THE EQUATIONS OF MOTION OF AN ARTIFICIAL SATELLITE IN NONSINGULAR VARIABLES

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Abstract. The equations of motion of an artificial satellite are given in nonsingular variables. Any term in the geopotential is considered as well as luni-solar perturbations up to an arbitrary power of r/r', r' being the geocentric distance of the disturbing body. Resonances with tesseral harmonics and with the Moon or Sun are also considered. By neglecting the shadow effect, the disturbing function for solar radiation is also developed in nonsingular variables for the long periodic perturbations. Formulas are developed for implementation of the theory in actual computations.

1. Definition of Nonsingular Elements

For $I \neq \pi$, $e < 1$, the following set of elements is nonsingular

$$\begin{aligned} a &= \text{semi major axis} \\ \lambda &= M + \omega + \Omega \\ \xi &= e \cos \tilde{\omega} \quad (\tilde{\omega} = \omega + \Omega) \\ \eta &= e \sin \tilde{\omega} \\ P &= \sin I/2 \cos \Omega \\ Q &= \sin I/2 \sin \Omega. \end{aligned} \tag{1.1}$$

Let $\gamma = \sqrt{1 - e^2}$, $c = \cos I/2$, $s = \sin I/2$. For this set of elements, Lagrange's planetary equations are

$$\begin{aligned} \dot{a} &= \frac{2}{na} R_\lambda \\ \dot{\lambda} &= n - \frac{2}{na} R_a + \frac{\gamma}{2na^2} (\xi R_\xi + \eta R_\eta) + \frac{1}{2na^2\gamma} (PR_P + QR_Q) \\ \dot{\xi} &= -\frac{\gamma}{na^2(1 + \gamma)} \xi R_\lambda - \frac{\gamma}{na^2} R_\eta - \frac{1}{2na^2\gamma} \eta(PR_P + QR_Q) \\ \dot{\eta} &= -\frac{\gamma}{na^2(1 + \gamma)} \eta R_\lambda + \frac{\gamma}{na^2} R_\xi + \frac{1}{2na^2\gamma} \xi(PR_P + QR_Q) \\ \dot{P} &= -\frac{1}{2na^2\gamma} PR_\lambda - \frac{1}{4na^2\gamma} R_Q + \frac{1}{2na^2\gamma} P(\eta R_\xi - \xi R_\eta) \\ \dot{Q} &= -\frac{1}{2na^2\gamma} QR_\lambda + \frac{1}{4na^2\gamma} R_p + \frac{1}{2na^2\gamma} Q(\eta R_\xi - \xi R_\eta). \end{aligned} \tag{1.2}$$

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If the perturbations are given by a disturbing function R and a disturbing (non-conservative) force $\mathbf{F}$, in the above equations we must consider

$$R_\alpha = \frac{\partial R}{\partial \alpha} + \mathbf{F}\cdot\frac{\partial \mathbf{r}}{\partial \alpha}, \tag{1.3}$$

where α is any element and $\mathbf{r}$ the radius vector. This form is necessary in order to take into account nonconservative perturbations like atmospheric drag, radiation pressure, etc. In rectangular inertial coordinates $\mathbf{F}=(X, Y, Z)$, $\mathbf{r}=(x, y, z)$ all we need are the derivatives of the Keplerian x, y, z with respect to the set of elements a, λ, ξ, η, P, Q.

Let $u=\omega+f$. Then we have

$$\begin{aligned}
&\frac{\partial x}{\partial a} = \frac{x}{a}, \qquad \frac{\partial y}{\partial a} = \frac{y}{a}, \qquad \frac{\partial z}{\partial a} = \frac{z}{a}\\
&\frac{\partial x}{\partial \lambda} = \frac{a}{\gamma}\left\{\frac{1}{r}\frac{\partial x}{\partial u} - \mathrm{e}\,(\sin\omega\cos\Omega + \cos\omega\sin\Omega\cos I)\right\}\\
&\frac{\partial y}{\partial \lambda} = \frac{a}{\gamma}\left\{\frac{1}{r}\frac{\partial y}{\partial u} - \mathrm{e}\,(\sin\omega\sin\Omega - \cos\omega\cos\Omega\cos I)\right\}\\
&\frac{\partial z}{\partial \lambda} = \frac{a}{\gamma}\left\{\frac{1}{r}\frac{\partial z}{\partial u} + \mathrm{e}\cos\omega\sin I\right\}\\
&\frac{\partial x}{\partial u} = -r\{\sin u\cos\Omega + \cos u\sin\Omega\cos I\}\\
&\frac{\partial y}{\partial u} = -r\{\sin u\sin\Omega - \cos u\cos\Omega\cos I\}\\
&\frac{\partial z}{\partial u} = r\cos u\sin I\\
&\frac{\partial x}{\partial \tilde{\omega}} = \frac{\partial x}{\partial u} - \frac{\partial x}{\partial \lambda}, \qquad \frac{\partial y}{\partial \tilde{\omega}} = \frac{\partial y}{\partial u} - \frac{\partial y}{\partial \lambda}, \qquad \frac{\partial z}{\partial \tilde{\omega}} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial \lambda}\\
&\frac{\partial x}{\partial \Omega} = -y - \frac{\partial x}{\partial u}, \qquad \frac{\partial y}{\partial \Omega} = x - \frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial \Omega} = -\frac{\partial z}{\partial u}\\
&\frac{\partial x}{\partial e} = \frac{\sin f}{\gamma^2}\frac{\partial x}{\partial u} - a\,(\cos\omega\cos\Omega - \sin\omega\sin\Omega\cos I)\\
&\frac{\partial y}{\partial e} = \frac{\sin f}{\gamma^2}\frac{\partial y}{\partial u} - a\,(\cos\omega\sin\Omega + \sin\omega\cos\Omega\cos I)\\
&\frac{\partial z}{\partial e} = \frac{\sin f}{\gamma^2}\frac{\partial z}{\partial u} - a\sin\omega\sin I\\
&\frac{\partial x}{\partial I} = r\sin u\sin\Omega\sin I\\
&\frac{\partial y}{\partial I} = -r\sin u\cos\Omega\sin I\\
&\frac{\partial z}{\partial I} = r\sin u\cos I.
\end{aligned} \tag{1.4}$$

The above relations are valid if x, y, z are expressed in terms of $(a, \lambda, \tilde{\omega}, e, \Omega, I)$. The derivatives with respect to ξ, η, P, Q are easily found to be

$$\begin{aligned}
\frac{\partial x}{\partial \xi} &= \frac{\partial x}{\partial e}\cos\tilde{\omega} - \frac{\partial x}{\partial \tilde{\omega}}\frac{1}{e}\sin\tilde{\omega} \\
\frac{\partial x}{\partial \eta} &= \frac{\partial x}{\partial e}\sin\tilde{\omega} + \frac{\partial x}{\partial \tilde{\omega}}\frac{1}{e}\cos\tilde{\omega} \\
\frac{\partial x}{\partial P} &= 2\frac{\partial x}{\partial I}\frac{1}{c}\cos\Omega - \frac{\partial x}{\partial \Omega}\frac{1}{s}\sin\Omega \\
\frac{\partial x}{\partial Q} &= 2\frac{\partial x}{\partial I}\frac{1}{c}\sin\Omega + \frac{\partial x}{\partial \Omega}\frac{1}{s}\cos\Omega
\end{aligned} \tag{1.5}$$

and analogous for y, z.

2. Development of Geopotential in Nonsingular Variables

We propose a form suitable for both numerical or analytical solution and necessary to recognize possible resonances occurring with tesseral harmonics.

We begin considering the usual form of representation

$$R = \sum_{l\geqslant 2}\sum_{m=0}^{l}\sum_{p=0}^{l}\sum_{q} R_{lmpq}, \tag{2.1}$$

where

$$R_{lmpq} = \frac{\mu a_e^l}{a^{l+1}} F_{lmp}(I) G_{lpq}(e)(A_{lm}\cos\psi_{lmpq} + B_{lm}\sin\psi_{lmpq}). \tag{2.2}$$

In the above expression

$$\begin{aligned}
\mu &= Gm_{\oplus}, \\
a_e &= \text{mean equatorial radius of } \oplus \\
F_{lmp}(I) &= \text{inclination functions} \quad \text{(Allan, 1965)}
\end{aligned} \tag{2.3}$$

$$G_{lpq}(e) = X_{l-2p+q}^{-l-1,\, l-2p} \quad \text{(Plummer, 1960)} \tag{2.4}$$

$$\begin{aligned}
A_{lm} &= \begin{cases} C_{lm}, & l-m \text{ even} \\ -S_{lm}, & l-m \text{ odd} \end{cases} \\
B_{lm} &= \begin{cases} S_{lm}, & l-m \text{ even} \\ C_{lm}, & l-m \text{ odd} \end{cases}
\end{aligned} \tag{2.5}$$

and

$$\psi_{lmpq} = (l-2p+q)\lambda - q\tilde{\omega} + (m+2p-l)\Omega - m\theta, \tag{2.6}$$

where θ is Greenwich sidereal time.

