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Quantum K-Systems

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Abstract. We generalize the classical notion of a K-system to a non-commutative dynamical system by requiring that an invariantly defined memory loss be 100%. We give some examples of quantum K-systems and show that they cannot contain any quasi-periodic subsystem.

1. Introduction

There seems to be general agreement [1–4] that classical K-systems exhibit those mixing and chaotic properties which are necessary for the foundation of statistical mechanics. Classically they can be characterized by the existence of a subalgebra $\mathscr{A} \subset \mathscr{M} =$ the algebra of observables with

(i) $\sigma^n \mathscr{A} \supset \mathscr{A} \, \forall n \in \mathbf{Z}^+$,
(ii) $\bigvee_{n \geqq 0} \sigma^n \mathscr{A} = \mathscr{M}$,
(iii) $\bigwedge_{n \geqq 0} \sigma^{-n} \mathscr{A} = c\mathbf{1}$.

Here σ is the time evolution and $\vee$ and $\wedge$ mean union and intersection of algebras. These conditions are met in particular if there exists a generating subalgebra $\mathscr{A}_0 \subset \mathscr{M}$ with $\bigvee_{-\infty < n < \infty} \sigma^n \mathscr{A}_0 = \mathscr{M}$, $\bigwedge_{n=1}^{\infty} \bigvee_{j=1}^{\infty} \sigma^{-n-j} \mathscr{A}_0 = c\mathbf{1}$. The difficulties of generalizing this for non-commutative algebras $\mathscr{M}$ comes from the fact that then even two finite-dimensional isomorphic subalgebras may generate algebraically an infinite-dimensional $\mathscr{M}$. For instance, if x and p satisfy $[x,p]=i$ and χ is a characteristic function of $[-1,1]$ and $\sigma:(x,p) \to (p,-x)$, then $\mathscr{A}_0 = (\chi(x), 1-\chi(x))$ and $\sigma\mathscr{A}_0$ generate the algebra $W = l_\alpha \otimes M_2$ and $\mathscr{A}_0 \wedge \sigma\mathscr{A}_0 = c\mathbf{1}$. Nevertheless, Emch [2] has proposed a notion of a non-commutative K-system and an associated dynamical entropy starting with the algebraic characterization given at the beginning (see also [3,4]). We have recently [5] given an alternative definition of the dynamical entropy of a non-commutative system and we propose a corresponding notion of a quantum K-system. We start with the classically equivalent characterization of a K-system by requiring that the tail $\bigwedge_{n=1}^{\infty} \bigvee_{j=1}^{\infty} \sigma^{-n-j}\mathscr{A}$ of any finite partition (finite subalgebra) $\mathscr{A}$ is trivial ($=c\mathbf{1}$). The triviality of the tail can be rephrased in terms of the entropy

$$\lim_{n\to\infty} \lim_{j\to\infty} \left[S\left(\mathscr{A} \bigvee_{k=n}^{j} \sigma^{-k}\mathscr{A} \right) - S\left(\bigvee_{k=n}^{j} \sigma^{-k}\mathscr{A} \right) \right] = S(\mathscr{A}),$$

and this can be used as a starting point for a non-commutative theory. In [5] we have introduced entropy functionals $H(\mathscr{A}_1,\ldots,\mathscr{A}_n)$ which have the desired properties and reduce to $S\left(\bigvee_{i=1}^{n} \mathscr{A}_i\right)$ in the commutative case. They have the intuitive meaning of the maximal information to be gained about the subalgebras $\mathscr{A}_i$ by a measurement of the total system. Using $H(\mathscr{A}_1,\ldots,\mathscr{A}_n)$ instead of $S\left(\bigvee_{i=1}^{n} \mathscr{A}_n\right)$ in the above criterion and replacing $\lim_{j\to\infty}$ by the mean we get a characterization of quantum K-systems which roughly says the following: The maximal information obtained about any $\mathscr{A}$ at previous times can never give the full information about $\mathscr{A}$ at present, in fact if these times were too far in the past all information gets lost.

In this note we will explore the consequences of such a definition of quantum K-systems. They show features which contradict what one is used to from finite quantum systems. Firstly, K-systems are ergodic in the sense that the only time-invariant elements of the algebra of observables are multiples of unity. Thus the Hamiltonian H which generates the time evolution cannot be an element of this algebra. Even more strikingly Zermelo's recurrence objection is completely rejected in the sense that there are no quasi-periodic elements $\neq c\mathbf{1}$.

We shall show that some infinite quantum system which are generalizations of classical K-systems do indeed have our K-property. Our examples of K-systems are of the type studied by Emch [2], Kümmerer and Schröder [3,4] but we have the advantage that we are not obliged to exhibit the expanding subalgebra $\mathscr{A}$. If one adds the assumption of strong asymptotic abelianess one can show [14] that the K-systems in the sense of Schröder [4] are also K-systems in our sense. Hopefully also the systems relevant for physics, namely bosons or fermions interacting with pair potentials are of this class, but we are far from having investigated all potentialities of this notion.

2. The Entropy Functionals

Our theory is based on finite-dimensional unital *-subalgebras and we shall abbreviate this cumbersome construction by "finite subalgebra." The theory can also be extended to nuclear C^*-algebras without finite subalgebras, but for simplicity of exposition we shall restrict ourselves to UHF-algebras $\mathscr{M}$. Furthermore, we shall only consider faithful states over $\mathscr{M}$ (which means $\omega(|a|^2) > |\omega(a)|^2 \,\forall a \in \mathscr{M}, a \neq c\mathbf{1}$).

