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WHEN IS THE TANGENT SPHERE BUNDLE CONFORMALLY FLAT?

Dedicated to Professor N. K. Stephanidis on his 65th Birthday

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1 INTRODUCTION

In [2] one of the authors showed that the standard contact metric structure on the tangent sphere bundle is locally symmetric if and only if the base manifold is flat or of dimension 2 and of constant curvature +1. In this paper we show that this structure is conformally flat if and only if the base manifold is a surface of constant Gaussian curvature 0 or +1. In the final section of the paper we give an additional result on conformally flat contact metric manifolds.

2 PRELIMINARIES

A differentiable (2n+1)-dimensional manifold M^{2n+1} is called a *contact manifold* if it carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M^{2n+1}. It is well known that given η there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$ called the *characteristic vector field* of the contact structure η. A Riemannian metric g is an *associated metric* to a contact structure η if there exists a tensor field ϕ of type (1,1) satisfying

$$\phi^2 = -Id + \eta \otimes \xi, \quad \eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y). \tag{2.1}$$

We refer to (η, g) or (ϕ, ξ, η, g) as a *contact metric structure.*

Denoting by $\mathcal{L}$ and R, Lie differentiation and the curvature tensor respectively, we define operators l and h by

$$lX = R(X, \xi)\xi, \quad h = \frac{1}{2}\mathcal{L}_\xi \phi \tag{2.2}$$

The tensor fields h and l are self-adjoint and satisfy

$$h\xi = 0, \quad l\xi = 0, \quad Trh = 0, \quad Trh\phi = 0, \quad h\phi + \phi h = 0.$$

A contact metric structure is *K-contact* if ξ is a Killing field; this is the case if and only if $h = 0$. If the structure is normal it is *Sasakian*; a Sasakian structure is K-contact but not conversely for dimensions > 3. Also on a contact metric manifold we have the following general formulas (see e.g. [1,2]).

$$\nabla_X \xi = -\phi X - \phi h X \quad (\text{and so} \nabla_\xi \xi = 0) \tag{2.3}$$

$$\frac{1}{2}(-l + \phi l \phi) = h^2 + \phi^2 \tag{2.4}$$

$$\nabla_\xi h = \phi - \phi h^2 - \phi l \tag{2.5}$$

$$\nabla_\xi \phi = 0 \tag{2.6}$$

where ∇ is the Riemannian connection of g.

A Riemannian manifold M^n is said to be *conformally flat* if it is locally conformally equivalent to a Euclidean space. On M^n we denote by Q the Ricci operator, by $R = TrQ$ the scalar curvature and by P the tensor field $P = -Q + \frac{R}{4} Id$. It is well known that a Riemannian manifold M^n is conformally flat if and only if

$$R(X,Y)Z = \frac{1}{n-2}(g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y)$$
$$-\frac{R}{(n-1)(n-2)}(g(Y,Z)X - g(X,Z)Y) \quad \text{for } n > 3 \tag{2.7}$$

and

$$(\nabla_X P)Y = (\nabla_Y P)X \quad \text{for } n = 3 \tag{2.8}$$

in which case (2.7) (with $n = 3$) holds.

3 REVIEW OF THE TANGENT SPHERE BUNDLE

Let M be an (n+1)-dimensional C^∞ manifold and $\bar{\pi} : TM \longrightarrow M$ its tangent bundle. If $(x^1, \ldots, x^{n+1})$ are local coordinates on M, set $q^i = x^i \circ \bar{\pi}$; then $(q^1, \ldots, q^{n+1})$ together with the fibre coordinates $(v^1, \ldots, v^{n+1})$ form local coordinates on TM. If X is a vector field on M, its *vertical lift* X^V on TM is the vector field defined by $X^V \omega = \omega(X) \circ \bar{\pi}$ where ω is a 1-form on M, which on the left side of this equation is regarded as a function on TM. For an affine connection D on M, the *horizontal lift* X^H of X is defined by $X^H \omega = D_X \omega$. The local expression for X^H in terms of the connection coefficients of D is

$$X^H = X^i \frac{\partial}{\partial q^i} - X^i v^j \Gamma^k_{ij} \frac{\partial}{\partial v^k}. \tag{3.1}$$

The *connection map* $K : TTM \longrightarrow TM$ is defined by

$$KX^H = 0,\ K(X_t^V) = X_{\bar{\pi}(t)},\ t \in TM.$$

TM admits an almost complex structure J defined by

$$JX^H = X^V,\ JX^V = -X^H.$$

Dombrowski [5] showed that J is integrable if and only if D has vanishing curvature and torsion.

