A NOTE ON RELATIVE MOTION IN THE GENERAL THREE-BODY PROBLEM

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Abstract. It is shown that the equations of the general three-body problem take on a very symmetric form when one considers only their relative positions, rather than position vectors relative to some given coordinate system. From these equations one quickly surmises some well known classical properties of the three-body problem such as the first integrals and the equilateral triangle solutions. Some new Lagrangians with relative coordinates are also obtained. Numerical integration of the new equations of motion is about 10 percent faster than with barycentric or heliocentric coordinates.

1. The Equations of Motion in Barycentric Coordinates

Let the coordinates of the three positive point-masses m_1, m_2 and m_3 be represented by the position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ in the barycentric rectangular coordinate system with three dimensions. The Lagrangian of the system is:

$$\mathscr{L} = \tfrac{1}{2}\left[m_1\dot{\mathbf{r}}_1^2 + m_2\dot{\mathbf{r}}_2^2 + m_3\dot{\mathbf{r}}_3^2\right] + \left[\frac{m_2 m_3}{|\mathbf{r}_3 - \mathbf{r}_2|} + \frac{m_1 m_3}{|\mathbf{r}_1 - \mathbf{r}_3|} + \frac{m_1 m_2}{|\mathbf{r}_2 - \mathbf{r}_1|}\right], \tag{1}$$

and the three position vectors satisfy the following equations:

$$\begin{aligned}
\ddot{\mathbf{r}}_1 &= -m_2\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} - m_3\frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3},\\
\ddot{\mathbf{r}}_2 &= -m_1\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} - m_3\frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3},\\
\ddot{\mathbf{r}}_3 &= -m_1\frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3} - m_2\frac{\mathbf{r}_3 - \mathbf{r}_2}{|\mathbf{r}_3 - \mathbf{r}_2|^3},
\end{aligned} \tag{2}$$

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 = 0. \tag{3}$$

2. The Equations of Motion in Relative Coordinates

If we are interested only in the relative motion of the three masses, it is then indicated to use as variables the three relative position vectors $\boldsymbol{\varrho}_1$, $\boldsymbol{\varrho}_2$ and $\boldsymbol{\varrho}_3$ defined by,

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(Figure 1):

$$\boldsymbol{\varrho}_1 = \mathbf{r}_3 - \mathbf{r}_2\,,$$
$$\boldsymbol{\varrho}_2 = \mathbf{r}_1 - \mathbf{r}_3\,, \tag{4}$$
$$\boldsymbol{\varrho}_3 = \mathbf{r}_2 - \mathbf{r}_1\,.$$
$$\mathbf{S} \equiv \boldsymbol{\varrho}_1 + \boldsymbol{\varrho}_2 + \boldsymbol{\varrho}_3 = 0\,. \tag{5}$$

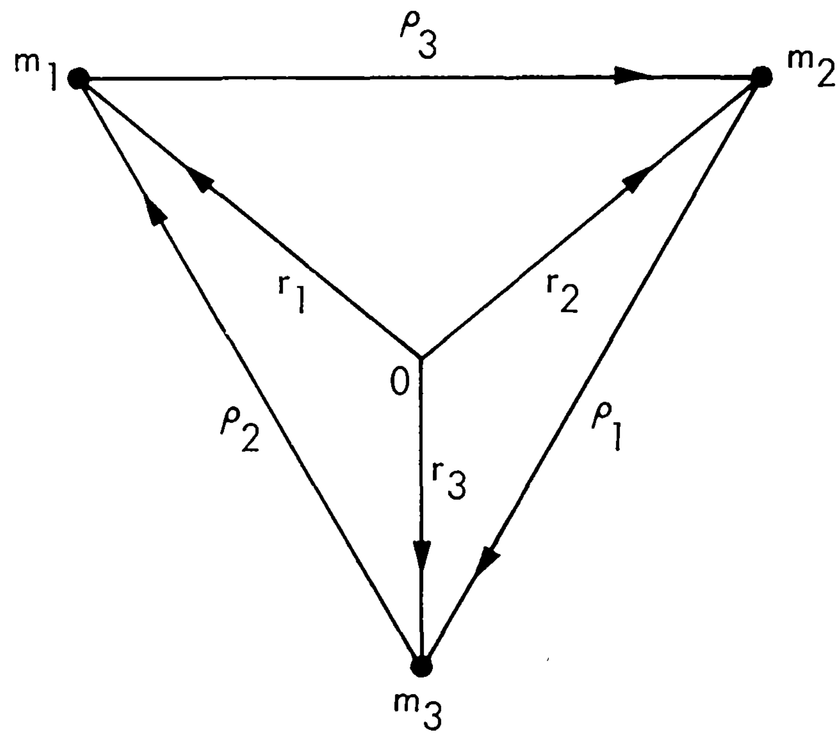

Fig. 1. Configuration of masses m_1, m_2 and m_3.