Before we embark on the theory of general K-systems we shall first recall the general definitions and deduce some useful estimates. Let ω be a faithful state over an UHF-algebra $\mathscr{M}$ and

$$\omega_{i_1,\ldots,i_n} > 0, \qquad \sum_{i_1,\ldots,i_n} \omega_{i_1,\ldots,i_n} = \omega,$$

a decomposition. For the multi-index $(i_1,\ldots,i_n)$ we shall use the shorthand I and define

$$\omega_{i_k}^{(k)} = \sum_{\substack{i_1,\ldots,i_n \\ i_k \text{ fixed}}} \omega_{i_1,\ldots,i_n}.$$

Furthermore let us denote the entropy function $-x\ln x$ by $\eta(x), \omega_{|\mathscr{A}} =$ the restriction of ω to $\mathscr{A} \subset \mathscr{M}$, $S(\varphi|\psi) =$ the relative entropy of φ and ψ, $S(\varphi_{|\mathscr{A}}|\psi_{|\mathscr{A}}) \equiv S(\varphi|\psi)_{\mathscr{A}}$. Now we are all set for

Definition (2.1).

$$H_\omega(\mathscr{A}_1,\ldots,\mathscr{A}_n) = \sup_{\sum_I \omega_I = \omega} \left[\sum_I \eta(\omega_I(1)) + \sum_{k=1}^{n} \sum_{i_k} S(\omega|\omega_{i_k}^{(k)})_{\mathscr{A}_k} \right].$$

Remarks (2.2).

1. Denote by $\dim \mathscr{A}_k$ the linear dimension of a maximal subalgebra of $\mathscr{A}_k$. Given $M = \max_{1 \leq k \leq n} \dim \mathscr{A}_k$ and ε there is a number $\delta(\varepsilon, M, n) > 0$ such that sup is reached within ε by a decomposition with $\omega_I(1) > \delta \forall I$ and thus $\# I \leq 1/\delta$ [5].
2. In general a decomposition can be written $\omega_I(a) = \omega(x'_I a) = \omega(\sigma_{i/2}^{\omega}(x_I)a)$ with $x'_I \in \mathscr{M}'$, $x_I \in \mathscr{M}''$, σ_t^{ω} the modular automorphism of ω. Since H_ω is strongly continuous in ω it is sufficient to take the sup over ω_I with x_I from a strongly dense subalgebra of $\mathscr{M}''$. Thus if $\mathscr{M}$ is a quasi-local algebra we may assume the x_I to be strictly local.

Properties of H (2.3).

(i) $H(\mathscr{A}_1,\ldots,\mathscr{A}_n) \geq 0$ and is symmetric in its arguments.
(ii) Monotonicity: $\mathscr{A}_i \supset \mathscr{B}_i \Rightarrow H_\omega(\mathscr{A}_1,\ldots,\mathscr{A}_n) \geq H_\omega(\mathscr{B}_1,\ldots,\mathscr{B}_n)$.
(iii) Subadditivity: $H_\omega(\mathscr{A}_1,\ldots,\mathscr{A}_n) \leq H_\omega(\mathscr{A}_1,\ldots,\mathscr{A}_k) + H_\omega(\mathscr{A}_{k+1},\ldots,\mathscr{A}_n) \quad \forall 1 \leq k \leq n$.
(iv) Invariance under repetitions:

$$H_\omega(\mathscr{A}_1,\mathscr{A}_1,\ldots,\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_n) = H_\omega(\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_n).$$

Lemma (2.4).

$$\begin{aligned} &H_\omega(\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_{n-1},\mathscr{A}) - H_\omega(\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_{n-1},\mathscr{B}) \\ &\quad \leq \sup_{\sum_i \omega_i = \omega} \sum_i (S(\omega|\omega_i)_{\mathscr{A}} - S(\omega|\omega_i)_{\mathscr{B}}) \equiv H_\omega(\mathscr{A}|\mathscr{B}). \end{aligned}$$

Proof. Use a decomposition which gives, within ε, $H_\omega(\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_{n-1},\mathscr{A})$ as a decomposition for $H_\omega(\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_{n-1},\mathscr{B})$. Then in the difference all terms cancel except the term $k = n$ in the last sum of (2.1).

Remarks (2.5).

1. If $\mathscr{A}$ and $\mathscr{B}$ are abelian, $H(\mathscr{A}|\mathscr{B})$ equals the corresponding classical quantity,

i.e., $H_\omega(\mathscr{A}|\mathscr{B}) = H_\omega(\mathscr{A} \vee \mathscr{B}) - H_\omega(\mathscr{B})$ (see Appendix 1). Classically H and S coincide, and we shall use both notations $S_\omega(\mathscr{A}) \equiv S(\omega_{|\mathscr{A}})$. If only one state ω is involved we might skip the subscript ω.

2. (2.3, iv) implies

$$\begin{aligned}&H(\mathscr{A}_1,\ldots,\mathscr{A}_{n-1},\mathscr{A}_n) - H(\mathscr{A}_1,\ldots,\mathscr{A}_{n-1})\\ &\quad = H(\mathscr{A}_1,\ldots,\mathscr{A}_{n-1},\mathscr{A}_n) - H(\mathscr{A}_1,\ldots,\mathscr{A}_{n-1},\mathscr{A}_{n-1}) \leqq H(\mathscr{A}_n|\mathscr{A}_{n-1}),\end{aligned}$$

thus

$$H(\mathscr{A}_1,\ldots,\mathscr{A}_n) \leqq H(\mathscr{A}_1) + \sum_{i=1}^{n-1} H(\mathscr{A}_{i+1}|\mathscr{A}_i)$$

and

$$H(\mathscr{A}_1,\ldots,\mathscr{A}_n) - H(\mathscr{B}_1,\ldots,\mathscr{B}_n) \leqq \sum_{i=1}^{n} H(\mathscr{A}_i|\mathscr{B}_i).$$

To complement these upper bounds by lower bounds we need more information about the possible decompositions ω_I:

Lemma (2.6). *Suppose that $x_I \in \mathscr{M}$ give, within ε, $H_\omega(\mathscr{B}_1,\ldots,\mathscr{B}_n)$ and that there exist $\mathscr{M} \ni y_j > 0$, $\sum_j y_j = 1$ such that*

(i) $[x_I, y_j] = 0 \, \forall I, j$,

(ii) $\left|\dfrac{\omega(x_I y_j)}{\omega(x_I)\omega(y_j)} - 1\right| < \varepsilon_1 \, \forall I, j$.

