THE THREE-BODY PROBLEM NEAR TRIPLE COLLISION

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Abstract. The behaviour of three gravitationally interacting particles in a plane, which approach each other almost on a central configuration, is studied. Linearization near a Lagrangean solution and matching methods lead to the following results: (i) After a close triple encounter in the planar problem of three bodies, one particle generally escapes with an arbitrarily large asymptotic velocity. (ii) Particular cases of actual triple collisions may be extended by the method of Easton.

1. Introduction

Triple collision is one of the crucial processes in the evolution of systems of gravitationally interacting particles. It is intimately connected with *escape* in the problem of three bodies; in systems with four and more interacting particles triple collision is essential for the existence of 'exotic' types of motion like infinite expansion in finite time (Mather and McGehee, 1975).

In general, a triple collision solution terminates (or originates) at the singularity, i.e., no *real-analytic continuation* exists past the singularity. This is a consequence of Siegel's (1941) representation of triple collision solutions by convergent power series. It is in agreement with our main result (Waldvogel, 1975) that, generally after a sufficiently close triple encounter, one particle escapes with an asymptotic velocity as high as desired. The same result, together with a complete description of the triple collision manifold, was obtained for the *one-dimensional* three-body problem by McGehee (1974, 1975).

In contrast, no such behaviour is present in collisions of only two point masses. It is well known that the solutions can always be analytically continued past a singularity due to a binary collision. Furthermore, this continuation agrees with Easton's extension, which is always possible.

In the following we restrict ourselves to the planar problems of three bodies. This problem is governed by the differential equations

$$m_j \frac{\mathrm{d}^2\mathbf{x}_j}{\mathrm{d}t^2} = \frac{\partial U}{\partial \mathbf{x}_j}, \quad U = \sum_{j<k} \frac{m_j m_k}{|\mathbf{x}_j - \mathbf{x}_k|}, \tag{1}$$

where m_j, $\mathbf{x}_j$ are the mass and position vector (with respect to the center of mass) of the jth particle ($j = 1, 2, 3$); thus we have

$$\sum_{j=1}^{3} m_j \mathbf{x}_j = 0. \tag{2}$$

First of all, the family of solutions which are in some sense *close* to a fixed Lagrangean

Celestial Mechanics **14** (1976) 287–300.

solution will be investigated by means of linearization. Then the behaviour near an almost-collision will be deduced using singular perturbation techniques (see Cole, 1968).

The main tool for applying these methods is the simple idea of scaling (Waldvogel, 1975). Let $\tilde{\mathbf{x}}_j$, $\tilde{t}$ be new coordinates and time, related to the old ones by the homothetic transformation

$$\mathbf{x}_j = \delta^2 \tilde{\mathbf{x}}_j, \quad t = \delta^3 \tilde{t}, \tag{3}$$

where δ is a scaling factor. If $\delta \ll 1$, the transformation (3) 'blows up' coordinates and time. The velocities $\dot{\mathbf{x}}_j$, the total energy h and the total angular momentum $\mathbf{C}$ are transformed as

$$\dot{\mathbf{x}}_j = \delta^{-1} \dot{\tilde{\mathbf{x}}}_j, \quad h = \delta^{-2} \tilde{h}, \quad \mathbf{C} = \delta \tilde{\mathbf{C}}, \tag{4}$$

whereas the equations of motion (1) remain *invariant*, which follows from the homogeneity relation

$$U(\lambda \mathbf{x}) = \lambda^{-1} U(\mathbf{x}) \tag{5}$$

of the force function U. Hence a given close triple encounter without actual collisions can always be transformed into a three-body motion where the smallest distance between two bodies is 1, provided the original minimum distance is known. However, if the three bodies approach approximately on a central configuration, this minimum distance is *not* known a priori, but depends on how far the considered solution is from a triple collision solution.

2. Linearized Theory of the Close Triple Encounter

The simplest explicit solutions of Equation (1) are the homothetic solutions with zero total energy ($h=0$):

$$\mathbf{x}_j = c \hat{\mathbf{x}}_j t^{2/3}. \tag{6}$$

Here c is the constant

$$c = \sqrt[3]{\frac{9}{2} m}, \quad m = m_1 + m_2 + m_3, \tag{7}$$

and $\hat{\mathbf{x}}_j$ are constant vectors satisfying certain algebraic conditions which are obtained by inserting (6) into (1). A set of vectors $\hat{\mathbf{x}}_j$ satisfying these conditions is called a *central configuration* (see Wintner, 1941). In the problem of three bodies two types exist: the *equilateral triangle* configuration (Lagrange) and the mass-dependent *collinear* configurations (Euler).

For simplicity, relative coordinates

$$\mathbf{y}_1 = \mathbf{x}_1 - \mathbf{x}_3, \quad \mathbf{y}_2 = \mathbf{x}_2 - \mathbf{x}_3 \tag{8}$$

are now introduced; after elimination of the center-of-mass integrals the equations of motion (1) may be written as

$$\ddot{y} = f(y), \quad (\dot{\ }) = \frac{\mathrm{d}}{\mathrm{d}t}, \tag{9}$$

where

$$y = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}, \quad f = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}$$

are 4-vectors with

$$\mathbf{f}_1 = -(m_1 + m_3)\frac{\mathbf{y}_1}{r_1^3} - m_2\frac{\mathbf{y}_2}{r_2^3} - m_2\frac{\mathbf{y}_1 - \mathbf{y}_2}{r_{12}^3}, \quad r_j = |\mathbf{y}_j| \tag{10}$$

$$\mathbf{f}_2 = -(m_2 + m_3)\frac{\mathbf{y}_2}{r_2^3} - m_1\frac{\mathbf{y}_1}{r_1^3} - m_1\frac{\mathbf{y}_2 - \mathbf{y}_1}{r_{12}^3}, \quad r_{12} = |\mathbf{y}_1 - \mathbf{y}_2|$$

and

$$f(\lambda y) = \lambda^{-2} f(y). \tag{11}$$

The solutions (6) are particular triple collision solutions with a collision at $t=0$; they are now denoted by

$$y_0(t) = c\hat{y}t^{2/3}, \quad \hat{y} = \begin{pmatrix} \hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_3 \\ \hat{\mathbf{x}}_2 - \hat{\mathbf{x}}_3 \end{pmatrix}, \tag{12}$$

and they will be used as a reference in order to investigate nearby solutions

$$y(t) = y_0(t) + \eta(t). \tag{13}$$

In a linear approximation, the perturbation $\eta(t)$ satisfies the variational equation