The exact D'Alembert characteristics are

$$F_{lmp} = s^{|\alpha|} J_{lmp}(c) \tag{2.7}$$

$$G_{lpq} = e^{|q|} K_{lpq}(\gamma), \tag{2.8}$$

where J_{lmp} is a polynomial in $c=\cos I/2$, K_{lpq} a power series in $\gamma=\sqrt{1-e^2}$, and $\alpha=m+2p-l$. For $q=2p-l$, K_{lpq} are expressible in closed form as functions of γ. We shall give expressions for J_{lmp} and K_{lpq} in due time. A particular term of R can thus be written in the following form

$$R_{lmpq} = \frac{\mu a_e^l}{a^{l+1}} J_{lmp}(c) K_{lpq}(\gamma) s^{|\alpha|} e^{|q|} (A_{lm} \cos \psi_{lmpq} + B_{lm} \sin \psi_{lmpq}). \tag{2.9}$$

We begin by considering

$$q > 0, \qquad \alpha = m + 2p - l > 0$$

so that we have

$$e^q \exp(iq\tilde{\omega}) = (\xi + i\eta)^q$$
$$e^q \exp(-iq\tilde{\omega}) = (\xi - i\eta)^q$$
$$s^\alpha \exp(i\alpha\Omega) = (P + iQ)^\alpha$$
$$s^\alpha \exp(-i\alpha\Omega) = (P - iQ)^\alpha.$$

Let us consider the expression

$$s^{|\alpha|} e^{|q|} \cos \psi_{lmpq}$$

which can be written as

$$\tfrac{1}{2} e^{|q|-q} s^{|\alpha|-\alpha} \{[(\xi + i\eta)^q (P - iQ)^\alpha + (\xi - i\eta)^q (P + iQ)^\alpha] \cos \theta_{lmpq} -$$
$$- i[(\xi + i\eta)^q (P - iQ)^\alpha - (\xi - i\eta)^q (P + iQ)^\alpha] \sin \theta_{lmpq}\},$$

where

$$\theta_{lmpq} = (l - 2p + q)\lambda - m\theta. \tag{2.10}$$

Obviously for $q>0$, $\alpha>0$, the exponents of e and s vanish, but we keep them in order to treat the case $q<0$, $\alpha<0$.

Let now

$$\mathbb{R}_{lmpq} = \text{Real}\,\{(\xi + i\eta)^q (P - iQ)^\alpha\}$$

$$\mathbb{I}_{lmpq} = \text{Imag}\,\{(\xi + i\eta)^q (P - IQ)^\alpha\} \tag{2.11}$$

so that

$$e^{|q|} s^{|\alpha|} \cos \psi_{lmpq} = \mathbb{R}_{lmpq} \cos \theta_{lmpq} + \mathbb{I}_{lmpq} \sin \theta_{lmpq}. \tag{2.12}$$

We now evaluate the expressions for $\mathbb{R}$ and $\mathbb{I}$. We have that

$$(\xi + i\eta)^q (P - iQ)^\alpha = \sum_{n=0}^{k} \sum_{u=u_1}^{u_2} (-1)^{n+u} \binom{q}{u} \binom{\alpha}{2n-u} \xi^{q-u} \eta^u P^{\alpha-2n+u} Q^{2n-u} +$$
$$+ i \sum_{n=0}^{k'} \sum_{u=u_1'}^{u_2'} (-1)^{n+u+1} \binom{q}{u} \binom{\alpha}{2n+1-u} \xi^{q-u} \eta^u P^{\alpha-2n-1+u} Q^{2n+1-u},$$

where

$$k = \left[\frac{q+\alpha}{2}\right], \qquad k' = \left[\frac{q+\alpha-1}{2}\right],$$
$$u_1 = \max(0, 2n-\alpha), \qquad u_2 = \min(2n, q) \tag{2.13}$$
$$u_1' = \max(0, 2n+1-\alpha), \qquad u_2' = \min(2n+1, q).$$

Thus

$$\mathbb{R}_{lmpq} = \sum_{n=0}^{k} \sum_{u=u_1}^{u_2} (-1)^{n+u} \binom{q}{u}\binom{\alpha}{2n-u} \xi^{q-u} \eta^u P^{\alpha-2n+u} Q^{2n-u} \tag{2.14}$$

and

$$\mathbb{I}_{lmpq} = \sum_{n=0}^{k'} \sum_{u=u_1'}^{u_2'} (-1)^{n+u+1} \binom{q}{u}\binom{\alpha}{2n+1-u} \xi^{q-u} \eta^u P^{\alpha-2n-1+u} Q^{2n+1-u}. \tag{2.15}$$

By a shift of $-\pi/2$ in the angle ψ_{lmpq} we also find

$$c^{|q|} s^{|\alpha|} \sin \psi_{lmpq} = \mathbb{R}_{lmpq} \sin \theta_{lmpq} - \mathbb{I}_{lmpq} \cos \theta_{lmpq}.$$

Thus, for $q>0$, $\alpha=m+2p-l>0$, we have

$$\begin{aligned} \mathbb{R}_{lmpq} = {} & \frac{\mu a_e^l}{a^{l+1}} J_{lmpq}(c) K_{lpq}(\gamma) e^{|q|-q} s^{|\alpha|-\alpha} \times \\ & \times \{\mathbb{R}_{lmpq}(A_{lm} \cos \theta_{lmpq} + B_{lm} \sin \theta_{lmpq}) + \\ & + \mathbb{I}_{lmpq}(A_{lm} \sin \theta_{lmpq} - B_{lm} \cos \theta_{lmpq})\}. \end{aligned} \tag{2.16}$$

Consider now $q<0$. We may write

$$\begin{aligned} (\xi + i\eta)^q = (\xi + i\eta)^{-|q|} = (\xi^2 + \eta^2)^{-|q|}(\xi - i\eta)^{|q|} = \\ = e^{-2|q|}(\xi - i\eta)^{|q|}. \end{aligned}$$

Thus for $q<0$, in the definition of $\mathbb{R}_{lmpq}$ and $\mathbb{I}_{lmpq}$ we must introduce the change

$$\begin{aligned} &\xi \to \xi/e^2 \\ &\eta \to -\eta/e^2 \\ &q \to -q = |q|. \end{aligned}$$

The corresponding factor in R_{lmpq} is

$$e^{|q|-q} = e^{2|q|}$$

so that, the above changes introducing a factor $e^{-2|q|}$, that factor disappears. Therefor we can leave out the factor $e^{|q|-q}$ for both $q>0$ and $q<0$ defining for $q<0$ the new $\mathbb{R}_{lmpq}$ and $\mathbb{I}_{lmpq}$ by the changes

$$\begin{aligned} &\xi \to \xi \\ &\eta \to -\eta \\ &q \to |q| = -q. \end{aligned} \tag{2.17}$$

A similar reasoning applies for $\alpha < 0$. We see therefore that the expressions valid for any q and α are

$$\mathbb{R}_{lmpq} = \sum_{n=0}^{k} \sum_{u=u_1}^{u_2} (-1)^{n+u} \delta_u \binom{|q|}{u} \binom{|\alpha|}{2n-u} \xi^{|q|-u} \eta^u P^{|\alpha|-2n+u} Q^{2n-u} \tag{2.18}$$

$$\mathbb{I}_{lmpq} = \sum_{n=0}^{k'} \sum_{u=u_1'}^{u_2'} (-1)^{n+u+1} \delta_u \binom{|q|}{u} \binom{|\alpha|}{2n+1-u} \xi^{|q|-u} \eta^u P^{|\alpha|-2n-1+u} Q^{2n+1-u}, \tag{2.19}$$

where

$$k = \left[\frac{|q| + |\alpha|}{2}\right], \qquad k' = \left[\frac{|q| + |\alpha| - 1}{2}\right] \tag{2.20}$$

$$\begin{aligned} u_1 &= \max(0, 2n - |\alpha|), \qquad u_2 = \min(2n, |q|) \\ u_1' &= \max(0, 2n + 1 - |\alpha|), \qquad u_2' = \min(2n + 1, |q|) \\ \delta_u &= 1, \quad \text{if } q, \alpha \text{ are both positive or negative} \\ \delta_u &= (-1)^u \text{ if } q \text{ or } \alpha \text{ is negative.} \end{aligned} \tag{2.21}$$

The final result is

$$\begin{aligned} R_{lmpq} = \frac{\mu a_e^l}{a^{l+1}} J_{lmp}(c) K_{lpq}(\gamma) \{ \mathbb{R}_{lmpq}(A_{lm} \cos \theta_{lmpq} + B_{lm} \sin \theta_{lmpq}) + \\ + \mathbb{I}_{lmpq}(A_{lm} \sin \theta_{lmpq} - B_{lm} \cos \theta_{lmpq}) \}. \end{aligned} \tag{2.22}$$

3. Elimination of Short Periodic Terms

(a) No resonance with tesseral harmonics.

In this case, the short periodic terms are eliminated by setting the coefficient of λ equal to zero, that is

$$q = 2p - l \tag{3.1}$$

so that

$$\begin{aligned} \theta_{lmpq} &= -m\theta \\ k &= \left[\frac{|2p - l| + |\alpha|}{2}\right] \\ k' &= \left[\frac{|2p - l| + |\alpha| - 1}{2}\right] \\ u_1 &= \max(0, 2n - |\alpha|), \qquad u_2 = \min(2n, |2p - l|) \\ u_1' &= \max(0, 2n + 1 - |\alpha|), \qquad u_2' = \min(2n + 1, |2p - l|). \end{aligned} \tag{3.2}$$

(b) Resonance with tesseral harmonics. We consider two types:

(b.1) $\quad s(\dot{M} + \dot{\omega}) = r(\dot{\Omega} - \dot{\theta})$ (Nodal resonance), $\qquad$ (3.3)

where s, r are mutually prime integers.