Then

$$\begin{aligned}&H_\omega(\mathscr{A},\mathscr{B}_1,\ldots,\mathscr{B}_n) - H_\omega(\mathscr{B}_1,\ldots,\mathscr{B}_n)\\ &\quad \geqq \sup_{\substack{a\in\mathscr{A}\\ \|a\|=1}} \frac{1}{2}\sum_j |\omega(y_j)\omega(a) - \omega(a y_j)|^2 \frac{1}{\omega(y_j)} - \varepsilon - \frac{\varepsilon_1}{1-\varepsilon_1}.\end{aligned}$$

If the y_j give, within ε, $H_\omega(\mathscr{A})$ then

$$H_\omega(\mathscr{A},\mathscr{B}_1,\ldots,\mathscr{B}_n) - H_\omega(\mathscr{B}_1,\ldots,\mathscr{B}_n) \geqq H_\omega(\mathscr{A}) - 2\varepsilon - \frac{\varepsilon_1}{1-\varepsilon_1}.$$

Proof. Consider the decomposition

$$\omega_{I,j}(a) = \omega(a\sigma^\omega_{i/2}(x_I y_j)).$$

We have

$$\sum_I \omega_{I,j} = \omega(\cdot\,\sigma^\omega_{i/2}(y_j)) \equiv \omega_j, \quad \sum_j \omega_{I,j} = \omega_I \quad \text{and} \quad \omega(x_I y_j) = \omega_{I,j}(1).$$

Thus, if we use $\omega_{I,j}$ as decomposition for $H(\mathscr{A},\mathscr{B}_1,\ldots,\mathscr{B}_n)$ all but the first term of the $\sum_k$ in (2.1) can be used to form $H_\omega(\mathscr{B}_1,\ldots,\mathscr{B}_n)$ and we get

$$\begin{aligned}&H_\omega(\mathscr{A},\mathscr{B}_1,\ldots,\mathscr{B}_n) - H_\omega(\mathscr{B}_1,\ldots,\mathscr{B}_n)\\ &\quad \geqq \sum_{I,j}\eta(\omega(x_I y_j)) - \sum_I \eta(\omega(x_I)) - \sum_j \eta(\omega(y_j)) + \sum_j \omega(y_j) S(\omega|\hat{\omega}_j)_{\mathscr{A}} - \varepsilon.\end{aligned}$$

Here $\hat{\omega}_j(a) \equiv \omega_j(a)/(\omega_j(1))$ is the normalized functional and we used the scaling $S(\varphi|\lambda\psi) = \lambda(S(\varphi|\psi) + \psi(1)\ln\lambda)$. The first three terms can be written

$$-\sum_{I,j} \omega(x_I y_j) \ln \frac{\omega(x_I y_j)}{\omega(x_I)\omega(y_j)} \geqq -\frac{\varepsilon_1}{1-\varepsilon_1},$$

since $\sum_{I,j} \omega(x_I y_j) = 1$ and $|\ln(1+x)| \leqq |x|/(1-|x|)$. For normalized functionals one knows [6]

$$S(\varphi|\psi) \geqq \tfrac{1}{2}\|\varphi - \psi\|^2$$

and

$$\|\psi_{|\mathscr{A}}\| = \sup_{\substack{a \in \mathscr{A} \\ \|a\|=1}} |\psi(a)|.$$

This estimate for the last term gives the first part of the claim (2.6). The second is immediate from

$$H_\omega(\mathscr{A}) = \sum_j \omega(y_j) S(\omega|\hat{\omega}_j)_{\mathscr{A}} + \varepsilon.$$

Corollary (2.7). *If $\mathscr{A} \neq c\mathbf{1}$, then $H_\omega(\mathscr{A}) > 0$.*

Proof. Take in (2.6) $\mathscr{B}_i = c\mathbf{1}$, $I = \{1\}$, $x_I = 1, 0 < y_1 = a \in \mathscr{A}, y_2 = 1 - a$. $H_\omega(\mathscr{A}) = 0$ would imply $\omega(a^2) = \omega(a)^2 \; \forall a \in \mathscr{A}$. But for a faithful state

$$\omega(a^2) - \omega(a)^2 = \omega((a - \omega(a))^2) = 0 \Rightarrow a = \omega(a)\cdot\mathbf{1}.$$

Let σ be an automorphism of $\mathscr{M}$ which leaves ω invariant, $\omega \circ \sigma = \omega$.

Definition (2.8).

$$h_\omega(\sigma, \mathscr{A}) = \lim_{k\to\infty} \frac{1}{k} H_\omega(\mathscr{A}, \sigma\mathscr{A}, \ldots, \sigma^{k-1}\mathscr{A}).$$

Remarks (2.9).

1. Because of (2.3, iii) we have $\lim\limits_{k\to\infty} = \inf\limits_k$, and since $H \geqq 0$ we know that the limit exists.
2. If $\mathscr{M}$ is abelian

$$H_\omega(\mathscr{A}, \sigma\mathscr{A}, \ldots, \sigma^{k-1}\mathscr{A}) = S_\omega\left(\bigvee_{j=0}^{k-1} \sigma^j \mathscr{A}\right)$$

and

$$h_\omega(\sigma, \mathscr{A}) = \lim_{k\to\infty}\left(S_\omega\left(\bigvee_{j=0}^{k} \sigma^j\mathscr{A}\right) - S_\omega\left(\bigvee_{j=0}^{k-1}\sigma^j\mathscr{A}\right)\right).$$

Thus (2.8) is a generalization of the classical definition and our results also cover this situation.

Properties of h (2.10).

(i) $\frac{1}{n} h_\omega(\sigma^n, \mathscr{A}) \leqq h_\omega(\sigma, \mathscr{A}) \leqq h_\omega(\sigma^n, \mathscr{A})$,

(ii) $h_\omega(\sigma^{-1}, \mathscr{A}) = h_\omega(\sigma, \mathscr{A})$,
(iii) $h_{\omega\circ\alpha^{-1}}(\alpha\sigma\alpha^{-1}, \alpha\mathscr{A}) = h_\omega(\sigma, \mathscr{A}) \,\forall\, \alpha \in \operatorname{Aut} \mathscr{M}$.

Proof.