$$\ddot{\eta} = J\eta, \tag{14}$$

where J is the Jacobi matrix of the vector function (10) at the reference solution (12). By (11) and (12) J may be written as

$$J = t^{-2} \cdot J_0 \tag{15}$$

with a *constant* matrix J_0. This implies that (14) has solutions of the form

$$\eta = \gamma_k \cdot t^{\mu_k}, \tag{16}$$

where γ_k is an eigenvector of J_0 and μ_k satisfies the quadratic equation

$$\mu_k^2 - \mu_k - \lambda_k = 0, \tag{17}$$

λ_k being the eigenvalue corresponding to γ_k. Analyzing the eigenvalue problem

$$J_0\gamma = \lambda\gamma \tag{18}$$

shows that for both types of central configurations and for all possible masses m_j the matrix J_0 has 4 *linearly independent* eigenvectors γ_1, γ_2, γ_3, γ_4 and 4 *real* (but not necessarily different) eigenvalues λ_1, λ_2, λ_3, λ_4.

Let μ_k, ν_k be the two solutions of (17); if they are real we assume $\mu_k \geqslant \nu_k$. Then we have

$$\begin{aligned} \mu_k + \nu_k &= 1, \quad \text{if} \quad \mu_k, \nu_k \text{ real}, \\ \operatorname{Re} \mu_k &= \operatorname{Re} \nu_k = \tfrac{1}{2}, \quad \text{if} \quad \mu_k, \nu_k \text{ complex}. \end{aligned} \tag{19}$$

A time shift in (12) shows that $\hat{y}$ is one of the eigenvectors γ_k; hence we denote $\gamma_1 = \hat{y}$. Furthermore, there follows

$$\lambda_1 = \tfrac{4}{9}, \quad \mu_1 = \tfrac{4}{3}, \quad \nu_1 = -\tfrac{1}{3}. \tag{20}$$

In a first order approximation the 8-parameter family of all solutions nearby the reference (12) is given by

$$y = ct^{2/3}\left\{\gamma_1 + \sum_{j=1}^{4} \gamma_j(a_j t^{\alpha_j} + b_j t^{\beta_j})\right\}, \tag{21}$$

where Siegel's exponents

$$\alpha_j = \mu_j - \tfrac{2}{3}, \quad \beta_j = \nu_j - \tfrac{2}{3} \quad (j = 1, 2, 3, 4) \tag{22}$$

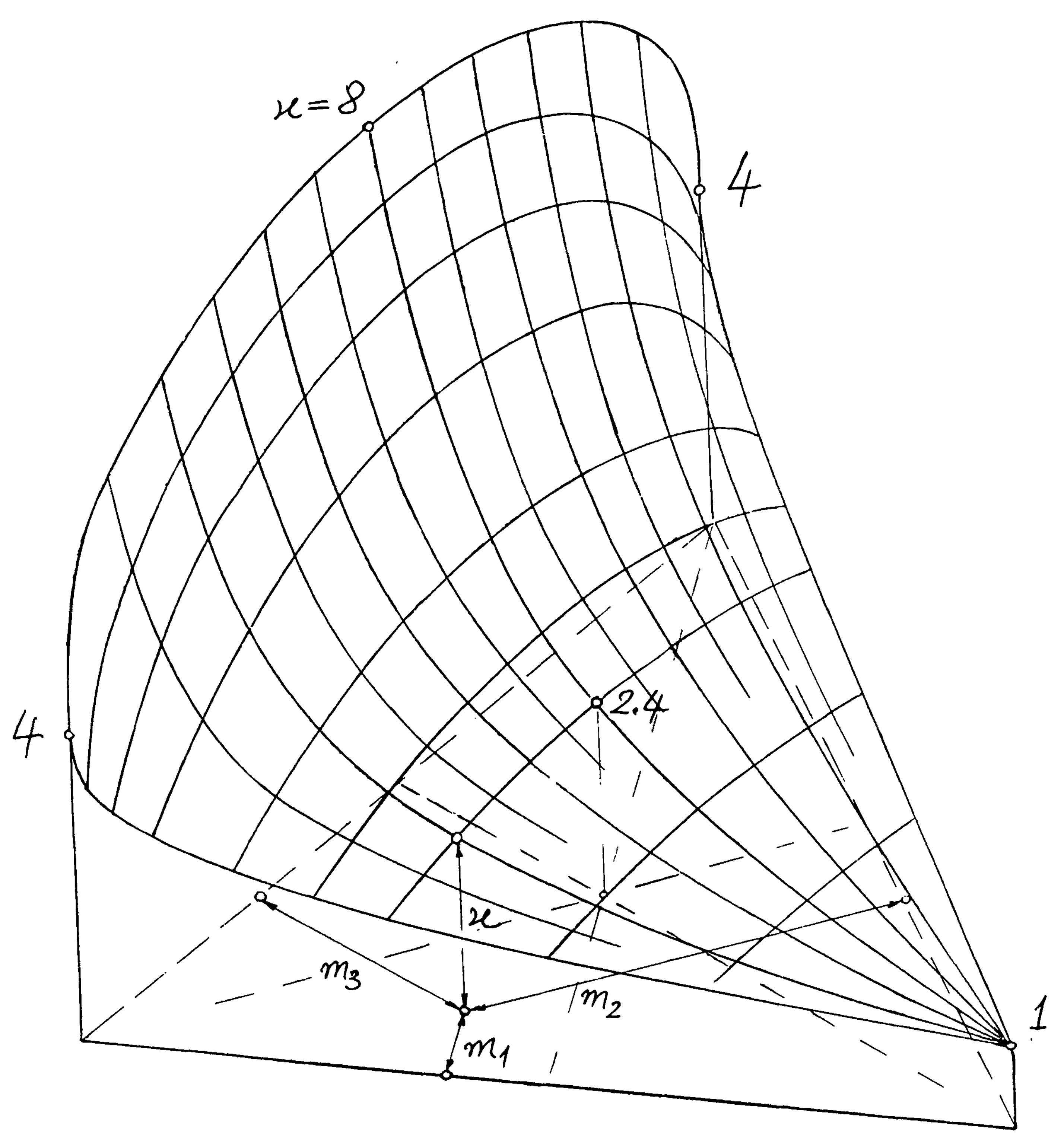

Fig. 1. The auxiliary parameter $\kappa(m_1, m_2, m_3)$ for the collinear central configuration with m_3 as the inner mass. The masses are represented in triangular coordinates.

have been introduced. The 8 small quantities a_j, b_j $(j=1, 2, 3, 4)$ are the parameters of the linear family.