In this case the short periodic terms are eliminated by retaining only integers l, p, m satisfying the conditions

$$\begin{aligned} &0 \leqslant p \leqslant l \\ &l - 2p = js \quad (j = \text{integer}) \\ &1 \leqslant m = jr \leqslant l \end{aligned} \tag{3.4}$$

so that

$$\begin{aligned} &\theta_{lmpq} = (js + q)\lambda - jr\theta \\ &|\alpha| = |j(r - s)| \\ &k = \left[\frac{|q| + |j(r - s)|}{2}\right], \qquad k' = \left[\frac{|q| + |j(r - s)| - 1}{2}\right] \\ &u_1 = \max(0, 2n - |j(r - s)|), \qquad u_2 = \min(2n, |js|) \\ &u_1' = \max(0, 2n + 1 - |j(r - s)|), \qquad u_2' = \min(2n + 1, |js|). \end{aligned} \tag{3.5}$$

(b.2) $\quad s\dot{\lambda} = r\dot{\theta} \quad$ (longitude resonance) (3.6)

where s, r are mutually prime integers. The short periodic terms are eliminated by retaining only integers l, p, m satisfying the conditions

$$\begin{aligned} &0 \leqslant p \leqslant l \\ &2p - l = q - js \quad (j = \text{integer}) \\ &1 \leqslant m = jr \leqslant l \end{aligned} \tag{3.7}$$

so that

$$\begin{aligned} &\theta_{lmpq} = j(s\lambda - r\theta) \\ &|\alpha| = |q + j(r - s)|. \end{aligned} \tag{3.8}$$

In both cases $|q|$ is a free integer indicating the maximum power of e retained in R_{lmpq}.

4. Development of Luni-Solar Potential in Nonsingular Variables

The disturbing function due to the Moon or Sun can be written as

$$R' = \beta' n'^2 r^2 \left(\frac{a'}{r'}\right)^3 \sum_{l \geqslant 2} \left(\frac{r}{r'}\right)^{l-2} P_l(\cos\psi'), \tag{4.1}$$

where

$\beta' = \dfrac{m'}{m' + m_\oplus}$, $\quad m'$ = mass of disturbing body,

n' = mean motion in longitude of the disturbing body,

a' = mean distance of disturbing body from Earth,

ψ' = geocentric elongation of the satellite from the perturbing body,

r' = geocentric distance of perturbing body.

Using equatorial coordinates (α, δ) and (α', δ') for the satellite and the perturbing body, we have

$$\cos \psi' = \sin \delta \sin \delta' + \cos \delta \cos \delta' \cos (\alpha - \alpha')$$

and

$$P_l (\cos \psi') = \sum_{m=0}^{l} \varepsilon_m \frac{(l-m)!}{(l+m)!} P_{lm} (\sin \delta) P_{lm} (\sin \delta') \cos m(\alpha - \alpha'),$$

where $\varepsilon_0 = 1$, $\varepsilon_m = 2$ for $m \neq 0$.

We define the harmonic coefficients

$$C'_{lm} = \left\{ \frac{\beta' n'^2}{a'^{l-2}} \left(\frac{a'}{r'}\right)^{l+1} \varepsilon_m \frac{(l-m)!}{(l+m)!} P_{lm} (\sin \delta') \right\} \cos m\alpha' \tag{4.2}$$

$$S'_{lm} = \left\{ \frac{\beta' n'^2}{a'^{l-2}} \left(\frac{a'}{r'}\right) \varepsilon_m \frac{(l-m)!}{(l+m)!} P_{lm} (\sin \delta') \right\} \sin m\alpha'$$

so that

$$R' = \sum_{l \geq 2} \sum_{m=0}^{l} R'_{lm}, \tag{4.3}$$

where

$$R'_{lm} = a^l \left(\frac{r}{a}\right)^l P_{lm} (\sin \delta) \{C'_{lm} \cos m\alpha + S'_{lm} \sin m\alpha\}. \tag{4.4}$$

By the usual transformation to orbital coordinates we have that

$$P_{lm} (\sin \delta) \{C'_{lm} \cos m\alpha + S'_{lm} \sin m\alpha\} =$$

$$= \sum_{p=0}^{l} F_{lmp}(I) \{A'_{lm} \cos \psi_{lmp} + B'_{lm} \sin \psi_{lmp}\}, \tag{4.5}$$

where

$$\psi_{lmp} = (l - 2p)(\omega + f) + m\Omega \tag{4.6}$$

and for

$$l - m \text{ even:} \quad A'_{lm} = C'_{lm}, \quad B'_{lm} = S'_{lm} \tag{4.7}$$

$$l - m \text{ odd:} \quad A'_{lm} = -S'_{lm}, \quad B'_{lm} = C'_{lm}$$

so that

$$R' = \sum_{l \geq 2} \sum_{m=0}^{l} \sum_{p=0}^{l} R'_{lmp}, \tag{4.8}$$

where

$$R'_{lmp} = a^l \left(\frac{a}{r}\right)^l F_{lmp}(I) \{A'_{lm} \cos \psi_{lmp} + B'_{lm} \sin \psi_{lmp}\}. \tag{4.9}$$

Using Hansen's coefficients $H_{lpq} = X^{l,\, l-2p}_{l-2p+q}$, we have that

$$\left(\frac{r}{a}\right)^l \begin{matrix}\cos\\ \sin\end{matrix}\, \psi_{lmp} = \sum_{q} H_{lpq}(e) \begin{matrix}\cos\\ \sin\end{matrix}\, \psi_{lmpq}, \tag{4.10}$$

where

$$\psi_{lmpq} = (l - 2p + q)\lambda - q\tilde{\omega} + (m + 2p - l)\Omega \tag{4.11}$$

and

$$H_{lpq} = e^{|q|}L_{lpq}(\gamma). \tag{4.12}$$

The functions L_{lpq} are power series in $\gamma=\sqrt{1-e^2}$ or, in case $q=2p-l$, they can be written in closed form in terms of γ. They will be given later in this work. We finally write

$$R' = \sum_{l\geq 2}\sum_{m=0}^{l}\sum_{p=0}^{l}\sum_{q} R'_{lmpq}, \tag{4.13}$$

where

$$R'_{lmpq} = a^l F_{lmp}(I)e^{|q|}L_{lpq}\{A'_{lm}\cos\psi_{lmpq} + B'_{lm}\sin\psi_{lmpq}\}. \tag{4.14}$$

Proceeding as in Section 2, in terms of the nonsingular variables, one finds

$$\begin{aligned} R'_{lmpq} = {} & a^l J_{lmp}(c)L_{lpq}(\gamma)\{\mathbb{R}_{lmpq}[A'_{lm}\cos\phi_{lpq} + B'_{lm}\sin\phi_{lpq}] + \\ & + \mathbb{I}_{lmpq}[A'_{lm}\sin\phi_{lpq} - B'_{lm}\cos\phi_{lpq}]\}, \end{aligned} \tag{4.15}$$

where

$$\phi_{lpq} = (l - 2p + q)\lambda. \tag{4.16}$$

If there is no resonance with the Moon or Sun, the short periodic terms are easily eliminated by setting $q=2p-l$. In this case the best form of the disturbing function for a numerical integration approach is given by

$$R'_{lmp(2p-l)} = a^l J_{lmp}L_{lp(2p-l)}\{\mathbb{R}_{lmp(2p-l)}A'_{lm} - \mathbb{I}_{lmp(2p-l)}B'_{lm}\}, \tag{4.17}$$

where A'_{lm}, B'_{lm} depend solely on the coordinates of the perturbing body as defined by Equation (4.7). In case of longitude resonance with the perturbing body consider the expansions

$$P_{lm}(\sin\delta')\cos m\alpha' = \sum_{p'=0}^{l} F_{lmp'}(I')\{A''_{lm}\cos\theta_{lmp'} + B''_{lm}\sin\theta_{lmp'}\}$$

$$P_{lm}(\sin\delta')\sin m\alpha' = \sum_{p'=0}^{l} F_{lmp}(I')\{-B''_{lm}\cos\theta_{lmp'} + A''_{lm}\sin\theta_{lmp'}\},$$

where

$$\theta_{lmp'} = (l - 2p')(\omega' + f') + m\Omega'. \tag{4.18}$$

and for

$$l - m \text{ even:} \quad A''_{lm} = 1, \qquad B''_{lm} = 0$$

$$l - m \text{ odd:} \quad A''_{lm} = 0, \qquad B''_{lm} = 1.$$

Therefore we find

$$R'_{lmpq} = \sum_{p'=0}^{l} R'_{lmpqp'}, \tag{4.19}$$

where

$$R'_{lmpqp'} = \beta' n'^2 \left(\frac{a'}{r'}\right)^{l+1} \frac{\varepsilon_m}{a'^{l-2}} \frac{(l-m)!}{(l+m)!} a^l J_{lmp} L_{lpq} \cdot$$
$$\cdot \{\mathbb{R}_{lmpq} \cos \phi_{lpqp'} + \mathbb{I}_{lmpq} \sin \phi_{lpqp'}\}, \tag{4.20}$$

where

$$\phi_{lmpqp'} = (l - 2p + q)\lambda - (l - 2p')(\omega' + f') - m\Omega'. \tag{4.21}$$