(i) From (2.3, ii) we deduce $H(\mathscr{A}, \mathscr{B}) \geqq H(\mathscr{A}, \mathbf{1}) = H(\mathscr{A})$ and by iteration

$$h_\omega(\sigma, \mathscr{A}) = \lim_{k\to\infty} \frac{1}{kn} H_\omega(\mathscr{A}, \sigma\mathscr{A}, \dots, \sigma^n\mathscr{A}, \dots, \sigma^{kn-1}\mathscr{A})$$

$$\geqq \frac{1}{n} \lim_{k\to\infty} \frac{1}{k} H_\omega(\mathscr{A}, \sigma^n\mathscr{A}, \dots, \sigma^{n(k-1)}\mathscr{A}) = \frac{1}{n} h_\omega(\sigma^n, \mathscr{A}).$$

Conversely, (2.3, iii) tells us

$$\lim_{k\to\infty} \frac{1}{kn} H_\omega(\mathscr{A}, \sigma\mathscr{A}, \dots, \sigma^n\mathscr{A}, \dots, \sigma^{nk-1}\mathscr{A})$$

$$\leqq \frac{1}{n} \lim_{k\to\infty} \frac{1}{k} \{H_\omega(\mathscr{A}, \sigma^n\mathscr{A}, \dots, \sigma^{n(k-1)}\mathscr{A}) + H_\omega(\sigma\mathscr{A}, \sigma^{n+1}\mathscr{A}, \dots, \sigma^{n(k-1)+1}\mathscr{A})$$

$$+ \dots + H_\omega(\sigma^{n-1}\mathscr{A}, \sigma^{2n-1}\mathscr{A}, \dots, \sigma^{kn-1}\mathscr{A})\} = h_\omega(\sigma^n, \mathscr{A}).$$

(ii) $H_\omega(\mathscr{A}, \sigma\mathscr{A}, \dots, \sigma^n\mathscr{A}) = H_\omega(\sigma^{-n}\mathscr{A}, \sigma^{-n+1}\mathscr{A}, \dots, \mathscr{A}) = H_\omega(\mathscr{A}, \sigma^{-1}\mathscr{A}, \dots, \sigma^{-n}\mathscr{A})$ by (2.3, i).

(iii) Follows because $\sup_{\sum_I \omega_I = \omega}$ is invariant under automorphisms of $\mathscr{M}$.

3. Quantum K-Systems

Proposition (3.1). *Between the properties*

(i) $h_\omega(\sigma, \mathscr{A}) > 0 \,\forall\, \mathscr{A} \neq c\mathbf{1}, \mathscr{A} \subset \mathscr{M}$.
(ii) $\lim_{n\to\infty} h_\omega(\sigma^n, \mathscr{A}) = H_\omega(\mathscr{A}) \,\forall\, \mathscr{A} \neq c\mathbf{1}, \mathscr{A} \subset \mathscr{M}$.
(iii) $\lim_{n\to\infty} \varliminf_{k\to\infty} [H_\omega(\mathscr{B}, \sigma^{n+j_1}\mathscr{A}, \dots, \sigma^{n+j_k}\mathscr{A}) - H_\omega(\sigma^{n+j_1}\mathscr{A}, \dots, \sigma^{n+j_k}\mathscr{A})] = H_\omega(\mathscr{B}) \,\forall\, \mathscr{A}, \mathscr{B} \subset \mathscr{M},\ j_i \geqq 0$.
(iv) $\lim_{n\to\infty} \varliminf_{k\to\infty} [H_\omega(\mathscr{B}, \sigma^{n+j_1}\mathscr{A}, \dots, \sigma^{n+j_k}\mathscr{A}) - H_\omega(\sigma^{n+j_1}\mathscr{A}, \dots, \sigma^{n+j_k}\mathscr{A})] = 0 \Rightarrow \mathscr{B} = c\mathbf{1}$ $\forall\, \mathscr{A} \subset \mathscr{M},\ j_i \geqq 0$,

there are the implications

$$\begin{array}{ccc} \text{(ii)} & \Rightarrow & \text{(i)} \\ \Uparrow & & \Uparrow \\ \text{(iii)} & \Rightarrow & \text{(iv)}. \end{array}$$

In the commutative case they are equivalent.

Proof.

(iii)$\Rightarrow$(ii) By (2.3, iii) we have

$$[H_\omega(\mathscr{A},\sigma^n\mathscr{A},\dots,\sigma^{kn}\mathscr{A})-H_\omega(\sigma^n\mathscr{A},\dots,\sigma^{kn}\mathscr{A})]\leqq H_\omega(\mathscr{A}).$$

If, for sufficiently big n, $\underline{\lim}_{k\to\infty}[\ ]\geqq H_\omega(\mathscr{A})-\varepsilon$, then also the mean $(1/k)H_\omega(\mathscr{A},\sigma^n\mathscr{A},\dots,\sigma^{(k-1)n}\mathscr{A})$ has to approach $H_\omega(\mathscr{A})$.

(ii)$\Rightarrow$(i) follows from (2.10, i).
(iii)$\Rightarrow$(iv) follows from (2.7).
(iv)$\Rightarrow$(i) (iv) says that for $\mathscr{A}\neq c\mathbf{1}$ there is some n such that $h_\omega(\sigma^n,\mathscr{A})>0$, thus $h_\omega(\sigma,\mathscr{A})\geqq(1/n)h_\omega(\sigma^n,\mathscr{A})>0$.

For the converse implications in the commutative case, see Appendix 2.

Remarks (3.2).

1. It seems that for realistic quantum systems and σ the time translation $h_\omega(\sigma,\mathscr{A})$ is more instructive than the dynamical entropy $h_\omega(\sigma)=\sup_{\mathscr{A}} h_\omega(\sigma,\mathscr{A})$, since the later will be infinite in 3 dimensions. Only when combined with space translations one can get a finite dynamical entropy of a 3-dimensional abelian group.
2. Generally $h_\omega(\sigma^n,\mathscr{A})\leqq H_\omega(\mathscr{A})$ (see (2.3, iii)) and not decreasing in n. Thus we know that $\lim_{n\to\infty}$ in (ii) exists. On the other hand, we have neither a proof nor a counterexample for the strong subadditivity which would insure the existence of $\lim_{k\to\infty}$ in (iii) and (iv). Thus we have to make do with the limit inferior.
3. We do not have a counterexample which shows that the conditions (3.1) are not generally equivalent but at present we do not venture a conjecture.