In view of the irrational exponents in (21) t can only take positive values. In the following we therefore let time go *backwards* towards $t=0$.

For a rigorous analysis the complete discussion of the eigenvalue problem of J_0 is necessary. Here we shall only state Siegel's (1941) classical results regarding the eigenvalues; the detailed discussion is left to a later paper.

In order to obtain Siegel's exponents the auxiliary parameter $\kappa(m_1, m_2, m_3)$ is calculated first. In the *equilateral case* we define

$$\kappa = \frac{1}{m}\sqrt{\tfrac{1}{2}[(m_1 - m_2)^2 + (m_2 - m_3)^2 + (m_3 - m_1)^2]}, \tag{23}$$

and there follows

$$0 \leqslant \kappa \leqslant 1$$

($\kappa=0$ corresponds to $m_1=m_2=m_3$, and $\kappa=1$ results if two of the masses vanish). In the *collinear cases* m_3 is assumed to lie between m_1 and m_2. The geometry of the central configuration is described by one of the ratios

$$u = \frac{r_1}{r_1 + r_2}, \quad v = \frac{r_2}{r_1 + r_2} \quad (u + v = 1);$$

they satisfy the well-known quintic equation

$$m_1 v^2(u^3 - 1) + m_2 u^2(1 - v^3) + m_3(u^3 - v^3) = 0. \tag{24}$$

From u, v we obtain

$$\kappa = \frac{m(m_1 u^{-3} + m_2 v^{-3} + m_3 u^{-3} v^{-3})}{[m_1 + m_2 + m_3(u^{-2} + v^{-2})]^2}, \tag{25}$$

and there follows

$$1 \leqslant \kappa \leqslant 8.$$

($\kappa=1$ corresponds to the vanishing of the two outer masses, and $\kappa=8$ results if $m_3=0$, $m_1=m_2$; if the two masses to the right or to the left vanish, $\kappa=4$). Figure 1 shows κ as a function of the masses m_1, m_2, m_3, for which triangular coordinates are used $(m_1+m_2+m_3=1)$.

For every set of masses and every type of central configuration two eigenvalues of J_0 agree with those of the two-body problem: $\lambda_1=\frac{4}{9}$, $\lambda_2=-\frac{2}{9}$. This follows from the existence of the Lagrangean solutions (see Waldvogel, 1976). The corresponding exponents are

$$\begin{aligned}
\alpha_1 &= \tfrac{2}{3} \text{ (energy variation)}\\
\alpha_2 &= 0 \text{ (coordinate rotation)}\\
\beta_2 &= -\tfrac{1}{3} \text{ (angular momentum variation)}\\
\beta_1 &= -1 \text{ (time shift),}
\end{aligned}$$

where the parenthesis refers to the effect of the corresponding perturbing term onto the reference solution. The remaining 2 eigenvalues and 4 exponents are

$$\lambda_3 = \frac{1}{9} + \frac{\kappa}{3}, \quad \lambda_4 = \frac{1}{9} - \frac{\kappa}{3}$$

$$\left.\begin{matrix}\alpha_3\\ \beta_3\end{matrix}\right\} = \tfrac{1}{6}(-1 \pm \sqrt{13 + 12\kappa})$$

$$\left.\begin{matrix}\alpha_4\\ \beta_4\end{matrix}\right\} = \tfrac{1}{6}(-1 \pm \sqrt{13 - 12\kappa}) \tag{26}$$

in the equilateral case and

$$\lambda_3 = \tfrac{4}{9}\kappa, \quad \lambda_4 = -\tfrac{2}{9}\kappa$$

$$\left.\begin{matrix}\alpha_3\\ \beta_3\end{matrix}\right\} = \tfrac{1}{6}(-1 \pm \sqrt{9 + 16\kappa})$$

$$\left.\begin{matrix}\alpha_4\\ \beta_4\end{matrix}\right\} = \tfrac{1}{6}(-1 \pm \sqrt{9 - 8\kappa}) \tag{27}$$