We now consider the expression

$$\left(\frac{a'}{r'}\right)^{l+1} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (l - 2p')f' = \sum_{q'} G_{lp'q'}(e') \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (l - 2p' + q')M' \tag{4.22}$$

so that, finally

$$R'_{lmpqp'q'} = \beta' n'^2 \frac{a^l}{a'^{l-2}} \varepsilon_m \frac{(l-m)!}{(l+m)!} J_{lmp}(c) L_{lpq}(\gamma) F_{lmp'}(I') \times$$
$$\times\; G_{lp'q'}(e') \{\mathbb{R}_{lmpq} \cos \phi_{lmpqp'q'} + \mathbb{I}_{lmpq} \sin \phi_{lmpqp'q'}\}, \tag{4.23}$$

where

$$\phi_{lmpqp'q'} = (l - 2p + q)\lambda - (l - 2p' + q')\lambda' + q'\tilde{\omega}' -$$
$$- (m + 2p' - l)\Omega' \tag{4.24}$$

and

$$R' = \sum_{l \geq 2} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{p'=0}^{l} \sum_{q} \sum_{q'} R'_{lmpqp'q'}. \tag{4.25}$$

The order of magnitude of any term $R_{lmpqp'q'}$ is given by

$$n^2 a^2 \beta' \left(\frac{n'}{n}\right)^2 \left(\frac{a}{a'}\right)^{l-2} \frac{(l-m)!}{(l+m)!} e^{|q|} e'^{|q'|} \left(\sin \frac{I}{2}\right)^{|m+2p-l|} \left(\sin \frac{I'}{2}\right)^{|m+2p'-l|}. \tag{4.26}$$

For a term R_{lmpq} in the geopotential, it is

$$n^2 a^2 \left(\frac{a_e}{a}\right)^l \sqrt{C_{lm}^2 + S_{lm}^2} e^{|q|} \left(\sin \frac{I}{2}\right)^{|m+2p-l|}. \tag{4.27}$$

5. Elimination of Short Periodic Terms

If no resonance occurs with the Moon or Sun, short periodic terms are eliminated by setting in $R'_{lmpqp'q'}$

$$q = 2p - l. \tag{5.1}$$

If resonance in longitude occurs of the type

$$s\dot{\lambda} = r\dot{\lambda}' \qquad (r, s \text{ integers}) \tag{5.2}$$

then short periodic terms are eliminated by retaining only integers l, p, q, p', q' satisfying the relation

$$(l - 2p + q)r = (l - 2p' + q')s. \tag{5.3}$$

6. The Inclination Functions F_{lmp} and J_{lmp}

They can be defined by

$$F_{lmp} = \sum_{j=j_1}^{j_2} F_{lmp}^j c^{3l-m-2p-2j} s^{m-l+2p+2j}, \tag{6.1}$$

where

$$F_{lmp}^j = (-1)^k \frac{(l+m)!}{2^l p!(l-p)!} (-1)^j \binom{2l-2p}{j} \binom{2p}{l-m-j} \tag{6.2}$$

$$k = \text{integral part of } \left[\frac{l-m}{2}\right],$$

$$j_1 = \max(0, -\alpha),$$

$$j_2 = \min(2l - 2p, l - m),$$

$$\alpha = m - l + 2p,$$

$$s = \sin\frac{I}{2}, \qquad c = \cos\frac{I}{2}.$$

Following definition (2.7) and noting that the exponent of c in (6.1) is $2l - \alpha - 2j$, we obtain

$$J_{lmp} = \sum_{j=j_1}^{j_2} F_{lmp}^j c^{2l-\alpha-2j} s^{\alpha-|\alpha|+2j} \tag{6.3}$$

or

$$J_{lmp} = \sum_{j=j_1}^{j_2} F_{lmp}^j c^{2l-\alpha-2j} (1 - c^2)^{j+(\alpha-|\alpha|)/2)}. \tag{6.4}$$

Note that J_{lmp} is a polynomial in $c = \cos I/2$, of degree $2l - |\alpha| \leqslant 2l$.

Recurrence relations for the computation of these functions will be discussed in a separate paper.

The partial derivatives necessary in the integration of Lagrange's equations are

$$\frac{\partial J_{lmp}}{\partial P} = \frac{\partial J_{lmp}}{\partial c}\frac{\partial c}{\partial P} = -2P\frac{\partial J_{lmp}}{\partial c} \tag{6.5}$$

$$\frac{\partial J_{lmp}}{\partial Q} = -2Q\frac{\partial J_{lmp}}{\partial c}. \tag{6.6}$$

If

$$J_{lmp}^j = (-1)^k \frac{(l+m)!}{2^l p!(l-p)!} F_{lmp}^j \tag{6.7}$$

then

$$J_{lmp} = \sum_{j=j_1}^{j_2} J_{lmp}^{j} c^{2l-\alpha-2j}(1 - c^2)^j \tag{6.8}$$

and

$$\frac{\partial J_{lmp}}{\partial c} = \sum_{j=j_1}^{j_2} J_{lmp}^{j} c^{2l-\alpha-2j-1}[(2l - \alpha)s^{2j} - 2js^{2j-2}]. \tag{6.9}$$

7. The Eccentricity Functions G_{lpq} and K_{lpq}

We begin considering the definition of Hansen's coefficients (Plummer, 1960)

$$\left(\frac{r}{a}\right)^n \exp(imf) = \sum_j X_j^{n,m} \exp(ijM), \tag{7.1}$$

where, for $j>m$:

$$X_j^{n,m} = (-e)^{|j-m|}2^{-n-1}(1 + \gamma)^{n+1-|j-m|} \sum_{k=0}^{\infty} \sum_{r=0}^{j-m+k} \sum_{t=0}^{k} \frac{(-1)^r}{r!t!} \times$$
$$\times \binom{n - m + 1}{j - m + k - r}\binom{n + m + 1}{k - t}\left(\frac{j}{2}\right)^{r+t}(1 + \gamma)^{r+t-k}(1 - \gamma)^k \tag{7.2}$$

and, for $j<m$:

$$X_j^{n,m} = (-e)^{|j-m|}2^{-n-1}(1 + \gamma)^{n+1-|j-m|} \sum_{k=0}^{\infty} \sum_{r=0}^{-j+m+k} \sum_{t=0}^{k} \frac{(-1)^t}{r!t!} \times$$
$$\times \binom{n + m + 1}{m - j + k - r}\binom{n - m + 1}{k - t}\left(\frac{j}{2}\right)^{r+t}(1 + \gamma)^{r+t-k}(1 - \gamma)^k, \tag{7.3}$$

where

$$\gamma = \sqrt{1 - e^2}. \tag{7.4}$$

The definition of the G_{lpq} functions is

$$\left(\frac{a}{r}\right)^{l+1} \exp[i(l - 2p)f] = \sum_q G_{lpq} \exp[i(l - 2p + q)M] \tag{7.5}$$

so that

$$G_{lpq} = X_{l-2p+q}^{-l-1,\,l-2p} \tag{7.6}$$

for $l=-n-1$, $2p=-n-m-1$, $q=j-m$, in Equation (7.1), or $n=-l-1$, $m=l-2p$, $j=l-2p+q=m+q$. We see therefore that G_{lpq} is factored by $e^{|q|}$ and the remaining factor (K_{lpq}) is a power series in $\gamma=\sqrt{1-e^2}$.

Making use of the expressions for Hansen's coefficients we find

$$G_{lpq} = e^{|q|}K_{lpq}, \tag{7.7}$$

where, for $q>0$:

$$K_{lpq} = (-1)^{|q|}2^l(1+\gamma)^{-l-|q|}\sum_{k=0}^{\infty}\sum_{r=0}^{|q|+k}\sum_{t=0}^{k}\frac{(-1)^r}{r!t!}\binom{2p-2l}{|q|+k-r}\times$$
$$\times\binom{-2p}{k-t}\left(\frac{l-2p+q}{2}\right)^{r+t}(1+\gamma)^{r+t-k}(1-\gamma)^k \tag{7.8}$$

and, for $q>0$:

$$K_{lpq} = (-1)^{|q|}2^l(1+\gamma)^{-l-|q|}\sum_{k=0}^{\infty}\sum_{r=0}^{|q|+k}\sum_{t=0}^{k}\frac{(-1)^t}{r!t!}\binom{-2p}{|q|+k-r}\times$$
$$\times\binom{2p-2l}{k-t}\left(\frac{l-2p+q}{2}\right)^{r+t}(1+\gamma)^{r+t-k}(1-\gamma)^k. \tag{7.9}$$