We see that there are two possible generalizations of the positivity of h and two of the triviality of the tail, the latter implying the former. To us the condition (ii) seems the most suggestive one and we propose

Definition (3.3). Let σ be an automorphism of an UHF-algebra $\mathscr{M}$ and ω a faithful invariant state. We define an invariant memory loss of $(\mathscr{M},\sigma,\omega)$ by

$$m_\omega(\sigma)=\inf_{\substack{\mathscr{A}\neq c\mathbf{1}\\ \dim\mathscr{A}<\infty}}\lim_{n\to\infty}h_\omega(\sigma^n,\mathscr{A})/H_\omega(\mathscr{A}).$$

Generally $0\leqq m_\omega(\sigma)\leqq 1$. We call $(\mathscr{M},\sigma,\omega)$ a K-system, if $m_\omega(\sigma)=1$.

Remarks (3.4).

1. Remember that $H_\omega(\mathscr{A})>0\,\forall\,\mathscr{A}\neq c\mathbf{1}$, (2.7), so that $m_\omega(\sigma)$ is well defined.
2. We cannot offer a non-commutative version of a theorem of Krieger [7] which implies

$$h_\omega(\sigma)>0\Leftrightarrow\sup_{\substack{\mathscr{A}\neq c\mathbf{1}\\ \dim\mathscr{A}<\infty}}\lim_{n\to\infty}\frac{h_\omega(\sigma^n,\mathscr{A})}{\ln\dim\mathscr{A}}=1.$$

3. Intuitively speaking is $m_\omega(\sigma)$ the minimal percentagewise information gain by measurements after long intervals. For K-systems every subalgebra has 100% memory loss.

4. In contradistinction to $h_\omega(\sigma)$ the invariant $m_\omega(\sigma)$ depends on the completion of the embedding algebra $\mathcal{M}$. If $\mathcal{M}_0$ is a σ-invariant algebraic inductive limit of a net of finite subalgebras, $\mathcal{M}$ its norm closure and $\mathcal{M}''$ its weak closure in π_ω then $h_\omega(\sigma)$ is the same for $(\mathcal{M}_0, \sigma, \omega_{|\mathcal{M}_0})$, $(\mathcal{M}, \sigma, \omega)$ and $(\mathcal{M}'', \sigma, \langle\Omega|\cdots|\Omega\rangle)$ [5]. That even in the abelian case this is not the case for $m_\omega(\sigma)$ is shown by the following surprise. Classically all conditions (3.1) are equivalent to K-clustering if we consider the system $(\mathcal{M}'', \sigma, \omega)$ where $\mathcal{M}''$ is the von Neumann algebra of ω-integrable functions. Now one knows that clustering is lost by mixing of states. On the other hand,

$$h_\omega(\mathcal{A}, \sigma) = \lim_{k\to\infty} \frac{1}{k} S_\omega\left(\bigvee_{i=1}^{k} \sigma^i \mathcal{A}\right)$$

is concave in ω since S_ω is. Thus

$$h_{\omega_{1,2}}(\mathcal{A}, \sigma) > 0 \Rightarrow h_{\lambda\omega_1 + (1-\lambda)\omega_2}(\mathcal{A}, \sigma) > 0,$$

and hence there can be only one invariant state. If we start with the algebra $\mathcal{M}$ of continuous functions for which there are several invariant states the K-property cannot extend for all of them to the strong closure $\mathcal{M}''$ but we only have the implications

$\mathcal{M}$ is a K-system $\Leftarrow \mathcal{M}''$ is a K-system $\Leftrightarrow \mathcal{M}''$ is clustering $\Leftrightarrow \mathcal{M}$ is clustering.

Nevertheless the K-property has some kind of stability which follows from the

Covariance of the Memory Loss (3.5).

$$m_\omega(\sigma) = m_\omega(\sigma^{-1}) = m_\omega(\sigma^n) = m_{\omega\circ\alpha^{-1}}(\alpha\sigma\alpha^{-1}),$$

where $\alpha \in \mathrm{Aut}\,\mathcal{M}$ and $n \in \mathbf{Z}^+$.

Proof. (i) The first and the last equalities follow from (2.10, ii and iii) the other from the definition of m.

The conditions (3.1) require that any finite subalgebra has to keep changing under the evolution σ. This seems to contradict the usual situation where all observables converge towards their thermal expectation values. This puzzle is resolved by noticing that the convergence is weak and only strongly converging elements form converging algebras. In fact, in a faithful state ω strong convergence of any $a \neq c\mathbf{1}$ to $\omega(a)$ is impossible because $\sigma^n a \to \omega(a)$ implies $\sigma^n a^2 \to \omega(a)^2$, but we have seen (for $a^* = a$) that $\omega(a^2) > \omega(a)^2$. If the cyclic vector $|\Omega\rangle \in \mathcal{H}_\omega$ corresponding to ω is the only invariant vector no $a \neq c\mathbf{1}$ can converge strongly to any operator: Strong convergence of $\sigma^n a = U^{-n} a U^n$ requires that $\forall\, \varepsilon > 0 \exists N$ with

$$\|(U^{-n} a U^n - U^{-m} a U^m |\Omega\rangle\| = \|(U^{m-n} - 1) a |\Omega\rangle\| < \varepsilon \quad \forall m, n \geqq N.$$

Thus $(U-1)a|\Omega\rangle = 0$ or $(a - c\mathbf{1})|\Omega\rangle = 0$ for some $c \in \mathbf{C}$. Since $|\Omega\rangle$ is separating this implies $a = c\mathbf{1}$. (The Fock vacuum is not separating for the CAR-algebra and there the annihilation operators converge indeed strongly to zero for the free time evolution [8].) Thus the absence of strongly converging operators is not characteristic for K-systems but implied by our general setting. For K-systems also quasiperiodic elements are excluded and thus all finite quantum systems are excluded too. But also III_λ-factors with σ their modular automorphism do not qualify as K-systems.

Definition (3.6). Let Q be the set of finite subalgebras $\mathscr{A}$ which are quasiperiodic in the sense that $\forall \varepsilon > 0 \exists n \in \mathbf{Z}^+$ and $\Theta \in \operatorname{Aut} \mathscr{A}$ such that

$$\|(\sigma^n \Theta a - a)|\Omega\rangle\| \leqq \varepsilon \|a\| \quad \forall a \in \mathscr{A}.$$

Theorem (3.7). *For K-systems Q is trivial* (i.e. $Q = \{\{c\mathbf{1}\}\}$).

Remarks (3.8).