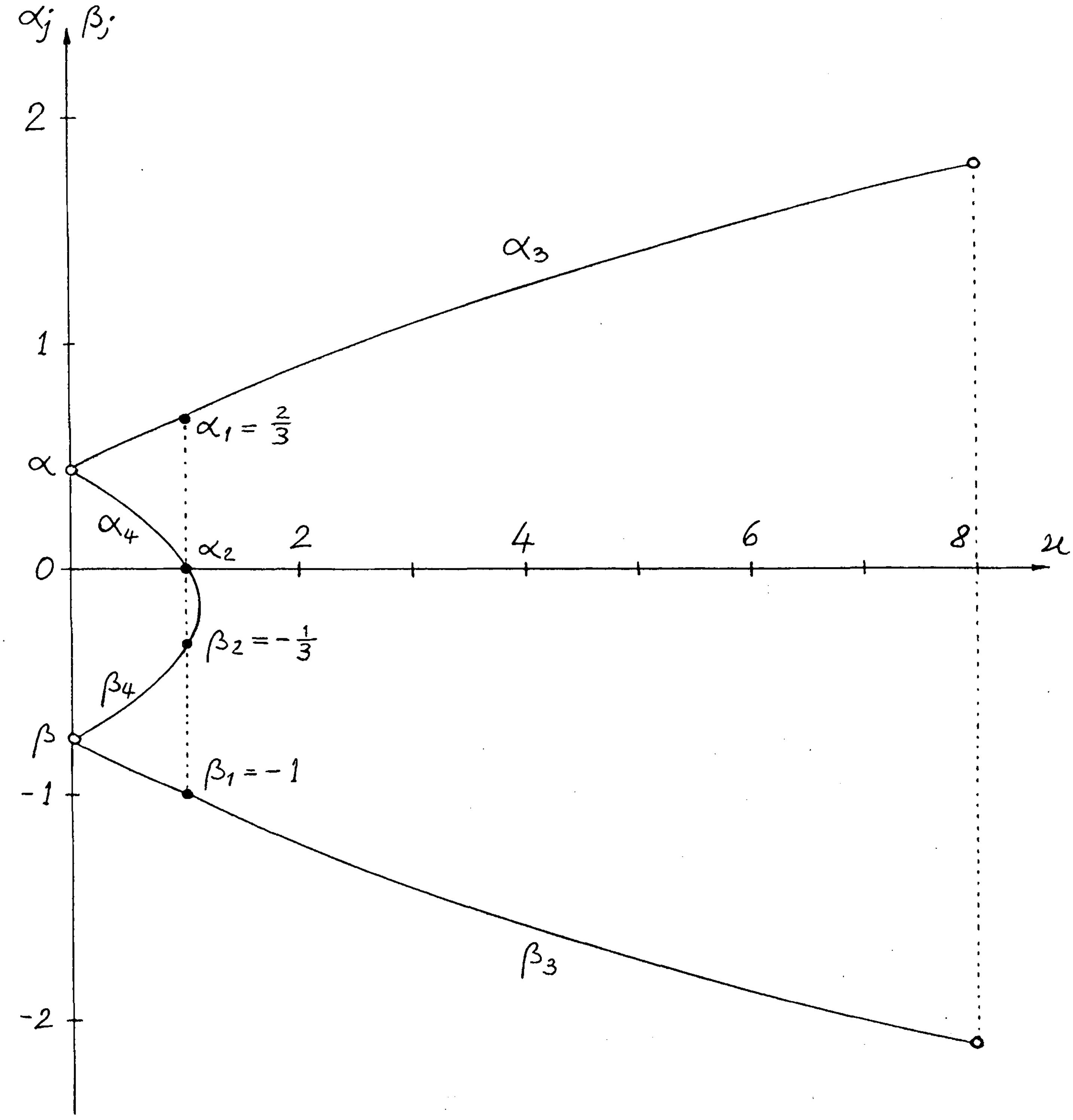

Fig. 2. Siegel's exponents α_j, β_j as functions of the parameter κ.

in the collinear cases (see Figure 2). For $\kappa > \frac{9}{8}$ the exponents α_4, β_4 are complex conjugate,

$$\left.\begin{matrix}\alpha_4\\ \beta_4\end{matrix}\right\} = -\tfrac{1}{6} \pm i\omega, \quad |\omega| = \tfrac{1}{6}\sqrt{8\kappa - 9} \leqslant \tfrac{1}{6}\sqrt{57}, \tag{28}$$

otherwise all the exponents are real and in the interval $[\frac{1}{6}(-1-\sqrt{137}), \frac{1}{6}(-1+\sqrt{137})]$.

The first order approximation is valid as long as all perturbing terms are small compared to the leading term. Since some of the exponents have negative real parts, Equation (21) will generally be meaningless for $t \to 0$. However, if

$$\begin{aligned} &b_1 = b_2 = b_3 = b_4 = 0 \quad \text{(triangular case)}\\ &b_1 = b_2 = b_3 = b_4 = a_4 = 0 \quad \text{(collinear cases)} \end{aligned} \tag{29}$$

the perturbation remains bounded with respect to the reference solution as $t \to 0$. Hence (29) represents the subfamily of *triple collision solutions*; it is described by the 4 parameters a_1, a_2, a_3, a_4 in the triangular case and by the 3 parameters a_1, a_2, a_3 in the collinear cases.

Here only first order approximations will be considered; the subfamily of collision solutions is near $t=0$ fully described by Siegel's convergent power series

$$y = ct^{2/3}P(a_1t^{2/3}, a_3t^{\alpha_3}, a_4t^{\alpha_4}), \tag{30}$$

where P is a Taylor series in all its arguments, with coefficients depending on the masses only. In the collinear cases the last argument must be omitted. A modification in the above series (logarithmic terms) is generally necessary if two of the exponents have an integer ratio (see Siegel, 1967). The parameter a_2 has been omitted from the series since this degree of freedom can be eliminated by rotating the coordinate system.

On the other hand, for $t \to \infty$ the perturbing terms with positive exponents exceed the contribution of the reference solution, and the linearized approximation is no longer valid either. However, in both of the subfamilies

$$\begin{aligned} &a_1 = a_3 = a_4 = 0 \qquad \text{(triangular case)}\\ &5 \text{ parameters: } b_1, a_2, b_2, b_3, b_4;\\ &a_1 = a_3 = 0 \qquad \text{(collinear cases)}\\ &6 \text{ parameters: } b_1, a_2, b_2, b_3, a_4, b_4 \end{aligned} \tag{31}$$

the perturbation remains bounded with respect to the reference even for $t \to \infty$. These families contain the *parabolic solutions*, which are characterized by

$$\mathbf{x}_j \to \infty, \quad \dot{\mathbf{x}}_j \to 0 \quad \text{as} \quad t \to \infty. \tag{32}$$

By means of the substitution $t=\tau^{-1}$ the situation of the parabolic motion as $t \to \infty$ is reduced to a situation analogous to triple collision as $\tau \to 0$. Hence the following *theorem* holds:

The parabolic solutions of the planar problem of three bodies have convergent power series about $t = \infty$ of the form

$$y = ct^{2/3}P(b_2 t^{-1/3}, b_3 t^{\beta_3}, b_4 t^{\beta_4}, a_4 t^{\alpha_5}), \tag{33}$$

where P is a Taylor series in all of its arguments, with coefficients depending on the masses only. In the triangular case the last argument is missing. A modification of the series is generally necessary if two of the exponents have an integer ratio.

The parameters a_2 and b_1 have been eliminated by an appropriate coordinate rotation and time shift. The parabolic solutions in the collinear cases with complex exponents $\alpha_4, \beta_4 (\kappa > \frac{9}{8})$ show an oscillatory behaviour since

$$\mathrm{Re}(a_4 t^{\alpha_4 + 2/3}) = \sqrt{t}\,|a_4| \cos\left(\omega \log \frac{t}{t_0}\right). \tag{34}$$

This term, however, is generally overshadowed by the leading term $O(t^{2/3})$. Only if this latter contribution vanishes, the oscillatory motion can be 'seen' directly. An example will be given in Section 4.

3. Matching

In general a solution near the reference (for instance given by initial conditions which nearly result in the reference solution) will neither be a pure collision solution nor a parabolic solution, but all the 8 perturbing terms of Equation (21) will be present.

We now consider the limit of a nearby solution *arbitrarily close* to the reference. This is done by introducing one single small parameter $\varepsilon \to 0$ and assuming all the parameters a_j, b_j to be of the exact order $O(\varepsilon)$, i.e. a_j, b_j are assumed to be proportional to ε. Now the solution (21) is subjected to the homothetic transformation (3):

$$\tilde{y} = c\tilde{t}^{2/3}\left\{\gamma_1 + \sum_{j=1}^{4} \gamma_1 (a_j \delta^{3\alpha_j} \tilde{t}^{\alpha_j} + b_j \delta^{3\beta_j} \tilde{t}^{\beta_j})\right\}, \tag{35}$$

hereby introducing new coordinates and time $\tilde{y}$, $\tilde{t}$. In the following limit process $\tilde{t}$ will be fixed. If in addition δ is fixed, while $\varepsilon \to 0$,

$$\tilde{y} \to c\tilde{t}^{2/3}\gamma_1,$$

i.e. the reference solution is obtained again. This is referred to as the *outer limit* (Cole 1968), whereas (21) corresponds to the intermediate solution in singular perturbation theory.