For $j=0$, $n<0$ and $n+m$ odd, we have:

$$X_0^{n,m} = e^{|j-m|}\gamma^{2n+3}\sum_{k=0}^{k'-1}\binom{-n-2}{2k+|m|}\binom{2k+|m|}{k}\times$$
$$\times 2^{-2k-|m|}(1-\gamma^2)^k, \qquad k' = \frac{-n-1-|m|}{2} \tag{7.10}$$

so that, for $q=2p-l$,

$$G_{lp(2p-l)} = e^{|2p-l|}K_{lp(2p-l)}, \tag{7.11}$$

where

$$K_{lp(2p-l)} = \gamma^{-2l+1}\sum_{k=0}^{p'-1}\binom{l-1}{2k+|2p-l|}\binom{2k+|2p-l|}{k}\times$$
$$\times 2^{-2k-|2p-l|}(1-\gamma^2)^k \tag{7.12}$$

and

$$p' = \frac{l-|2p-l|}{2}\ \text{(always integer)}. \tag{7.13}$$

The following derivatives are necessary:

$$\frac{\partial K_{lpq}}{\partial\xi} = \frac{\partial K_{lpq}}{\partial\gamma}\frac{\partial\gamma}{\partial\xi} = -\frac{\xi}{\gamma}\frac{\partial K_{lpq}}{\partial\gamma}$$
$$\frac{\partial K_{lpq}}{\partial\eta} = -\frac{\eta}{\gamma}\frac{\partial K_{lpq}}{\partial\gamma}, \tag{7.14}$$

where, for $q>0$:

$$\frac{\partial K_{lpq}}{\partial \gamma} = \frac{(-l-|q|)}{1+\gamma} K_{lpq} + (-1)^{|q|} 2^l (1+\gamma)^{-l-|q|} \sum_{k=0}^{\infty} \sum_{r=0}^{|q|+k} \sum_{t=0}^{k} \times$$
$$\times \frac{(-1)^r}{r!t!} \binom{2p-2l}{|q|+k-r} \binom{-2p}{k-t} \left(\frac{l-2p+q}{2}\right)^{r+t} \times$$
$$\times (1+\gamma)^{r+t-k-1} [(r+t-k)(1-\gamma)^k - k(1+\gamma)(1-\gamma)^{k-1}]; \tag{7.15}$$

for $q<0$:

$$\frac{\partial K_{lpq}}{\partial \gamma} = \frac{(-l-|q|)}{1+\gamma} K_{lpq} + (-1)^{|q|} 2^l (1+\gamma)^{-l-|q|} \sum_{k=0}^{\infty} \sum_{r=0}^{|q|+k} \sum_{t=0}^{k} \times$$
$$\times \frac{(-1)^t}{r!t!} \binom{-2p}{|q|+k-r} \binom{2p-2l}{k-t} \left(\frac{l-2p+q}{2}\right)^{r+t} \times$$
$$\times (1+\gamma)^{r+t-k-1} [(r+t-k)(1-\gamma)^k - k(1+\gamma)(1-\gamma)^{k-1}]; \tag{7.16}$$

and for $q=2p-l$:

$$\frac{\partial K_{lp(2p-l)}}{\partial \gamma} = \frac{-2l+1}{\gamma} K_{lp(2p-l)} - 2\gamma^{-2l+2} \sum_{k=0}^{p'-1} \binom{l-1}{2k+|2p-l|} \times$$
$$\times \binom{2k+|2p-l|}{k} 2^{-2k-|2p-l|} k(1-\gamma^2)^{k-1}. \tag{7.17}$$

8. The Eccentricity Functions H_{lpq} and L_{lpq}

In terms of Hansen's coefficients we have the definition

$$H_{lpq} = X_{l-2p+q}^{l,\, l-2p}. \tag{8.1}$$

According to Equation (4.10):

$$\left(\frac{r}{a}\right)^l \exp\,[i(l-2p)f] = \sum_q H_{lpq} \exp\,[i(l-2p+q)M]. \tag{8.2}$$

Also

$$H_{lpq} = e^{|q|} L_{lpq}(\gamma), \tag{8.3}$$

where

$$\gamma = \sqrt{1-e^2}.$$

Comparing with

$$\left(\frac{r}{a}\right)^n \exp\,(imf) = \sum_j X_j^{n,m} \exp\,(ijM) \tag{8.4}$$

the relationships are in Equation (8.1):

$$n = l$$
$$m = l - 2p$$
$$j = l - 2p + q$$

or

$$l = n$$
$$2p = n - m$$
$$q = j - m.$$

It is seen that, in opposition to the G_{lpq} functions $n-m$ should be even and $n>0$. Equations (7.2) and (7.3) are still valid, so that for $q>0$:

$$L_{lpq} = (-1)^{|q|}2^{-l-1}(1+\gamma)^{l+1-|q|} \sum_{k=0}^{\infty} \sum_{r=0}^{|q|+k} \sum_{t=0}^{k} \frac{(-1)^r}{r!t!} \binom{2p+1}{|q|+k-r} \times$$
$$\times \binom{2l-2p+1}{k-t} \left(\frac{l-2p+q}{2}\right)^{r+t} (1+\gamma)^{r+t-k}(1-\gamma)^k \tag{8.5}$$

and for $q<0$:

$$L_{lpq} = (-1)^{|q|}2^{-2l-1}(1+\gamma)^{l+1-|q|} \sum_{k=0}^{\infty} \sum_{r=0}^{|q|+k} \sum_{t=0}^{k} \frac{(-1)^t}{r!t!} \binom{2l-2p+1}{|q|+k-r} \times$$
$$\times \binom{2p+1}{k-t} \left(\frac{l-2p+q}{2}\right)^{r+t} (1+\gamma)^{r+t-k}(1-\gamma)^k. \tag{8.6}$$

For $j=0$ and $n>0$ we have

$$X_0^{nm} = \frac{(-e)^{|m|}(1+\gamma)^{n+1-|m|}}{2^{n+1}} \sum_{k=0}^{n+1-|m|} \binom{n+1-|m|}{k} \times$$
$$\times \binom{n+1+|m|}{|m|+k} \left(\frac{1-\gamma}{1+\gamma}\right)^k \tag{8.7}$$

so that

$$H_{lp(2p-l)} = \frac{(-e)^{|2p-l|}(1+\gamma)^{l+1-|2p-l|}}{2^{l+1}} \sum_{k=0}^{l+1-|2p-l|} \binom{l-|2p-l|+1}{k} \times$$
$$\times \binom{l+|2p-l|+1}{|2p-l|+k} \left(\frac{1-\gamma}{1+\gamma}\right)^k \tag{8.8}$$

and therefore

$$L_{lp(2p-l)} = \frac{(-1)^{|2p-l|}(1+\gamma)^{l+1-|2p-l|}}{2^{l+1}} \sum_{k=0}^{l+1-|2p-l|} \binom{l-|2p-l|+1}{k} \times$$
$$\times \binom{l+|2p-l|+1}{|2p-l|+k} \left(\frac{1-\gamma}{1+\gamma}\right)^k. \tag{8.9}$$

For the derivatives, we have, as in Equation (7.14),

$$\frac{\partial L_{lpq}}{\partial \xi} = -\frac{\xi}{\gamma}\frac{\partial L_{lpq}}{\partial \gamma}$$
$$\frac{\partial L_{lpq}}{\partial \eta} = -\frac{\eta}{\gamma}\frac{\partial L_{lpq}}{\partial \gamma}. \tag{8.10}$$

Indicating by a prime the derivative with respect to γ, we find that,

$$L'_{lpq} = \frac{l+1-|q|}{1+\gamma} L_{lpq} + (-1)^{|q|} 2^{-l-1} (1+\gamma)^{l+1-|q|} \sum_{k=0}^{\infty} \sum_{r=0}^{|q|+k} \sum_{t=0}^{k} \times$$
$$\times C_{lpq}^{krt} (1+\gamma)^{r+t+k-1} [(r+t+k)(1-\gamma)^k -$$
$$- k(1+\gamma)(1-\gamma)^{k-1}], \tag{8.11}$$

where, for $q>0$

$$C_{lpq}^{krt} = \frac{(-1)^r}{r!t!} \binom{2p+1}{|q|+k-r} \binom{2l-2p+1}{k-t} \left(\frac{l-2p+q}{2}\right)^{r+t} \tag{8.12}$$

and for $q<0$

$$C_{lpq}^{krt} = \frac{(-1)^t}{r!t!} \binom{2l-2p+1}{|q|+k-r} \binom{2p+1}{k-t} \left(\frac{l-2p+q}{2}\right)^{r+t}. \tag{8.13}$$

If $q=2p-l$,

$$L'_{lp(2p-l)} = \frac{(l+1-|2p-l|)}{1+\gamma} L_{lp(2p-l)} - \frac{(-1)^{|2p-l|}(1+\gamma)^{l-1-|2p-l|}}{2^l} \times$$
$$\times \sum_{k=1}^{l+1-|2p-l|} k \binom{l-|2p-l|+1}{k} \binom{l+|2p-l|+1}{|2p+l|+k} \times$$
$$\times \left(\frac{1-\gamma}{1+\gamma}\right)^{k-1}. \tag{8.14}$$

Recurrence relations for the Hansen coefficients and their derivatives will be presented in a separate paper.

9. The Functions $\mathbb{R}_{lmpq}$ and $\mathbb{I}_{lmpq}$

They are defined by (2.18) and (2.19). Expressions for the derivatives are easily found using Equations (2.11) in the form

$$2\mathbb{R}_{lmpq} = (\xi + i\eta)^q (P - iQ)^\alpha + (\xi - i\eta)^q (P + iQ)^\alpha \tag{9.1}$$

$$2i\mathbb{I}_{mpq} = (\xi + i\eta)^q (P - iQ)^\alpha - (\xi - i\eta)^q (P + iQ)^\alpha \tag{9.2}$$

together with (2.17).