If now G is a Riemannian metric on M and D its Levi-Civita connection, we define a Riemannian metric $\bar{g}$ on TM called the *Sasaki metric* (not to be confused with a Sasakian structure), by

$$\bar{g}(X,Y) = G(\bar{\pi}_*X, \bar{\pi}_*Y) + G(KX, KY)$$

where X and Y are vector fields on TM. Since $\bar{\pi}_* \circ J = -K$ and $K \circ J = \bar{\pi}_*$, $\bar{g}$ is Hermitian for the almost complex structure J.

On TM define a 1-form β by $\beta(X)_t = G(t, \bar{\pi}_*X)$, $t \in TM$ or equivalently by the local expression $\beta = \sum G_{ij}v^i dq^j$. Then $d\beta$ is a symplectic structure on TM and in particular $2d\beta$ is the fundamental 2-form of the almost Hermitian structure $(J, \bar{g})$. Thus TM has an almost Kähler structure which is Kählerian if and only if (M, G) is flat (Dombrowski [5], Tachibana and Okumura [11]).

Let $\mathbf{R}$ denote the curvature tensor of G, $\bar{\nabla}$ the Levi-Civita connection of $\bar{g}$ and $\bar{R}$ the curvature tensor of $\bar{g}$. Then $\bar{\nabla}$ and $\bar{R}$ are given by [8]

$$(\bar{\nabla}_{X^H}Y^H)_t = (D_XY)_t^H - \frac{1}{2}(\mathbf{R}(X,Y)t)_t^V,$$

$$(\bar{\nabla}_{X^H}Y^V)_t = -\frac{1}{2}(\mathbf{R}(Y,t)X)_t^H + (D_XY)_t^V,$$

$$(\bar{\nabla}_{X^V}Y^H)_t = -\frac{1}{2}(\mathbf{R}(X,t)Y)_t^H,$$

$$\bar{\nabla}_{X^V}Y^V = 0 \tag{3.2}$$

and

$$\bar{R}(X^V, Y^V)Z^V = 0$$

$$(\bar{R}(X^V, Y^V)Z^H)_t = \big(\mathbf{R}(X,Y)Z + \frac{1}{4}\mathbf{R}(t,X)\mathbf{R}(t,Y)Z - \frac{1}{4}\mathbf{R}(t,Y)\mathbf{R}(t,X)Z\big)_t^H,$$

$$(\bar{R}(X^H, Y^V)Z^V)_t = -\big(\frac{1}{2}\mathbf{R}(Y,Z)X + \frac{1}{4}\mathbf{R}(t,Y)\mathbf{R}(t,Z)X\big)_t^H,$$

$$(\bar{R}(X^H, Y^V)Z^H)_t = \big(\frac{1}{2}\mathbf{R}(X,Z)Y + \frac{1}{4}\mathbf{R}(\mathbf{R}(t,Y)Z,X)t\big)_t^V + \frac{1}{2}\big((D_X\mathbf{R})(t,Y)Z\big)_t^H,$$

$$(\bar{R}(X^H,Y^H)Z^V)_t = \big(\mathbf{R}(X,Y)Z + \frac{1}{4}\mathbf{R}(\mathbf{R}(t,Z)Y,X)t - \frac{1}{4}\mathbf{R}(\mathbf{R}(t,Z)X,Y)t\big)^V_t$$
$$+\frac{1}{2}\big((D_X\mathbf{R})(t,Z)Y - (D_Y\mathbf{R})(t,Z)X\big)^H_t,$$
$$(\bar{R}(X^H,Y^H)Z^H)_t = \frac{1}{2}\big((D_Z\mathbf{R})(X,Y)t\big)^V_t + \big(\mathbf{R}(X,Y)Z + \frac{1}{4}\mathbf{R}(t,\mathbf{R}(Z,Y)t)X$$
$$+\frac{1}{4}\mathbf{R}(t,\mathbf{R}(X,Z)t)Y + \frac{1}{2}\mathbf{R}(t,\mathbf{R}(X,Y)t)Z\big)^H_t. \tag{3.3}$$