1. Q contains all σ-invariant finite subalgebras and a K-system can have none of those. The adjective finite is essential, there may be infinite-dimensional invariant subalgebras. For instance, in the CAR-algebra elements of the form $\sum a^*_{f_1} \cdots a^*_{f_k} a_{g_1} \cdots a_{g_k}, k \geqq n$ are for all $n \in \mathbf{Z}^+$ *-algebras. They are invariant under all evolutions which conserve the particle number and some of them lead to K-systems.
2. We have to insist on *-subalgebras because the finite algebra generated by an annihilation operator a_f in the Fock vacuum $|\Omega\rangle, a_f|\Omega\rangle = 0$ would qualify in (3.6) for any quasifree automorphism some of which may lead to a K-system.
3. Since there are classical K-systems on compact manifolds (3.7) might seem to contradict Poincaré's recurrence theorem. However, as has been pointed out previously (see f.i. [9]), Zermelo's recurrence objection does not hold for L^∞-functions as observables. Though almost all orbits in any neighbourhood keep coming back to it, they do it at different times such that functions never come close to their original form.

For the proof of (3.7) we need

Lemma (3.9). *Assume that for* $\sigma \in \operatorname{Aut} \mathscr{M}$ *we have*

$$\|(a - \sigma(a))|\Omega\rangle\| \leqq \varepsilon \|a\| \quad \forall a \in \mathscr{A}, \quad \dim \mathscr{A} = d.$$

Then there exists $c(d)$ such that

$$H_\omega(\mathscr{A}|\sigma\mathscr{A}) \leqq -c(d)\varepsilon \ln \varepsilon.$$

Proof. From the arguments which lead to (2.2, 1) we can also in the sup in (2.4) restrict ourselves to decompositions with $\omega_i(1) > \delta$ to get $H_\omega(\mathscr{A}|\sigma\mathscr{A})$ within ε_1. This number δ depends only on d and ε_1 and with the continuity of S used in the proof of (2.6) we reach the conclusion as

$$H_\omega(\mathscr{A}|\sigma\mathscr{A}) \leqq S_\omega(\mathscr{A}) - S_\omega(\sigma\mathscr{A}) + \sum_i \omega_i(1)[S_{\hat{\omega}_i}(\mathscr{A}) - S_{\hat{\omega}_i}(\sigma\mathscr{A})].$$

Now $\hat{\omega}_{i|\sigma\mathscr{A}} = \hat{\omega}_i \circ \sigma_{|\mathscr{A}}$ and

$$\|\hat{\omega}_{i|\mathscr{A}} - \hat{\omega}_i \circ \sigma_{|\mathscr{A}}\| \leqq \sup_{\|a\|=1} \frac{\langle\Omega|x_i(a - \sigma a)|\Omega\rangle}{\langle\Omega|x_i|\Omega\rangle} \leqq \varepsilon \frac{\langle\Omega|x_i^2|\Omega\rangle^{1/2}}{\langle\Omega|x_i|\Omega\rangle} \leqq \varepsilon\delta^{-1/2}$$

as $\mathscr{M}' \ni x_i < 1$. Since $\sum_i \omega_i(1) = 1$ we can appeal to the continuity of S to complete the proof.

Proof of (3.7). First of all the H's depend only on the algebras. Thus for $\Theta_i \in \mathrm{Aut}\,\mathscr{A}_i$ we have $H_\omega(\mathscr{A}_1,\ldots,\mathscr{A}_n)=H_\omega(\Theta_1\mathscr{A}_1,\ldots,\Theta_n\mathscr{A}_n)$ and we might ignore Θ. Secondly the invariance of ω under σ says $H_\omega(\sigma^k\mathscr{A}\,|\,\sigma^{k+1}\mathscr{A})=H_\omega(\mathscr{A}\,|\,\sigma\mathscr{A})$ and using (2.5, 2) we conclude

$$\lim_{n\to\infty}\lim_{r\to\infty}\frac{1}{r+1}H(\mathscr{A},\sigma^n\mathscr{A},\ldots,\sigma^{nr}\mathscr{A})\leqq\lim_{n\to\infty}H(\mathscr{A}\,|\,\sigma^n\mathscr{A})=0.$$

Thus a nontrivial Q would violate even the weaker condition (3.1, i).

So far our theory is based on an invariant state ω, but in physics one considers the dynamical system $(\mathscr{M},\sigma)$ as the primary object and quantities appearing only in $\pi_\omega(\mathscr{M})''$ as mathematical artefacts. To get a characteristic which does not refer to a particular state but depends only on the topological structure of $(\mathscr{M},\sigma)$, we introduce

Definition (3.10). The topological memory loss is $m(\sigma)=\inf_\omega m_\omega(\sigma)$, where inf goes over all faithful extremal invariant states. We call $(\mathscr{M},\sigma)$ a topological K-system, if $m(\sigma)=1$.

Remark (3.11). For a von Neumann algebraic system $(\mathscr{M}'',\sigma)$ there will be only one invariant state and there is no distinction between $m(\sigma)$ and $m_\omega(\sigma)$. However, in physics we have a C^*-algebraic system $(\mathscr{M},\sigma)$ with many inequivalent representations and there the distinction makes sense.

Invariance of the Topological Memory Loss (3.12).

$$m(\sigma)=m(\sigma^{-1})=m(\sigma^n)=m(\alpha\sigma\alpha^{-1}),$$

where $\alpha\in\mathrm{Aut}\,\mathscr{M}$, $n\in\mathbf{Z}^+$.

4. Examples of Quantum K-Systems

As in [5, 10] we shall first examine the generalization of the Bernoulli shift of the classical theory, i.e. the shift of the quasilocal CAR-algebra.

Theorem (4.1). *$(\mathscr{M},\sigma)$ with $\mathscr{M}$ the C^*-algebra generated by even powers of a_f and* $\sigma a_f=a_{\sigma f}, (\sigma f)(\vec{x})=f(\vec{x}+\vec{s})$, $s\in\mathbf{R}^\nu\backslash\{0\}$ is a topological K-system.