The second-order differential equations of motion in relative coordinates can now be obtained directly from the substitution of the previous equations of motion (2) in (4). We obtain the remarkably simple and elegant result:

$$\ddot{\boldsymbol{\varrho}}_1 = -\mu \frac{\boldsymbol{\varrho}_1}{\varrho_1^3} + m_1 \left(\frac{\boldsymbol{\varrho}_1}{\varrho_1^3} + \frac{\boldsymbol{\varrho}_2}{\varrho_2^3} + \frac{\boldsymbol{\varrho}_3}{\varrho_3^3} \right),$$
$$\ddot{\boldsymbol{\varrho}}_2 = -\mu \frac{\boldsymbol{\varrho}_2}{\varrho_2^3} + m_2 \left(\frac{\boldsymbol{\varrho}_1}{\varrho_1^3} + \frac{\boldsymbol{\varrho}_2}{\varrho_2^3} + \frac{\boldsymbol{\varrho}_3}{\varrho_3^3} \right), \tag{6}$$
$$\ddot{\boldsymbol{\varrho}}_3 = -\mu \frac{\boldsymbol{\varrho}_3}{\varrho_3^3} + m_3 \left(\frac{\boldsymbol{\varrho}_1}{\varrho_1^3} + \frac{\boldsymbol{\varrho}_2}{\varrho_2^3} + \frac{\boldsymbol{\varrho}_3}{\varrho_3^3} \right),$$

where ϱ_i is the length of the vector $\boldsymbol{\varrho}_i$, and $\mu = m_1 + m_2 + m_3$.

Besides the theoretical properties that will be described below, the equations of motion (6) have a very important practical importance. In the numerical integration of orbits it has been found that the relative coordinates are much faster than either heliocentric or barycentric coordinates. A careful Fortran programming shows that with the equations of motion (6), there is a 10% saving of arithmetic operations in the calculation of the accelerations, given the position vectors. This is of course important if one remembers that in one orbit the acceleration has to be computed thousands of times, and that in many investigations, literally thousands of orbits are computed.

In agreement with Deprit and Delie (1961), we will introduce the quantity m_0 and three other quantities n_i (associated masses) satisfying the following relations:

$$m_0^2 = \frac{m_1 m_2 m_3}{\mu}. \tag{7}$$

$$n_1 = \frac{m_2 m_3}{\mu}, \qquad n_2 = \frac{m_1 m_3}{\mu}, \qquad n_3 = \frac{m_1 m_2}{\mu}, \tag{8}$$

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = \frac{\mu}{m_0^2}. \tag{9}$$

The equations of motion (6) may then be written in the form

$$\begin{aligned} n_1 \ddot{\boldsymbol{\varrho}}_1 &= -\frac{\mu n_1}{\varrho_1^3} \boldsymbol{\varrho}_1 + m_0^2 \left(\frac{\boldsymbol{\varrho}_1}{\varrho_1^3} + \frac{\boldsymbol{\varrho}_2}{\varrho_2^3} + \frac{\boldsymbol{\varrho}_3}{\varrho_3^3} \right), \\ n_2 \ddot{\boldsymbol{\varrho}}_2 &= -\frac{\mu n_2}{\varrho_2^3} \boldsymbol{\varrho}_2 + m_0^2 \left(\frac{\boldsymbol{\varrho}_1}{\varrho_1^3} + \frac{\boldsymbol{\varrho}_2}{\varrho_2^3} + \frac{\boldsymbol{\varrho}_3}{\varrho_3^3} \right), \\ n_3 \ddot{\boldsymbol{\varrho}}_3 &= -\frac{\mu n_3}{\varrho_3^3} \boldsymbol{\varrho}_3 + m_0^2 \left(\frac{\boldsymbol{\varrho}_1}{\varrho_1^3} + \frac{\boldsymbol{\varrho}_2}{\varrho_2^3} + \frac{\boldsymbol{\varrho}_3}{\varrho_3^3} \right). \end{aligned} \tag{10}$$

Equations (10) can be looked upon as representing the equations of motion of the associated masses n_i with position vectors $\boldsymbol{\varrho}_i$. Each associated mass is acted upon by a force of attraction due to a fixed mass μ located at the origin as well as a perturbing force (the last three terms in Equations (10)!) which may be thought of as a force that three unit-masses placed at the location of the associated masses n_i would exert on a mass m_0^2 placed at the origin. It is also easy to see that the equations of motion (6) satisfy the constraint equation $\mathbf{S} = \dot{\mathbf{S}} = \ddot{\mathbf{S}} = 0$ given in (5).