The tangent sphere bundle $\pi : T_1M \longrightarrow M$ is the hypersurface of TM defined by $\sum G_{ij}v^iv^j = 1$. The vector field $N = v^i\frac{\partial}{\partial v^i}$ is a unit normal, as well as the position vector for a point t. The Weingarten map A of T_1M with respect to the normal N is given by $AU = -U$ for any vertical vector U and $AX = 0$ for any horizontal vector X (see e.g. [1,p.132]). Thus many computations on T_1M involving horizontal vector fields can be done directly on TM. In particular let g' denote the metric on T_1M induced from $\bar{g}$ on TM, R' its curvature tensor and α the second fundamental form. The Gauss equation is then

$$R'(X,Y,Z,W) = \bar{R}(X,Y,Z,W) + \alpha(Y,Z)\alpha(X,W) - \alpha(X,Z)\alpha(Y,W). \tag{3.4}$$

Define ϕ', ξ' and η' on T_1M by

$$\xi' = -JN,\ \ JX = \phi'X + \eta'(X)N.$$

η' is the contact form on T_1M induced from the 1-form β on TM as one can easily check. However $g'(X,\phi'Y) = 2d\eta'(X,Y)$, so strictly speaking (ϕ',ξ',η',g') is not a contact metric structure. Of course the difficulty is easily rectified and

$$\eta = \frac{1}{2}\eta',\ \xi = 2\xi',\ \phi = \phi',\ g = \frac{1}{4}g'$$

is taken as the standard contact metric structure on T_1M. In local coordinates

$$\xi = 2v^i(\frac{\partial}{\partial x^i})^H.$$

On TM the vector field $v^i(\frac{\partial}{\partial x^i})^H$ is the so-called geodesic flow. Further results on TM and T_1M, including other metrics and techniques may be found in [9]. For more on the tangent sphere bundle specifically see [6,15].

4 CONFORMALLY FLAT TANGENT SPHERE BUNDLES

THEOREM 4.1: Let M be an (n+1)-dimensional Riemannian manifold and T_1M its tangent sphere bundle with the above Riemannain structure. Then T_1M is conformally flat if and only if M is a surface of constant Gaussian curvature 0 or +1.

PROOF: Case 1: $n \geq 2$. Since conformal flatness is invariant under a homothetic change of metric, we will work with the metric g' described in the preceding section. Our computations will make use of the condition for conformal flatness (2.7), the Gauss equation (3.4) of T_1M in TM and form of the curvature tensor on TM (3.3). For vertical vector fields U, V, W we have

$$\begin{aligned} 0 &= (g'(V,W)g'(U,U) - g'(U,W)g'(V,U)) \\ &-\frac{1}{2n-1}(g'(V,W)g'(Q'U,U) - g'(U,W)g'(Q'V,U) \\ &+g'(Q'V,W)g'(U,U) - g'(Q'U,W)g'(V,U)) \\ &+\frac{R'}{2n(2n-1)}(g'(V,W)g'(U,U) - g'(U,W)g'(V,U)). \end{aligned}$$

Thus for $n \geq 3$, choosing $\{U, V, W\}$ orthonormal we see that

$$g'(Q'V, W) = 0 \tag{4.1}$$

for any orthonormal pair $\{V, W\}$. Similarly computing $R'(U, V, V, X)$ for an orthonormal pair of vertical vectors $\{U, V\}$ and a horizontal vector X, we get

$$g'(Q'U, X) = 0 \tag{4.2}$$

for all $n \geq 2$. In particular for $n \geq 3$, we see that $Q'V$ is collinear with V for any vertical vector V.

Now let X be tangent to M and consider its horizontal lift X^H; also let $\{U, V\}$ be an orthonormal pair of vertical vectors and $t \in T_1M$. Then computing $R'(U, X^H, V, X^H)_t$ using (3.3) and (4.1) we have for $n \geq 3$

$$G(\mathbf{R}(t, KU)X, \mathbf{R}(t, KV)X) = 0. \tag{4.3}$$

Similarly let $\{X, Y, Z\}$ be tangent to M, U a vertical vector and $t \in T_1M$; then using (4.2), compute $R'(X^H, Y^H, Z^H, U)_t$ to obtain

$$G((D_Z\mathbf{R})(X, Y)t, KU) = 0, \tag{4.4}$$

i.e. M is locally symmetric. Taking $\{X, Y\}$ tangent to M and $\{U, V\}$ vertical vectors, computation of $R'(U, V, X^H, Y^H)_t$ yields

$$G(\mathbf{R}(KU, KV)X, Y) + \frac{1}{4}G(\mathbf{R}(t, KU)\mathbf{R}(t, KV)X, Y) - \frac{1}{4}G(\mathbf{R}(t, KV)\mathbf{R}(t, KU)X, Y) = 0. \tag{4.5}$$