The *inner limit* is obtained by linking δ to ε:

$$\delta = \delta(\varepsilon), \quad \delta \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

It is sufficient to choose the order function $\delta(\varepsilon)$ as

$$\delta(\varepsilon) = O(\varepsilon^{\varrho}), \quad \varrho > 0. \tag{36}$$

Then, the coefficients of $\tilde{t}^{\alpha_j}$, $\tilde{t}^{\beta_j}$ in (35) are of the orders $O(\varepsilon^{1+3\alpha_j\varrho})$, $O(\varepsilon^{1+3\beta_j\varrho})$, respectively. The limit $\varepsilon \to 0$ exists if

$$1 + 3\alpha_j\varrho \geqslant 0, \quad 1 + 3\beta_j\varrho \geqslant 0 \quad (j = 1, 2, 3, 4),$$

and a 'distinguished limit' is obtained if the equality holds in at least one of these conditions; hence

$$\varrho = \frac{-\frac{1}{3}}{\min\limits_j (\alpha_j, \beta_j)}. \tag{37}$$

In order to simplify the matching procedure, we assume $b_3 \neq 0$, and the parameter b_1 is to be eliminated by an appropriate time shift; then we have

$$\varrho = -\frac{1}{3\beta_3} > 0. \tag{38}$$

By particularly choosing

$$\delta = |b_3|^{-1/3\beta_3}, \tag{39}$$

hence transforming

$$t = |b_3|^{-1/\beta_3}\tilde{t}, \tag{40}$$

we obtain

$$\lim_{\varepsilon \to 0} \tilde{y} = c\tilde{t}^{2/3}\{\gamma_1 + \operatorname{sign}(b_3)\gamma_3\tilde{t}^{\beta_3}\}. \tag{41}$$

For any fixed $t > 0$ the above limit implies $\tilde{t} \to \infty$ according to the time transformation (40). Consequently, Equation (41) describes the asymptotic behaviour of the inner solution for $\tilde{t} \to \infty$, which, according to (4), is a three-body motion with zero energy ($\tilde{h} = 0$) and zero angular momentum ($\tilde{\mathbf{C}} = 0$). The inner solution itself may be computed by backwards numerical integration of the differential Equations (9). The initial conditions are obtained by substituting a sufficiently large value of $\tilde{t}$ into (41).

Only the two cases corresponding to the two possible signs of b_3 have to be treated for each type of central configuration in order to know all possible inner solutions (i.e. all triple collisions between m_1, m_2, m_3). Only in a boundary layer near $b_3 = 0$ the situation is more complicated.

The final evolution of the inner solution for $\tilde{t} \to -\infty$ can be hyperbolic-elliptic (escape) or parabolic, according to Chazy's (1929) classification, the latter case being of measure zero.

Let $\tilde{\mathbf{v}}$ (independent of ε) be the asymptotic velocity of the escaping body in the inner solution. Then, by (4) and (39) the escape velocity immediately after the triple encounter is

$$\mathbf{v} = |b_3|^{1/3\beta_3} \cdot \tilde{\mathbf{v}}, \tag{42}$$

which is arbitrarily large if only $|b_3|$ is sufficiently small.

In order to achieve a good agreement between numerical examples and the theory, reference solutions with non-zero energy must be considered as well. Also, it may be necessary to extend matching to slightly higher orders. This will be done in a later paper.

4. The Equal Mass Case

If $m_1 = m_2 = m_3$, the equations of section 2 yield in the equilateral configuration ($\kappa = 0$):

$$\begin{aligned} \alpha_3 &= \alpha_4 = \alpha = \frac{-1 + \sqrt{13}}{6} = 0.43425\,85459 \\ \beta_3 &= \beta_4 = \beta = \frac{-1 - \sqrt{13}}{6} = -0.76759\,18792; \quad \alpha\beta = -\tfrac{1}{3} \end{aligned} \tag{43}$$

and in the collinear configurations ($\kappa = 2.4$):

$$\begin{aligned} &\left.\begin{matrix}\alpha_3\\ \beta_3\end{matrix}\right\} = \frac{-1 \pm \sqrt{47.4}}{6} = \left\{\begin{matrix}0.98079\,42985\\ -1.31412\,76319\end{matrix}\right. \\ &\left.\begin{matrix}\alpha_4\\ \beta_4\end{matrix}\right\} = \frac{-1 \pm i\sqrt{10.2}}{6}, \quad \omega = \sqrt{10.2}/6 = 0.53229\,06474 \end{aligned} \tag{44}$$

We first consider the equilateral case. Even in this case of coinciding eigenvalues, the matrix J_0 is diagonalizable since it has 4 linearly independent eigenvectors $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ (then Siegel's series exists in the form (30)). The vectors γ_3 and γ_4, however, are not uniquely determined, but span a 2-dimensional eigenplane. Every vector in this plane is an eigenvector of J_0. We choose

$$\begin{aligned} \gamma_1^T &= \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2}\right) \\ \gamma_2^T &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\ \gamma_3^T &= \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \\ \gamma_4^T &= \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}\right). \end{aligned} \tag{45}$$

Equation (21) now becomes

$$\begin{aligned} y = ct^{2/3}\{&\gamma_1(1 + a_1 t^{2/3}) + (a_3\gamma_3 + a_4\gamma_4)t^{\alpha} + \\ &+ \gamma_2 b_2 t^{-1/3} + (b_3\gamma_3 + b_4\gamma_4)t^{\beta}\}, \end{aligned} \tag{46}$$

where b_1 and a_2 have been eliminated by an appropriate time shift and coordinate rotation. For simplicity we put

$$b_3 = b\cos\varphi, \quad b_4 = b\sin\varphi \quad (b > 0); \tag{47}$$

then the transformation

$$y = \delta^2 \tilde{y}, \quad t = \delta^3 \tilde{t}, \quad \delta = b^{-1/3\beta} = b^{\alpha} \tag{48}$$

yields the following results from the inner limit process:

$$\lim_{\varepsilon \to 0} \tilde{y} = c\tilde{t}^{2/3}\{\gamma_1 + (\gamma_3 \cos \varphi + \gamma_4 \sin \varphi)\tilde{t}^{\beta}\}. \tag{49}$$

Due to the degeneracy of the eigenvalue problem, a one-parameter family (parameter φ) of inner solutions has been obtained.