The equations of motion (10) also have a remarkable similarity with the equations of the two-body problem. In fact (10) may be considered as 3 two-body problems connected by a binding force which keeps the constraint (5) satisfied at all times. This will also be clearly shown in Section 4 with the aid of a Lagrange multiplier. Let us also mention that several attempts to reduce the three-body problem to a two-body form have previously been made, for instance by Lagrange himself (Nahon, 1963).

3. The Lagrangian in Relative Coordinates

Let us now also develop the Lagrangian in relative coordinates. In order to do this we have to solve the system (3)+(4) for the position vectors $\mathbf{r}_i$:

$$\begin{aligned} \mathbf{r}_1 &= (m_3 \boldsymbol{\varrho}_2 - m_2 \boldsymbol{\varrho}_3)/\mu, \\ \mathbf{r}_2 &= (m_1 \boldsymbol{\varrho}_3 - m_3 \boldsymbol{\varrho}_1)/\mu, \\ \mathbf{r}_3 &= (m_2 \boldsymbol{\varrho}_1 - m_1 \boldsymbol{\varrho}_2)/\mu. \end{aligned} \tag{11}$$

We will now consider the equations (11) as defining a change of variables from $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ to $\boldsymbol{\varrho}_1$, $\boldsymbol{\varrho}_2$ and $\boldsymbol{\varrho}_3$. This change of variables can be used to transform the

Lagrangian (1). However, in making the substitution (11) in (1), the constraint (5) should not be used to simplify the new Lagrangian. Before transforming the Lagrangian (1), the following differences are first derived from (11):

$$\begin{aligned} \mathbf{r}_3 - \mathbf{r}_2 &= \boldsymbol{\varrho}_1 - m_1 \mathbf{S}/\mu, \\ \mathbf{r}_1 - \mathbf{r}_3 &= \boldsymbol{\varrho}_2 - m_2 \mathbf{S}/\mu, \\ \mathbf{r}_2 - \mathbf{r}_1 &= \boldsymbol{\varrho}_3 - m_3 \mathbf{S}/\mu, \end{aligned} \tag{12}$$

where S has been defined in (5).

The Lagrangian (1) of the problem may then be written as:

$$\begin{aligned} \mathscr{L} = &\frac{1}{2\mu^2} \left[m_1 (m_3 \dot{\boldsymbol{\varrho}}_2 - m_2 \dot{\boldsymbol{\varrho}}_3)^2 + m_2 (m_1 \dot{\boldsymbol{\varrho}}_3 - m_3 \dot{\boldsymbol{\varrho}}_1)^2 + \right. \\ &\left. + m_3 (m_2 \dot{\boldsymbol{\varrho}}_1 - m_1 \dot{\boldsymbol{\varrho}}_2)^2 \right] + \\ &+ \left[\frac{m_2 m_3}{|\boldsymbol{\varrho}_1 - m_1 \mathbf{S}/\mu|} + \frac{m_1 m_3}{|\boldsymbol{\varrho}_2 - m_2 \mathbf{S}/\mu|} + \frac{m_1 m_2}{|\boldsymbol{\varrho}_3 - m_3 \mathbf{S}/\mu|} \right]. \end{aligned} \tag{13}$$

The equations of motion can now be derived from this Lagrangian. The fact that S depends on all three coordinates $\boldsymbol{\varrho}_i$ must be taken into account when the partial derivatives of $\mathscr{L}$ are taken, but the equations can be simplified afterwards by setting S equal to zero (Broucke, 1971).