Distinction must be made of the cases $q>0$, $q<0$, $\alpha>0$, $\alpha<0$. We easily find that

$$\begin{aligned}
\frac{\partial \mathbb{R}_{lmpq}}{\partial \xi} &= |q|\mathbb{R}_{lmpq'} \\
\frac{\partial \mathbb{R}_{lmpq}}{\partial \eta} &= -q\mathbb{I}_{lmpq'} \\
\frac{\partial \mathbb{I}_{lmpq}}{\partial \xi} &= |q|\mathbb{I}_{lmpq'} \\
\frac{\partial \mathbb{I}_{lmpq}}{\partial \eta} &= q\mathbb{R}_{lmpq'},
\end{aligned} \tag{9.3}$$

where

$$q' = \begin{cases} q-1 & (q>0) \\ q+1 & (q<0) \end{cases}$$

$$\begin{aligned}
\frac{\partial \mathbb{R}_{lmpq}}{\partial P} &= |\alpha|\mathbb{R}_{lm'pq} \\
\frac{\partial \mathbb{R}_{lmpq}}{\partial Q} &= -\alpha\mathbb{I}_{lm'pq} \\
\frac{\partial \mathbb{I}_{lmpq}}{\partial P} &= |\alpha|\mathbb{I}_{lm'pq} \\
\frac{\partial \mathbb{I}_{lmpq}}{\partial Q} &= \alpha\mathbb{R}_{lm'pq},
\end{aligned} \tag{9.4}$$

where

$$m' = \begin{cases} m-1 & (\alpha>0) \\ m+1 & (\alpha<0) \end{cases}$$

$$\alpha = m + 2p - l.$$

In order to obtain recurrence relations we observe that Equations (9.1) and (9.2) may also be written as

$$\begin{aligned}
(\xi + i\eta)^q (P - iQ)^\alpha &= \mathbb{R}_{q\alpha} + i\mathbb{I}_{q\alpha} \\
(\xi - i\eta)^q (P + iQ)^\alpha &= \mathbb{R}_{q\alpha} - i\mathbb{I}_{q\alpha},
\end{aligned} \tag{9.5}$$

where the indices of $\mathbb{R}$, $\mathbb{I}$ have been compressed for simplicity of notation, and when q and/or α are negative the changes (2.17) must be considered.

We have the following recurrence relations:

$$\begin{aligned}
\mathbb{R}_{l,m,p,q+1} &= \xi\mathbb{R}_{lmpq} - \eta\mathbb{I}_{lmpq} \\
\mathbb{I}_{l,m,p,q+1} &= \eta\mathbb{R}_{lmpq} + \xi\mathbb{I}_{lmpq}
\end{aligned} \quad (q \geqslant 0) \tag{9.6.1}$$

$$\begin{aligned}(\xi^2+\eta^2)\mathbb{R}_{l,m,p,q-1} &= \xi\mathbb{R}_{lmpq} - \eta\mathbb{I}_{lmpq}\\ (\xi^2+\eta^2)\mathbb{I}_{l,m,p,q-1} &= \eta\mathbb{R}_{lmpq} + \xi\mathbb{I}_{lmpq}\end{aligned} \quad (q<0) \tag{9.6.2}$$

$$\begin{aligned}(\xi^2+\eta^2)\mathbb{R}_{l+1,m,p,q} &= \xi\mathbb{R}_{lmpq} - \eta\mathbb{I}_{lmpq}\\ (\xi^2+\eta^2)\mathbb{I}_{l+1,m,p,q} &= \eta\mathbb{R}_{lmpq} + \xi\mathbb{I}_{lmpq}\end{aligned} \quad (\alpha \geqslant 0) \tag{9.6.3}$$

$$\begin{aligned}(\xi^2+\eta^2)\mathbb{R}_{l+1,m,p,q} &= \xi\mathbb{R}_{lmpq} + \eta\mathbb{I}_{lmpq}\\ (\xi^2+\eta^2)\mathbb{I}_{l+1,m,p,q} &= -\eta\mathbb{R}_{lmpq} + \xi\mathbb{I}_{lmpq}\end{aligned} \quad (\alpha < 0) \tag{9.6.4}$$

$$\begin{aligned}\mathbb{R}_{l,m+1,p,q} &= \xi\mathbb{R}_{lmpq} + \eta\mathbb{I}_{lmpq}\\ \mathbb{I}_{l,m+1,p,q} &= -\eta\mathbb{R}_{lmpq} + \xi\mathbb{I}_{lmpq}\end{aligned} \quad (\alpha \geqslant 0) \tag{9.6.5}$$

$$\begin{aligned}\mathbb{R}_{l,m+1,p,q} &= \xi\mathbb{R}_{lmpq} - \eta\mathbb{I}_{lmpq}\\ \mathbb{I}_{l,m+1,p,q} &= \eta\mathbb{R}_{lmpq} + \xi\mathbb{I}_{lmpq}\end{aligned} \quad (\alpha < 0) \tag{9.6.6}$$

$$\begin{aligned}\mathbb{R}_{l,m,p+1,q} &= (\xi^2-\eta^2)\mathbb{R}_{lmpq} + 2\xi\eta\mathbb{I}_{lmpq}\\ \mathbb{I}_{l,m,p+1,q} &= (\xi^2-\eta^2)\mathbb{I}_{lmpq} - 2\xi\eta\mathbb{R}_{lmpq}\end{aligned} \quad (\alpha \geqslant 0) \tag{9.6.7}$$

$$\begin{aligned}\mathbb{R}_{l,m,p+1,q} &= (\xi^2-\eta^2)\mathbb{R}_{lmpq} - 2\xi\eta\mathbb{I}_{lmpq}\\ \mathbb{I}_{l,m,p+1,q} &= (\xi^2-\eta^2)\mathbb{I}_{lmpq} + 2\xi\eta\mathbb{R}_{lmpq}\end{aligned} \quad (\alpha < 0) \tag{9.6.8}$$

Equations (9.6.1) through (9.6.8) are all recurrence relations necessary for $\mathbb{R}$ and $\mathbb{I}$.

For the derivatives, no recurrence relations are necessary in view of Equations (9.3) and (9.4).

10. Short Table for J_{lmp}

l	m	p	$\lvert\alpha\rvert = \lvert m+2p-l\rvert$	$J_{lmp}(c)$, $c=\cos\frac{I}{2}$
2	0	0	2	$-\frac{3}{2}c^2$
2	0	1	0	$-\frac{1}{2}+3c^2-3c^4$
2	0	2	2	$-\frac{3}{2}c^2$
2	1	0	1	$3c^3$
2	1	1	1	$3c-6c^3$
2	1	2	3	$-3c$
2	2	0	0	$3c^4$
2	2	1	2	$6c^2$
2	2	2	4	3
3	0	0	3	$-\frac{5}{2}c^3$
3	0	1	1	$-\frac{3}{2}c+\frac{15}{2}c^3-\frac{15}{2}c^5$
3	0	2	1	$\frac{3}{2}c-\frac{15}{2}c^3+\frac{15}{2}c^5$
3	0	3	3	$\frac{5}{2}c^3$
3	1	0	2	$-\frac{15}{2}c^4$

l	m	p	$\lvert\alpha\rvert = \lvert m+2p-l\rvert$	$J_{lmp}(c), \quad c=\cos\frac{I}{2}$
3	1	1	0	$-9c^2+30c^4-\frac{45}{2}c^6$
3	1	2	2	$-\frac{3}{2}+\frac{15}{4}c^2-\frac{45}{8}c^4$
3	1	3	4	$-\frac{15}{2}c^2$
3	2	0	1	$15c^3$
3	2	1	1	$30c^3-45c^5$
3	2	2	3	$15c-45c^3$
3	2	3	5	$-15c$
3	3	0	0	$15c^6$
3	3	1	2	$45c^4$
3	3	2	4	$45c^2$
3	3	3	6	15
4	0	0	4	$\frac{35}{8}c^4$
4	0	1	2	$\frac{15}{4}c^2-\frac{35}{2}c^4+\frac{35}{2}c^6$
4	0	2	0	$\frac{3}{8}-\frac{15}{2}c^2+\frac{135}{4}c^4-\frac{105}{2}c^6$
4	0	3	2	$\frac{15}{4}c^2-\frac{35}{2}c^4+\frac{35}{2}c^6$
4	0	4	4	$\frac{35}{8}c^4$
4	1	0	3	$-\frac{35}{2}c^5$
4	1	1	1	$-25c^3+\frac{175}{2}c^5-70c^7$
4	1	2	1	$-\frac{15}{2}c+\frac{135}{2}c^3-\frac{315}{2}c^5+105c^7$
4	1	3	3	$\frac{15}{2}c-\frac{105}{2}c^3+70c^5$
4	1	4	5	$\frac{35}{2}c^3$
4	2	0	2	$-\frac{105}{2}c^6$
4	2	1	0	$-\frac{45}{2}c^4+45c^6-30c^8$
4	2	2	2	$-\frac{75}{2}c^2+315c^4-315c^6$
4	2	3	4	$-\frac{15}{2}+105c^2-210c^4$
4	2	4	6	$-\frac{105}{2}c^2$
4	3	0	1	$105c^7$
4	3	1	1	$315c^5-420c^7$
4	3	2	3	$315c^3-630c^5$
4	3	3	5	$-105c+420c^3$
4	3	4	7	$-105c$
4	4	0	0	$105c^8$
4	4	1	2	$420c^6$
4	4	2	4	$630c^4$
4	4	3	6	$420c^2$
4	4	4	8	105