In Figure 3 and Figure 4 the members $\varphi = 30°$ and $\varphi = 45°$ of this family are shown. In both cases m_1, m_2, m_3 approach on almost equilateral triangles, and m_3 escapes after the triple encounter with finite velocity, which becomes arbitrarily large in the outer variables. The numerical integration was done by regularizing all three binaries (Waldvogel, 1972).

The case $\varphi = 30°$ is exactly (up to notation) the one investigated by V. Szebehely (1974) by numerical integration of extremely close triple encounters. Figure 3 shows an infinitely close encounter in infinite magnification.

In the case $\varphi = 60°$ the body m_1 describes a rectilinear path and escapes after the triple encounter. Hence there exists an exceptional value φ_0, $45° < \varphi_0 < 60°$, such that the final evolution for $\tilde{t} \to -\infty$ is again parabolic. This *doubly parabolic* solution is shown in Figure 5 and Figure 6; the value

$$\varphi_0 = 0.995\,775 \tag{50}$$

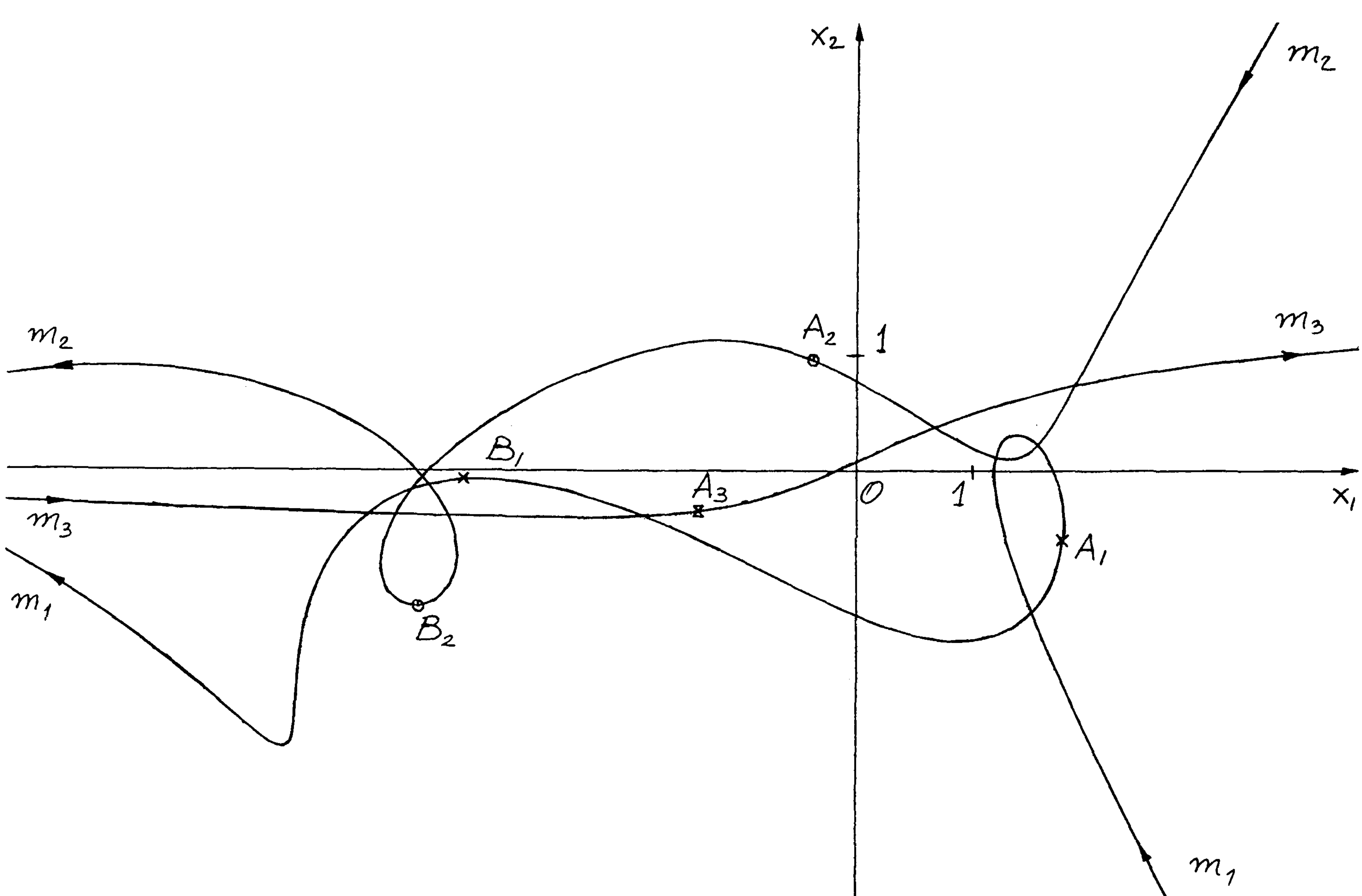

Fig. 3. Triangular parabolic solution in the equal mass case; $\varphi = 30°$; m_3 escapes. Example investigated by Szebehely (1975).