However the Lagrangian (13) has an unusual property which has to be mentioned. Due to the non-independence of the variables $\boldsymbol{\varrho}_i$, this Lagrangian has a Hessian determinant $|g_{ij}|$ which is zero. We have thus here an example of a Lagrangian which is perfectly useful and which has a singular matrix g_{ij}. This Lagrangian would thus not be appropriate for the definition of the canonical momenta and the Hamiltonian corresponding to the variables $\boldsymbol{\varrho}_i$, (at least in the classical sense). This property of the Lagrangian is all the more remarkable because the full rank of the Hessian matrix can be restored by adding to the Lagrangian a quantity which has a zero Lagrangian derivative, for instance an appropriate multiple of $\dot{\mathbf{S}}^2$. We find the simplified (but equivalent!) Lagrangian:

$$\mathscr{L} = \sum_{i=1}^{3} n_i \left[\frac{\dot{\boldsymbol{\varrho}}_i^2}{2} + \frac{\mu}{|\boldsymbol{\varrho}_i - m_i \mathbf{S}/\mu|} \right]. \tag{14}$$

4. The Lagrange Multiplier of the Problem

It turns out that the above Lagrangian (14) can be simplified by replacing S by zero, but then it is necessary to use a Lagrange multiplier λ to take this constraint into account. Another valid Lagrangian of the three-body problem is thus:

$$\mathscr{L} = \sum_{i=1}^{3} n_i \left[\frac{\dot{\boldsymbol{\varrho}}_i^2}{2} + \frac{\mu}{\varrho_i} \right] + \lambda \mathbf{S}.$$

The equations of motion derived from it are:

$$n_i \ddot{\boldsymbol{\varrho}}_i = -\frac{\mu n_i}{\varrho_i^3} \boldsymbol{\varrho}_i + \boldsymbol{\lambda}.$$

Using now the constraint $\mathbf{S}=0$ to determine the value of the Lagrange multiplier will give

$$\boldsymbol{\lambda} = m_0^2 \sum_{i=1}^{3} \frac{\boldsymbol{\varrho}_i}{\varrho_i^3},$$

which is the additional term that is found in the equations of motion (10).

5. The First Integrals in Relative Coordinates

It will easily be found that the energy E, the angular momentum $\mathbf{C}$, the polar moment of Inertia J and the Lagrange identity can be expressed and derived in relative coordinates:

$$E = \sum_{i=1}^{3} n_i \left[\frac{\dot{\boldsymbol{\varrho}}_i^2}{2} - \frac{\mu}{\varrho_i}\right], \qquad J = \sum_{i=1}^{3} n_i \boldsymbol{\varrho}_i^2$$

$$\mathbf{C} = \sum_{i=1}^{3} n_i [\boldsymbol{\varrho}_i \times \dot{\boldsymbol{\varrho}}_i], \qquad \ddot{J} = 2(E + T).$$

6. Lagrange's Equilateral Triangle Solutions

An immediate consequence of the equations of motion (6) is the equilateral triangle solution due to Lagrange. Let $|\boldsymbol{\varrho}_1| = |\boldsymbol{\varrho}_2| = |\boldsymbol{\varrho}_3| = \varrho$ initially and let the three bodies remain in a fixed plane. From the constraint $\mathbf{S}=0$ the equations of motion (6) will uncouple and become the familiar equations of Keplerian motion. From the theory of the two-body problem we have then for all three values $i = i, 2; 3$:

$$\varrho^2 \dot{\phi}_i = k_i = \text{const.}; \qquad \varrho \dot{\phi}_i^2 = \mu/\varrho^2,$$

where ϕ_i is the angle that $\boldsymbol{\varrho}_i$ makes with a fixed line, say the x-axis. Thus all three associated masses n_i must have the same angular momentum k_i given by the equation

$$k_i^2 = \mu \varrho.$$

They must also have the same angular velocity $\dot{\phi}_i$ given by

$$\dot{\phi}_i^2 = \frac{\mu}{\varrho^3}.$$

This result is well-known (Wintner, 1947, formula (40), page 304), and is in fact Kepler's third law for the equilateral triangle solution.

The above considerations refer to three circular orbits but they can be generalized to elliptic orbits, corresponding then to the *similar* conic sections with arbitrary eccentricity. We obtain the equilateral triangle solution whose sides vary with time. In fact this is an obvious consequence of our equations of motion (6), because these equations uncouple if $|\boldsymbol{\varrho}_1| = |\boldsymbol{\varrho}_2| = |\boldsymbol{\varrho}_3| = \varrho$ even if ϱ is a function of time. From the results of the two-body problem in polar coordinates we have then:

$$\ddot{\varrho} - \varrho\dot{\phi}_i^2 = -\frac{\mu}{\varrho^2}; \qquad \varrho^2\dot{\phi}_i = k_i = \text{const.},$$

and the angular velocity $\dot{\phi}_i$ is proportional to $1/\varrho^2$.

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