11. Short Table for K_{lpq} ($K_{lpq}=K_{l,\,l-p,\,-q}$)

l	p	q	l	p	q	K_{lpq}
2	0	−2	2	2	2	0
2	0	−1	2	2	1	$-\frac{1}{2}+\frac{1}{16}e^2+\cdots$
2	0	0	2	2	0	$1-\frac{5}{2}e^2+\frac{13}{16}e^4+\cdots$
2	0	1	2	2	−1	$\frac{7}{2}-\frac{123}{16}e^2+\cdots$
2	0	2	2	2	−2	$\frac{17}{2}-\frac{115}{6}e^2+\cdots$
2	1	−2	2	1	2	$\frac{9}{4}+\frac{7}{4}e^2+\cdots$
2	1	−1	2	1	1	$\frac{3}{2}+\frac{27}{16}e^2+\cdots$
2	1	0	2	1	0	$(1-e^2)^{-3/2}$
3	0	−2	3	3	2	$\frac{1}{8}+\frac{1}{48}e^2+\cdots$
3	0	−1	3	3	1	$-1+\frac{5}{4}e^2+\cdots$
3	0	0	3	3	0	$1-6e^2+\frac{423}{64}e^4+\cdots$
3	0	1	3	3	−1	$5-22e^2+\cdots$
3	0	2	3	3	−2	$\frac{127}{8}-\frac{3065}{48}e^2+\cdots$
3	1	−2	3	2	2	$\frac{11}{8}+\frac{49}{16}e^2+\cdots$
3	1	−1	3	2	1	$(1-e^2)^{-5/2}$
3	1	0	3	2	0	$1+2e^2+\frac{239}{64}e^4+\cdots$
3	1	1	3	2	−1	$3+\frac{11}{4}e^2+\cdots$
3	1	2	3	2	−2	$\frac{53}{8}+\frac{39}{16}e^2+\cdots$
4	0	−2	4	4	2	$\frac{1}{2}-\frac{1}{3}e^2+\cdots$
4	0	−1	4	4	1	$-\frac{3}{2}+\frac{75}{16}e^2+\cdots$
4	0	0	4	4	0	$1-11e^2+\frac{199}{8}e^4+\cdots$
4	0	1	4	4	−1	$\frac{13}{2}-\frac{765}{16}e^2+\cdots$
4	0	2	4	4	−2	$\frac{51}{2}-\frac{321}{2}e^2+\cdots$
4	1	−2	4	3	2	$\frac{3}{4}(1-e^2)^{-7/2}$
4	1	−1	4	3	1	$\frac{1}{2}+\frac{33}{16}e^2+\cdots$
4	1	0	4	3	0	$1+e^2+\frac{65}{16}e^4+\cdots$
4	1	1	4	3	−1	$\frac{9}{2}-\frac{3}{16}e^2+\cdots$
4	1	2	4	3	−2	$\frac{53}{4}-\frac{179}{24}e^2+\cdots$
4	2	−2	4	2	2	$5+\frac{155}{12}e^2+\cdots$
4	2	−1	4	2	1	$\frac{5}{2}+\frac{135}{16}e^2+\cdots$
4	2	0	4	2	0	$(1+\frac{3}{2}e^2)(1-e^2)^{-7/2}$

12. Short Table for L_{lpq} ($L_{lpq}=L_{l,\,l-p,\,-q}$)

l	p	q	l	p	q	L_{lpq}
2	0	−2	2	2	2	$\frac{5}{2}$
2	0	−1	2	2	1	$-3+\frac{39}{24}e^2+\cdots$
2	0	0	2	2	0	$1-\frac{5}{2}e^2+\frac{23}{16}e^4+\cdots$
2	0	1	2	2	−1	$1-\frac{19}{8}e^2+\cdots$
2	0	2	2	2	−2	$1-\frac{5}{2}e^2+\cdots$
2	1	−2	2	1	2	$-\frac{1}{4}+\frac{1}{12}e^2+\cdots$
2	1	−1	2	1	1	$-1+\frac{1}{8}e^2+\cdots$
2	1	0	2	1	0	$1+\frac{3}{2}e^2$
3	0	−2	3	3	2	$\frac{57}{8}-\frac{65}{16}e^2+\cdots$
3	0	−1	3	3	1	$-\frac{9}{2}+\frac{33}{4}e^2+\cdots$
3	0	0	3	3	0	$1-6e^2+\frac{591}{64}e^4+\cdots$
3	0	1	3	3	−1	$\frac{3}{2}-\frac{57}{8}e^2+\cdots$
3	0	2	3	3	−2	$\frac{15}{8}-\frac{135}{16}e^2+\cdots$
3	1	−2	3	2	2	$\frac{11}{8}+\frac{7}{48}e^2+\cdots$
3	1	−1	3	2	1	$-\frac{5}{2}-\frac{15}{8}e^2$
3	1	0	3	2	0	$1+2e^2-\frac{123}{192}e^4+\cdots$
3	1	1	3	2	−1	$-\frac{1}{2}+e^2+\cdots$
3	1	2	3	2	−2	$-\frac{3}{8}+\frac{11}{16}e^2+\cdots$
4	0	−2	4	4	2	$14-\frac{137}{6}e^2+\cdots$
4	0	−1	4	4	1	$-6-\frac{93}{4}e^2+\cdots$
4	0	0	4	4	0	$1-11e^2+\frac{253}{8}e^4+\cdots$
4	0	1	4	4	−1	$2-\frac{63}{4}e^2+\cdots$
4	0	2	4	4	−2	$3-21e^2+\cdots$
4	1	−2	4	3	2	$\frac{21}{4}+\frac{21}{8}e^2$
4	1	−1	4	3	1	$-4-3e^2+\cdots$
4	1	0	4	3	0	$1+e^2-\frac{129}{48}e^4+\cdots$
4	1	1	4	3	−1	$\frac{3}{2}e^2-\frac{9}{4}e^4+\cdots$
4	1	2	4	3	−2	$-\frac{1}{4}+\frac{37}{24}e^2+\cdots$
4	2	−2	4	2	2	$\frac{1}{2}-\frac{7}{12}e^2+\cdots$
4	2	−1	4	2	1	$-2-\frac{9}{4}e^2+\cdots$
4	2	0	4	2	0	$1+5e^2+\frac{15}{8}e^4$

13. Short Tables for $\mathbb{R}_{lmpq}$, $\mathbb{I}_{lmpq}$ $(\alpha = m+2p-l)$

q	α	$\mathbb{R}_{lmpq}$	$\mathbb{I}_{lmpq}$
0	0	1	0
0	1	P	$-Q$
0	2	P^2-Q^2	$-2PQ$
1	0	ξ	η
1	1	$\xi P+\eta Q$	$\eta P-\xi Q$
1	2	$\xi(P^2-Q^2)+2\eta PQ$	$\eta(P^2-Q^2)-2\xi PQ$
2	0	$\xi^2-\eta^2$	$2\xi\eta$
2	1	$P(\xi^2-\eta^2)+2Q\xi\eta$	$Q(\xi^2-\eta^2)+2P\xi\eta$
2	2	$(\xi^2-\eta^2)(P^2-Q^2)+4\xi\eta PQ$	$-2(\xi^2-\eta^2)(PQ+2(P^2-Q^2)\xi\eta$
0	-1	P	Q
0	-2	P^2-Q^2	$2PQ$
1	-1	$\xi P-\eta Q$	$\eta P+\xi Q$
1	-2	$\xi(P^2-Q^2)-2\eta PQ$	$\eta(P^2-Q^2)+2\xi PQ$
2	-1	$P(\xi^2-\eta^2)-2Q\xi\eta$	$-Q(\xi^2-\eta^2)+2P\xi\eta$
2	-2	$(\xi^2-\eta^2)(P^2-Q^2)-4\xi\eta PQ$	$2(\xi^2-\eta^2)PQ+2(P^2-Q^2)\xi\eta$
-2	0	$\xi^2-\eta^2$	$-2\xi\eta$
-2	1	$P(\xi^2-\eta^2)-2Q\xi\eta$	$Q(\xi^2-\eta^2)-2P\xi\eta$
-2	2	$(\xi^2-\eta^2)(P^2-Q^2)-4\xi\eta PQ$	$-2(\xi^2-\eta^2)PQ-2(P^2-Q^2)\xi\eta$
-1	0	ξ	$-\eta$
-1	1	$\xi P-\eta Q$	$-\eta P-\xi Q$
-1	2	$\xi(P^2-Q^2)-2\eta PQ$	$-\eta(P^2-Q^2)-2\xi PQ$
-2	-1	$P(\xi^2-\eta^2)+2Q\xi\eta$	$-Q(\xi^2-\eta^2)-2P\xi\eta$
-2	-2	$(\xi^2-\eta^2)(P^2-Q^2)+4\xi\eta PQ$	$2(\xi^2-\eta^2)PQ-2(P^2-Q^2)\xi\eta$
-1	-1	$\xi P+\eta Q$	$-\eta P+\xi Q$
-1	-2	$\xi(P^2-Q^2)+2\eta PQ$	$-\eta(P^2-Q^2)+2\xi PQ$