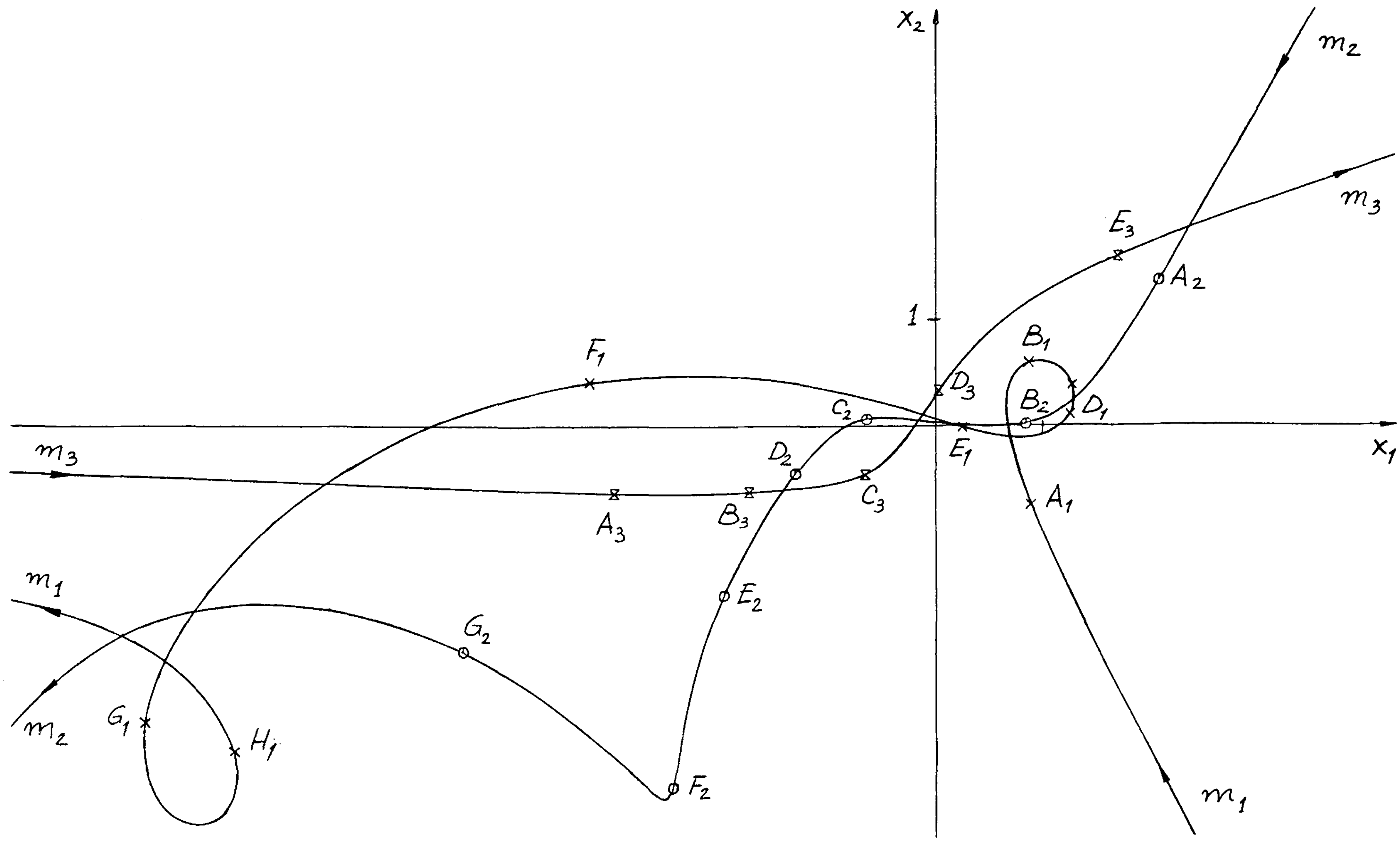

Fig. 4. Triangular parabolic solution in the equal mass case; $\varphi=45°$; m_3 escapes.

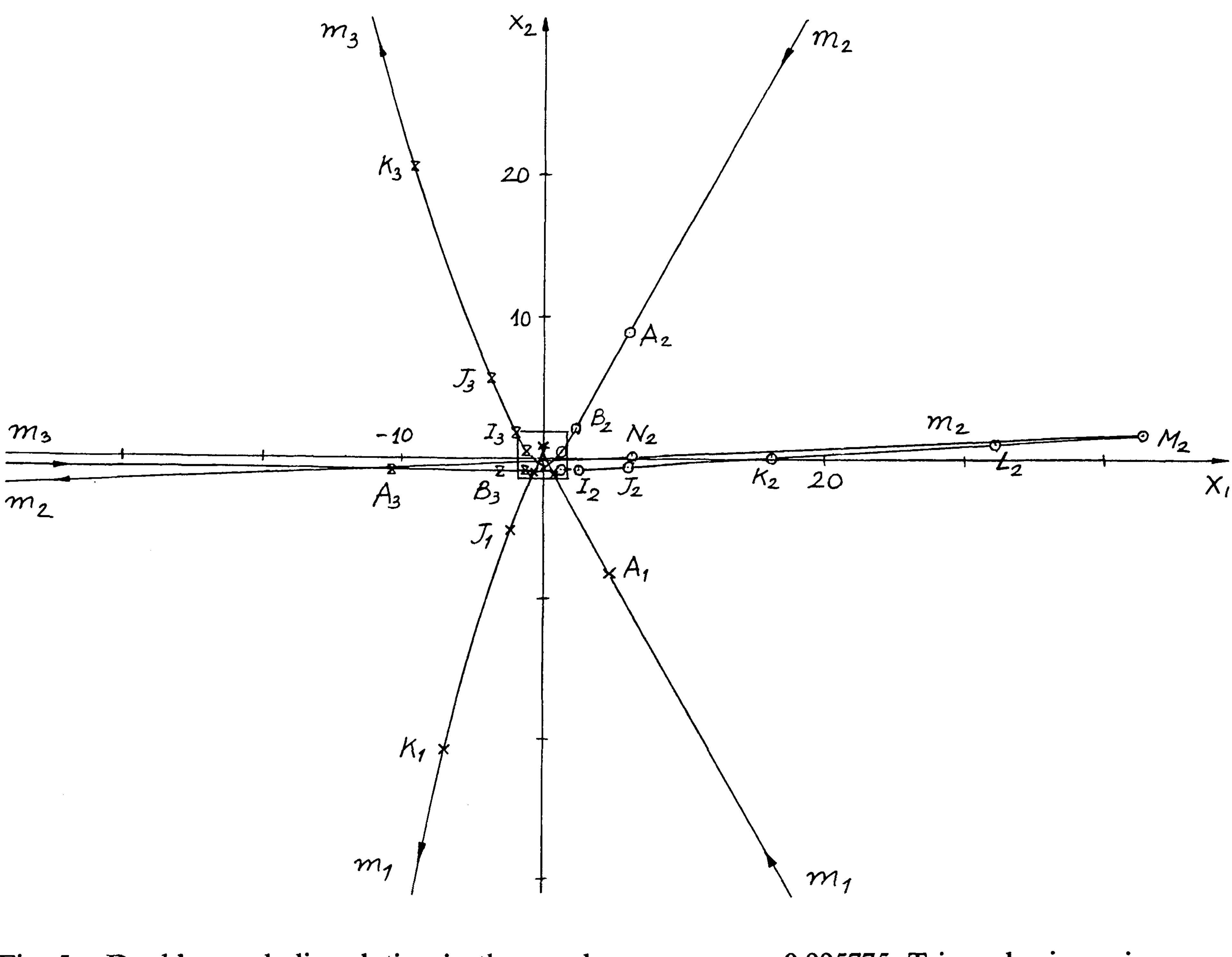

Fig. 5. Doubly parabolic solution in the equal mass case, $\varphi=0.995775$. Triangular incoming configuration, collinear outgoing configuration with m_2 as the inner mass.