Proof. Since for all faithful states ω $H_\omega(\mathscr{A})>0\,\forall\,\mathscr{A}\neq c\mathbf{1}$ it suffices to verify that $\forall\,\varepsilon>0\,\exists\,n$ with $h_\omega(\sigma^n,\mathscr{A})>H_\omega(\mathscr{A})-\varepsilon$. According to (2.2, 2) we may choose the x_i in $\omega_i(a)=\omega(\sigma^\omega_{i/2}(x_i)a)$ strictly local such that $[\sigma^{nk}x_i,x_j]=0\,\forall\,i,j,k>0$ for n sufficiently big. Therefore $x_{I_k}=\prod_{l=1}^{k}\sigma^{n(l-1)}x_{i_l}$ is a candidate for x_I in the decomposition for $H_\omega(\mathscr{A},\sigma^n\mathscr{A},\ldots,\sigma^{n(k-1)}\mathscr{A})$. We estimate

$$\begin{aligned}h_\omega(\sigma^n,\mathscr{A})&=\lim_{k\to\infty}\frac{1}{k}H_\omega(\mathscr{A},\sigma^n\mathscr{A},\ldots,\sigma^{n(k-1)}\mathscr{A})\\&\geqq\lim_{k\to\infty}\frac{1}{k}\sum_{I_k}\eta(\omega(x_{I_k}))-\sum_i\eta(\omega(x_i))+H_\omega(\mathscr{A})-\varepsilon,\end{aligned}$$

if the x_i give $H_\omega(\mathscr{A})$ within ε. Now consider the abelian algebra $\mathscr{M}_a = \bigotimes_{k=-\infty}^{\infty} J_k$, each J_k being $\{1,2,\ldots,r\}$, $r = \# I_1 < \infty$. The shift $\sigma_a J_k = J_{k+1}$ is an automorphism of $\mathscr{M}_a$ and $\omega(x_I)$ a state ω_a over $\mathscr{M}_a$ with $\omega_a \circ \sigma_a = \omega_a$. The quasilocal structure of $\mathscr{M}$ and the extremal invariance of ω imply already the following clustering [11]: For all strictly local $a \in \mathscr{M}$ and $\varepsilon > 0 \, \exists \, \Lambda \subset \mathbf{R}^\nu$ such that

$$|\omega(xa) - \omega(x)\omega(a)| \leqq \varepsilon \|x\| \quad \forall x \in \mathscr{A}_{\Lambda^c}.$$

This implies that $(\mathscr{M}_a, \sigma_a, \omega_a)$ is K-mixing (see (Appendix B, (iv)) and therefore a classical K-system. For them the properties (3.1) imply

$$\lim_{n\to\infty} \lim_{k\to\infty} \frac{1}{k} \sum_{I_k} \eta(\omega(x_{I_k})) = \lim_{n\to\infty} h_{\omega_a}(\sigma_a^n, I_1) = H_{\omega_a}(I_1) = \sum_i \eta(\omega(x_i)).$$

Corollary (4.2). *$(\mathscr{M}, \alpha\sigma\alpha^{-1})$ is for all $\alpha \in \operatorname{Aut}\mathscr{M}$ a topological K-system.*

Proof. Follows from (3.12).

Examples (4.3).

1. Consider $\nu = 1$ and a quasifree time evolution $\tau_t^0 a_f = a_{f_t}$, $\tilde{f}_t(k) = e^{-i\varepsilon(k)}\tilde{f}(k)$ with $\tilde{f}$ the Fourier transform of f. If ε is a strictly monotonic function with $1/\varepsilon'(k)$ integrable, this automorphism is conjugate to the shift which reads in Fourier space $\tilde{f}(k) \to e^{isk}\tilde{f}(k)$. $\tilde{f}(k) \to g(\varepsilon) \equiv 1/\sqrt{\varepsilon'(\varepsilon)}\, f(k(\varepsilon))$ is a unitary map $L^2(\mathbf{R}, dk) \to L^2(\mathbf{R}, d\varepsilon)$ and $\alpha^{-1} a_f = a_g$ is an automorphism of $\mathscr{M}$ such that $\alpha\sigma_t\alpha^{-1} = \tau_t^0$. Thus $(\mathscr{M}, \tau)$ is a topological K-system.
2. Introduce in example 1) an external potential such that the Møller operator

$$\Omega_+ = \lim_{t\to\infty} e^{iHt} e^{-iH_0 t}$$

exists and is complete. (H_0, H generate τ^0, respectively τ). If H has no bound state then $(\mathscr{M}, \tau)$ are a topological K-system since $\Omega_t e^{-iH_0 t} \Omega_t^{-1} = e^{-iHt}$ and thus τ and τ^0 are conjugate. If there is a bound state f_b then $(\mathscr{M}, \tau, \omega)$ is for no ω a K-system since a_{f_b} generate a finite invariant subalgebra.

Unfortunately, so far we are not able to control the tail properties in this generality. We can show only in a special case that the strongest condition (3.1, iii) is not empty.

Proposition (4.4). *Let σ be the shift on a quantum lattice system, τ the tracial state and $\mathscr{A}$ strictly local. Then*

$$\lim_{n\to\infty} \varliminf_{k\to\infty} [H_\tau(\mathscr{B}, \sigma^n\mathscr{A}, \sigma^{n+j_1}\mathscr{A}, \ldots, \sigma^{n+j_k}\mathscr{A}) - H_\tau(\sigma^n\mathscr{A}, \sigma^{n+j_1}\mathscr{A}, \ldots, \sigma^{n+j_k}\mathscr{A}] = H_\tau(\mathscr{B})$$

$\forall j_i \geqq 0$, $\mathscr{B}$ any finite subalgebra of $\mathscr{M}$.

Proof. In the tracial state any subalgebra is invariant under modular automorphism and thus $\forall \bar{\mathscr{A}} \subset \mathscr{M}$ exists the canonical conditional expectation $\gamma: \mathscr{M} \to \bar{\mathscr{A}}$ which conserves $\tau: \tau = \tau_{|\bar{\mathscr{A}}} \circ \gamma$. Similarly in the decomposition for $H_\tau(\mathscr{A}_1, \ldots, \mathscr{A}_n)$ one has $x_I \in \left(\bigvee_{k=1}^{n} \mathscr{A}_k\right)'' \equiv \bar{\mathscr{A}}$ since $\tau_I(a) = \tau(x_I a) = \tau(\gamma(x_I)a) \forall a \in \bar{\mathscr{A}}$.