Note that the H_{lpq}, G_{lpq} were obtained directly from Cayley's tables considering that

$$H_{lpq} = X_j^{n,m}\begin{cases} n = l \\ m = l-2p \\ j = m+q \end{cases}$$

$$G_{lpq} = X_j^{n,m}\begin{cases} m = -l-1 \\ m = l-2p \\ j = m+q. \end{cases}$$

14. Solar Radiation

Let the force per unit mass of the satellite be σ and the unit vector from the Earth to the Sun's position be $\hat{r}_\odot$. The force due to radiation pressure is then, per unit mass,

$$\mathbf{F} = -\sigma\hat{r}_\odot$$

and the disturbing function, in absence of shadow is

$$R^p = -\sigma\hat{r}_\odot\cdot\mathbf{r},$$

where $\mathbf{r}$ is the radius vector of the satellite. If ψ' is the geocentric elongation of this from the Sun

$$R^p = -\sigma r \cos\psi.$$

Transforming to orbital elements and observing that the average value of $(r/a)\cos f$ with respect to the satellite's mean anomaly is $-\frac{3}{2}e$, we find, for the long periodic part of R^p,

$$\bar{R} = \tfrac{3}{2}\sigma a\{2\sqrt{1 - P^2 - Q^2}\,(\eta P - \xi Q)\sin\varepsilon\sin u' + \\ + [\xi(1 - 2Q^2) + 2\eta PQ]\cos u' + [\eta(1 - 2P^2) + \\ + 2\xi PQ]\cos\varepsilon\sin u'\}, \tag{14.2}$$

where

$$u' = f_\odot + \omega_\odot$$
$$\varepsilon = \text{obliquity of the ecliptic.}$$

If the eccentricity of the Sun is neglected, $\lambda_\odot$ can be substituted for u' and

$$\lambda_\odot = 279\overset{\circ}{.}041 + 0\overset{\circ}{.}985\,647\,4(t - 1975.0),$$

where t is days.

The partial derivatives are easily found

$$\frac{\partial\bar{R}}{\partial a} = \frac{1}{a}\bar{R}$$

$$\frac{\partial\bar{R}}{\partial\lambda} = 0$$

$$\frac{\partial\bar{R}}{\partial\xi} = \tfrac{3}{2}\sigma a\{-2\sqrt{1 - P^2 - Q^2}\,Q\sin\varepsilon\sin u' + \\ + (1 - 2Q^2)\cos u + 2PQ\cos\varepsilon\sin u'\}$$

$$\frac{\partial\bar{R}}{\partial\eta} = \tfrac{3}{2}\sigma a\{2\sqrt{1 - P^2 - Q^2}\,P\sin\varepsilon\sin u' + \\ + 2PQ\cos u' + (1 - 2P^2)\cos\varepsilon\sin u'\}$$

$$\frac{\partial \bar{R}}{\partial P} = 3\sigma a\{-(1 - P^2 - Q^2)^{-1/2}[\eta(1 - Q^2) - \xi PQ] \sin \varepsilon \sin u' + \\ + \eta Q \cos u' + (\xi Q - 2\eta P) \cos \varepsilon \sin u'\}$$

$$\frac{\partial \bar{R}}{\partial Q} = 3\sigma a\{-(1 - P^2 - Q^2)^{-1/2}[\xi(1 - P^2) + \eta PQ] \sin \varepsilon \sin u' + \\ + (\eta P - 2\xi Q) \cos u' + \xi P \cos \varepsilon \sin u'\}. \tag{14.3}$$

If short periodic terms are to be included, the elliptic expansions necessary to develop (14.1) are simply given by

$$\frac{r}{a} \cos f = -\tfrac{3}{2}e + \sum_{k=1}^{\infty} \frac{1}{k} [J_{k-1}(ke) - J_{k+1}(ke)] \cos kM$$

$$\frac{r}{a} \sin f = 2\sqrt{1 - e^2} \sum_{k=1}^{\infty} \frac{1}{k} [J_{k-1}(ke) + J_{k+1}(ke)] \sin kM,$$

where $J_k(ke)$ is Bessel's function.

15. Linear Perturbations by Tesseral Harmonics

It is supposed that mean values $\bar{\xi}$, $\bar{\eta}$, $\dot{\bar{\xi}}$, $\dot{\bar{\eta}}$, $\bar{P}$, $\bar{Q}$, $\dot{\bar{P}}$, $\dot{\bar{Q}}$, $\bar{a}$, $\dot{\bar{\lambda}}$ are available either from observations or as given by the leading zonal harmonics. Under these circumstances one can easily write the perturbations in the nonsingular elements due to a particular term R_{lmpq} pertaining to a tesseral harmonic (l, m). This is readily achieved by transforming the well known results in Keplerian elements to the nonsingular elements.

Let us define:

$$\begin{aligned} \bar{a} &= \text{mean semimajor axis}, \\ \bar{n} &= \text{mean motion}, \\ \alpha &= m + 2p - l, \\ K'_{lpq} &= \partial K_{lpq}/\partial \gamma, \qquad \gamma = \sqrt{1 - e^2}, \\ J'_{lmp} &= \partial J_{lmp}/\partial c, \qquad c = \cos \frac{I}{2}, \\ \theta_{lmpq} &= (l - 2p + q)\lambda - m\theta, \end{aligned} \tag{15.1}$$

and

$$S_{lmpq} = \mathbb{R}_{lmpq}(A_{lm} \cos \theta_{lmpq} + B_{lm} \sin \theta_{lmpq}) + \\ + \mathbb{I}_{lmpq}(A_{lm} \sin \theta_{lmpq} - B_{lm} \cos \theta_{lmpq}), \tag{15.2}$$

$$T_{lmpq} = \mathbb{R}_{lmpq}(A_{lm} \sin \theta_{lmpq} - B_{lm} \cos \theta_{lmpq}) - \\ - \mathbb{I}_{lmpq}(A_{lm} \cos \theta_{lmpq} + B_{lm} \sin \theta_{lmpq}), \tag{15.3}$$

$$D_{lmpq} = (l - 2p + q)\dot{\bar{\lambda}} - q \frac{\bar{\xi}\dot{\bar{\eta}} - \bar{\eta}\dot{\bar{\xi}}}{\bar{\xi}^2 + \bar{\eta}^2} + \alpha \frac{\bar{P}\dot{\bar{Q}} - \bar{Q}\dot{\bar{P}}}{\bar{P}^2 - \bar{Q}^2} - m\dot{\theta}. \tag{15.4}$$

The linear perturbations due to R_{lmpq} are given by

$$\delta\lambda_{lmpq} = \bar{n}\left(\frac{a_e}{\bar{a}}\right)^l D^{-1}_{lmpq}K_{lpq}\left\{\left[\frac{|\alpha|}{2\gamma} + 2(l+1)\right]J_{mp} - \right.$$
$$\left. - \frac{1-c^2}{c} J'_{lmp}\right\}T_{lmpq} \tag{15.5}$$

$$\delta a_{lmpq} = 2\bar{n}\bar{a}\left(\frac{a_e}{\bar{a}}\right)^l (l - 2p + q) D^{-1}_{lmpq} J_{lmp} K_{lpq} S_{lmpq} \tag{15.6}$$

$$\bar{\xi}\delta\xi_{lmpq} + \bar{\eta}\delta\eta_{lmpq} = \bar{n}\left(\frac{a_e}{\bar{a}}\right)^l D^{-1}_{lmpq}\gamma[(l-2p+q)\gamma - (l-2p)] \times$$
$$\times J_{lmp}K_{lpq}S_{lmpq} \tag{15.7}$$

$$\bar{P}\delta P_{lmpq} + \bar{Q}\delta Q_{lmpq} = \tfrac{1}{4}\bar{n}\left(\frac{a_e}{\bar{a}}\right)^l D^{-1}_{lmpq}\frac{1}{\gamma}[(l-2p)(c^2 - s^2) - m] \times$$
$$\times J_{lmp}K_{lpq}S_{lmpq} \tag{15.8}$$

$$\bar{\xi}\delta\eta_{lmpq} - \bar{\eta}\delta\xi_{lmpq} = \bar{n}\left(\frac{a_e}{\bar{a}}\right)^l D^{-1}_{lmpq}\left[\left(\gamma|q| + \frac{e^2|\alpha|}{2\gamma}\right)J_{lmp}K_{lpq} - \right.$$
$$\left. - e^2 J_{lmp}K'_{lpq} - \frac{e^2(1-c^2)}{2\gamma c} J'_{lmp}K_{lpq}\right]T_{lmpq} \tag{15.9}$$

$$\bar{P}\delta Q_{lmpq} - \bar{Q}\delta P_{lmpq} = \tfrac{1}{4}\bar{n}\left(\frac{a_e}{\bar{a}}\right)^l D^{-1}_{lmpq}\frac{1}{\gamma}\left(|\alpha|J_{lmp} - \right.$$
$$\left. - \frac{1-c^2}{c} J'_{lmp}\right)K_{lpq}T_{lmpq}. \tag{15.10}$$

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