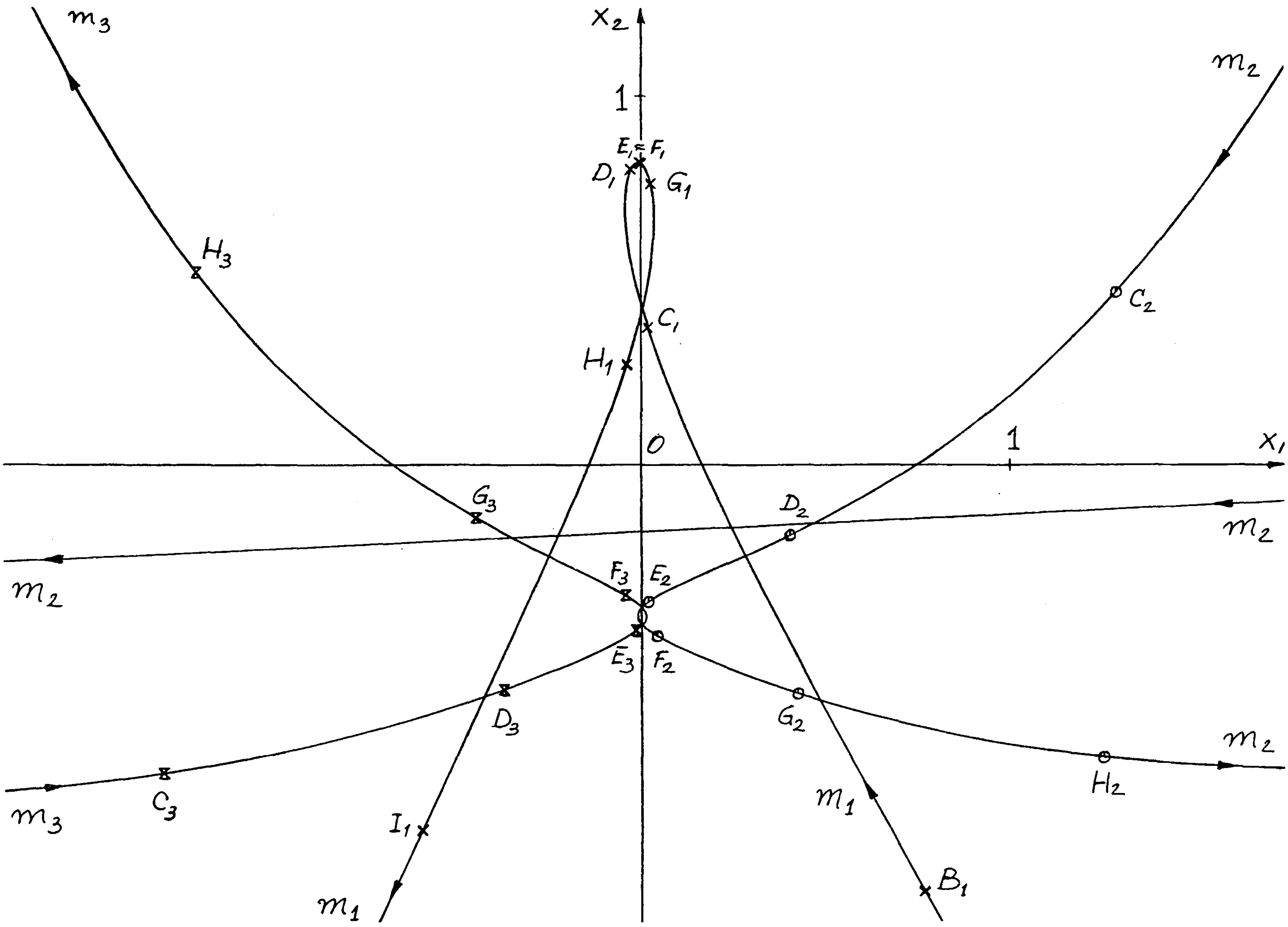

Fig. 6. Detail of Figure 5 in 25-fold magnification.

was found by repeated numerical integration to large negative values of $\tilde{t}$. Since the family of parabolic solutions has higher dimensionality in the collinear cases (six) than in the triangular case (five), the three bodies are expected to approach a collinear central configuration as $\tilde{t} \to -\infty$. In fact, m_2 appears to oscillate with increasing amplitude and increasing period around the center of mass, whereas m_1 and m_3 quickly evade to the bottom and to the top. The distance of these two bodies from the origin grows as $O(\tilde{t}^{2/3})$. In the expansion of the motion of m_2, however, the term $O(\tilde{t}^{2/3})$ is missing due to the symmetry of the central configuration. Hence the distance of m_2 from the origin grows as

$$O\left(\sqrt{-\tilde{t}} \cos\left(\omega \log \frac{\tilde{t}}{\tilde{t}_0}\right)\right), \quad \tilde{t} \to -\infty.$$

If the inner solution corresponding to a close triple encounter is doubly parabolic, the matching procedure can be applied for $\tilde{t} \to -\infty$ in the same way as for $\tilde{t} \to \infty$. Hence, after such an encounter a second outer solution becomes valid, and no immediate escape with high velocity takes place. Since this is true even for $\varepsilon \to 0$, the doubly parabolic solutions provide the *extension by Easton's* (1971) *method* of certain triple collision solutions. Beyond the collision the motion is a 'triple explosion' (triple collision in time reversal).

This extension is completely different from the real-analytic continuation of triple collision solutions that is possible for particular mass ratios (e.g., in the triangular case $m_1 = m_2 = 28$, $m_3 = 19$), where all the exponents are rational numbers with odd denominators. Even if analytic continuation is possible, however, it has no connection with the behaviour of nearby solutions, where the high velocities of the close encounter are retained afterwards.

On the other hand, Easton's extension becomes possible only when the collision solution is embedded into a family of close encounters such that the inner solution is doubly parabolic. In this family the extended solution is uniformly close to nearby solutions.

Acknowledgement

The author wishes to thank Mr Othmar Mäser for the careful preparation of the programs used for computing and plotting the solutions shown in Section 4.

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