi = 0(\delta^7),\ R_\eta = 0(\delta^9) + 0(\delta^5)0(\delta^4 + \delta^4 + \delta) = 0(\delta^6),\ R_h = 0(\delta^9). \tag{44}$$

Differentiating partially in (43a) we get

$$R_{\xi\zeta} = 3K^2 K_\zeta U' x_\xi + K^3 x_\zeta U'' x_\xi + K^3 U' x_{\xi\zeta},\ (\zeta = \xi, \eta \quad \text{or} \quad h), \tag{45a}$$

$$R_{\eta\zeta} = -15\int_0^U [-2(h+\eta+z)]^{-7/2}\,\mathrm{d}z + 3K^5 U' x_\zeta + 3K^2 K_\zeta U' x_\zeta +$$
$$+ K^3 x_\zeta U'' x_\eta + K^3 U' x_{\eta\zeta},\ (\zeta = \eta \quad \text{or} \quad h);\ K_\zeta = K_\zeta^0 + R_\zeta, \tag{45b}$$

where the second partial derivatives of $x = x(s, \phi, -K^0 - R, \eta)$ can be derived from (43b) and will contain the first and second partial derivatives of R. Solving for the latter, and using (44), $U'' = 0(1)$, (35), the estimates in (37) to (42), and our remark concerning the second derivatives of x after (41), we obtain from (45a)

$$R_{\xi\xi} = 0(\delta^{2+7+2+2}) + 0(\delta^{3+2+0+2}) + 0(\delta^{3+2+2}) = 0(\delta^7),$$
$$R_{\xi\eta} = 0(\delta^{2+3+2+2}) + 0(\delta^{3+1+0+2}) + 0(\delta^{3+2+1}) = 0(\delta^6),$$
$$R_{\xi h} = 0(\delta^{2+3+2+2}) + 0(\delta^{3+4+0+2}) + 0(\delta^{3+2+4}) = 0(\delta^9), \tag{46a}$$

and from (45b)

$$R_{\eta\eta} = 0(\delta^{11}) + 20(\delta^{5+2+1}) + 0(\delta^{3+1+0+1}) + 0(\delta^{3+2+0}) = 0(\delta^5),$$
$$R_{\eta h} = 0(\delta^{11}) + 0(\delta^{5+2+4}) + 0(\delta^{2+3+2+1}) + 0(\delta^{3+4+1}) + + 0(\delta^{3+2+3}) = 0(\delta^8). \tag{46b}$$

The estimates (44) and (46a, *b*) hold on the domain described by (34), and R is a real-analytic function there (with no restriction on ξ other than being real; and $\rho > 0$ an as yet unspecified parameter and δ sufficiently small).

We can now integrate (30) to obtain the functions in (31b). Applying the method of successive approximations on $0 \leqq s \leqq s^* = \pi |k^*| = 0(\delta^{-3})$, it follows by (44) that every iterate (ξ_n, η_n) satisfies

$$|\xi_n(s)| < c_2\delta^3, \; |\eta_n(s) - b^*| < c_3\delta^4$$

and thus (34); hence the solution of (30) exists on $0 \leqq s \leqq s^*$ and satisfies

$$\xi(s;h) = 0(\delta^3), \; \eta(s;h) = b^* + 0(\delta^4), \quad \text{(on (34))}. \tag{47}$$

Thus all the foregoing estimations hold along this entire solution curve also, and we can integrate (33) on this curve using (46). Again applying successive approximations we obtain, if δ is sufficiently small,

$$\xi_h(s;h) = 0(\delta^5), \; \eta_h(s;h) = 0(\delta^6), \quad \text{(on (34))}. \tag{48}$$

6. Proof of the Main Result and Conclusions

We can now show that (25) has a solution. Namely, by (24), (31), (35) and (47)

$$g(h^*) - f(h^*) = Z(s^*;h^*) = 0(\delta^3) + s^*0(\delta^{5+4}) = 0(\delta^3), \tag{49}$$

since $s^* = \pi |k^*| = \pi m^*\delta^{-3}$; and by (32) and (48), for $|h - h^*| < \rho$

$$g'(h) - f'(h) = Z_h(s^*;h) = 0(\delta^5) + s^*0(\delta^{5+6} + \delta^{7+4}) = 0(\delta^5). \tag{50}$$

By (26) and (49), since $a = \delta^2$,

$$|g(h^*) - f(h^*)| \cdot |f'(h^*)|^{-1} < c_4\delta = \gamma\rho \quad \text{for} \quad \gamma = \delta, \rho = c_4, \tag{51}$$

where c_4 is a constant. And by (50) and (27)

$$|g'(h) - f'(h^*)| \cdot |f'(h^*)|^{-1} < c_5\delta^3 + 60\delta^2\rho < \delta = = \gamma, \quad \text{for} \quad |h - h^*| < \rho = c_4. \tag{52}$$

Clearly, with this choice of γ and ρ we have now satisfied the conditions in (20a, b) and the last condition in (34), if δ is sufficiently small. Hence the existence theorem in Section 4 implies the above statement on solvability of the periodicity condition (25).

We have thus proven the following result for the circular collinear restricted $(N + 1)$-body problem with arbitrarily given masses of the primaries:

The equations of motion (in the form (5) say) admit infinitely many 1-parameter families of periodic solutions with periods $T = T(\varepsilon^*, k^*, m^*)$ close to $2\pi m^*$ for every positive integer m^*. Each such solution describes motion (of the particle of zero mass) close to a rotating keplerian *elliptic* orbit (the 'generating orbit') about any one of the primaries, closing after k^*-m^* revolutions in a coordinate system rotating uniformly with the N primaries. The approximating elliptic generating orbits described in (16a) can be chosen in advance. So ε^* in (0, 1) becomes the continuous family parameter, and the different families of periodic motions correspond to the rational numbers m^*/k^*, which determine period and winding number of the closed generating orbits in the rotating coordinate system (k^* negative for inertially retrograde, positive for direct orbits). The associated periodic solutions $x = x(t; \varepsilon^*, k^*, m^*)$ of (5) exist, if only

$$|k^*| \geqq C = C(\varepsilon^*, m^*),\ (0 < \varepsilon^* < 1, m^* > 0), \tag{53}$$

where C is a constructable function of the indicated parameters. Thus the 'Jacobi integral' $G(x,y) = h$ with G from (6) has on these solutions large negative values $h = h(\varepsilon^*, k^*, m^*)$, and the instantaneous major axis are always small. The trajectories are symmetric to the line carrying the primaries in the rotating coordinate system.

We remark finally that it is possible to derive nearly *circular* periodic solutions of (5) with small period by similar means as applied in this paper. In that case the variables of Poincaré instead of Delaunay are to be used; and the verification of the assumptions of the existence theorem in Section 4 becomes much easier than in the present case.

Periodic motions similar to the ones described in this paper, but for the (unrestricted) n-body problem, will be considered in another paper.

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