Thus in the decomposition for $H_\tau(\sigma^n\mathscr{A}, \sigma^{n+j_1}\mathscr{A}, \ldots, \sigma^{n+j_k}\mathscr{A})$ we can take the $x_I \in \left(\bigcup_{j=1}^{\infty} \sigma^{n+j}\mathscr{A}\right)''$. $H(\mathscr{B})$ can be obtained, within ε, by strictly local y_j. Thus, for sufficiently big n, x_I and y_j commute and $\tau(x_I y_j) = \tau(x_I)\tau(y_j) \forall I, j$. Then all condition of (2.6) are met and this proves Proposition (4.4).

Appendix A

In the classical theory one defines

$$H_\omega(\mathscr{A}|\mathscr{B}) = H_\omega(\mathscr{A} \vee \mathscr{B}) - H_\omega(\mathscr{B}). \tag{A.1}$$

We have to prove that it coincides with

$$H_\omega(\mathscr{A}|\mathscr{B}) = \sup_{\sum \omega_i = \omega} \sum_i (S(\omega|\omega_i)_{\mathscr{A}} - S(\omega|\omega_i)_{\mathscr{B}}) \tag{A.2}$$

in the abelian situation. Using in (A.2) for the ω_i the minimal projectors P_j of $\mathscr{A}, \omega_j(a) = \omega(P_j a)$ the right-hand side becomes $S_\omega(\mathscr{A} \vee \mathscr{B}) - S_\omega(\mathscr{B})$. There only remains to show that no other decomposition $\omega_i(a) = \omega(Q_i a)$ can give more. Now

$$\begin{aligned}\sum_i (S(\omega|\omega_i)_{\mathscr{A}} - S(\omega|\omega_i)_{\mathscr{B}}) &= \sum_j \eta(\omega(P_j)) - \sum_{i,j} \eta(\omega(P_j Q_i)) - \sum_k \eta(\omega(R_k)) + \sum_{i,k} \eta(\omega(R_k Q_i)) \\ &= S_\omega(\mathscr{A}) - S_\omega(\mathscr{A} \vee \mathbf{C}) - S_\omega(\mathscr{B}) + S_\omega(\mathscr{B} \vee \mathbf{C}).\end{aligned}$$

Here R_k are the minimal projectors in $\mathscr{B}$ and formally we considered $\omega(Q_i P_j R_k)$ as the state over the probability space (i, j, k) with $\mathbf{C}$ the elements depending only on i. Now monotonicity and strong subadditivity say

$$\begin{aligned}&S_\omega(\mathscr{A}) - S_\omega(\mathscr{A} \vee \mathbf{C}) - S_\omega(\mathscr{B}) + S_\omega(\mathscr{B} \vee \mathbf{C}) \\ &\quad \leqq S_\omega(\mathscr{A}) - S_\omega(\mathscr{A} \vee \mathbf{C}) - S_\omega(\mathscr{B}) + S_\omega(\mathscr{A} \vee \mathscr{B} \vee \mathbf{C}) \\ &\quad \leqq S_\omega(\mathscr{A} \vee \mathscr{B}) - S_\omega(\mathscr{B}).\end{aligned}$$

Appendix B

Classically a K-system $(\mathscr{M}, T, \omega)$ is characterized by the following equivalent conditions [12]

(i) $\exists \mathscr{M}_0 \subset \mathscr{M}$ with

1. $T\mathscr{M}_0 \supset \mathscr{M}_0$,
2. $\bigvee_{n=-\infty}^{\infty} T^n \mathscr{M}_0 = \mathscr{M}$,
3. $\bigwedge_{n=-\infty}^{\infty} T^n \mathscr{M}_0 = c\mathbf{1}$.

(ii) $\bigwedge_{n=0}^{-\infty} \bigvee_{k=-\infty}^{n} T^k \mathscr{A} = c\mathbf{1}$ for all finite $\mathscr{A} \subset \mathscr{M}$.

(iii) $h(T, \mathscr{A}) > 0 \forall \mathscr{A}, 1 < \dim \mathscr{A} < \infty$.

(iv) T is K-mixing. This means for all finite $\mathscr{A} \subset \mathscr{M}$, $A \in \mathscr{M}$ and $\varepsilon > 0 \exists N$ such that

$$|\omega(A\sigma^n B) - \omega(A)\omega(B)| < \varepsilon \|B\| \quad \forall B \in \bigvee_{k=0}^{\infty} \sigma^k \mathscr{A}, \quad n > N.$$

We add now some more equivalences. ($\mathscr{A}$ and $\mathscr{B}$ are finite subalgebras.)

(v) $\lim_{n\to\infty} h(T^n, \mathscr{A}) = H(\mathscr{A})$,

(vi) $\lim_{n\to\infty} H\left(\mathscr{B}\middle|\bigvee_{k=n}^{\infty} T^k \mathscr{A}\right) = H(\mathscr{B})$,

(vii) $\lim_{n\to\infty} H\left(\mathscr{B}\middle|\bigvee_{k=n}^{\infty} T^k \mathscr{A}\right) = 0 \Rightarrow \mathscr{B} = c\mathbf{1}$.

To show the equivalence we appeal to [12, 13]

Lemma.

(a) *$H(\mathscr{B}|\mathscr{A})$ is continuous for monotonic limits in both arguments,*
(b) *$H(\mathscr{B}|\mathscr{A}) = 0 \Leftrightarrow \mathscr{B} \subset \mathscr{A}$.*

It says (ii)$\Leftrightarrow$(vii), (ii)$\Rightarrow$(vi). *Next we argue that* (vi)$\Rightarrow$(v), *because*

$$H(\mathscr{A}) = \lim_{n\to\infty} H\left(\mathscr{A}\middle|\bigvee_{k=n}^{\infty} T^k \mathscr{A}\right) \leqq \lim_{n\to\infty} H\left(\mathscr{A}\middle|\bigvee_{s=1}^{\infty} T^{ns} \mathscr{A}\right) = \lim_{n\to\infty} h(T^n, \mathscr{A}) \leqq H(\mathscr{A}).$$

Finally (v)$\Rightarrow$(iii) *because* $h(T, \mathscr{A}) \geqq (1/n) h(T^n, \mathscr{A})$.

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