For $n \geq 3$ we first have from (4.3) that

$$G(\mathbf{R}(t, KU)t, \mathbf{R}(t, KV)t) = 0;$$

thus for $\{t, X, Y\}$ orthonormal on M

$$G(\mathbf{R}(t, \mathbf{R}(t, X)t)t, Y) = 0. \tag{4.6}$$

Define $L_t : [t]^\perp \longrightarrow [t]^\perp$ by

$$L_t X = \mathbf{R}(t, X)t.$$

L_t is a symmetric operator on $[t]^\perp$ and by (4.6)

$$L_t^2 X = \alpha_t X, \quad \alpha_t \geq 0$$

so the eigenvalues of L_t are $\pm\sqrt{\alpha_t}$.

We now consider the case that M is irreducible. By (4.4) M is locally symmetric, so for $n = 2$, M is 3-dimensional and of constant curvature. For $n \geq 3$, note that the sectional curvature of an irreducible locally symmetric space does not change sign. Therefore L_t has only one eigenvalue and hence $\mathbf{R}(t, X)t$ is collinear with X. Thus $G(\mathbf{R}(t, X)t, Y) = 0$ for any orthonormal triple $\{t, X, Y\}$ and hence M is of constant curvature. So for M irreducible and $n \geq 2$, M is of constant curvature c and by a homothetic change we may assume that $c = +1$, 0 or -1.

Recall that the contact metric structure on T_1M is Sasakian if and only if the base manifold is of constant curvature +1 (Tashiro [15]) and that every conformally flat K-contact manifold is of constant curvature +1 (Tanno [12,13]). Moreover the contact metric structure on T_1M is locally symmetric if and only if the base manifold is flat or is 2-dimensional and of constant curvature +1 [2] We remark here that in dimension 3 a contact metric manifold is locally symmetric if and only if it is of constant curvature 0 or +1 [4]; thus we have the converse direction of Theorem 5.1. Now we have that for $n \geq 2$, c cannot be +1 and since $S^n \times E^{n+1}$ is not conformally flat, c cannot be 0. If $c = -1$, set $\{X = KV, Y = KU, t\}$ as an orthonormal triple on M; then (4.5) gives $0 = -\frac{5}{4}$, a contradiction.

Now suppose that M is reducible. If M is flat we have a contradiction as above. So in (4.5) choose $X = KV$ orthogonal to $Y = KU$ tangent to a non-flat factor and t tangent to a different factor. Then the last two terms in (4.5) vanish and we have $G(\mathbf{R}(X, Y)Y, X) = 0$, a contradiction.

Case 2: $n = 1$. Since the condition for conformal flatness is given by (2.8), we first compute the Ricci operator Q' of g' on T_1M for a surface M. Let $\{X_1, X_2\}$ be the orthonormal pair $\{t, KU\}$ on M where U is vertical on T_1M, then summing on $i = 1, 2$ we compute using (3.3) and (3.4) as before,

$$g'(Q'U, U)_t = R'(U, X_i^H, X_i^H, U)_t = -G(\frac{1}{4}\mathbf{R}(\mathbf{R}(t, KU)X_i, X_i)t, KU)$$

$$= \frac{1}{4}G(\mathbf{R}(t, KU)X_i, \mathbf{R}(t, KU)X_i) = \frac{1}{4}|\mathbf{R}(t, KU)t|^2 + \frac{1}{4}|\mathbf{R}(t, KU)KU|^2 = \frac{1}{2}\mathbf{K}^2$$

where $\mathbf{K}$ is the Gaussian curvature of M. Similarly for a horizontal lift X^H,

$$g'(Q'U, X^H) = -\frac{1}{2}G((D_{X_i}\mathbf{R})(t, KU)X_i, X)$$

$$= -\frac{1}{2}G((X_i\mathbf{K})(G(X_i, KU)t - G(X_i, t)KU), X).$$

To continue our computation we introduce the following horizontal vectors on T_1M. Z will denote the geodesic flow, i.e. $Z = \frac{1}{2}\xi$, and $Z^\perp$ the horizontal vector corresponding to KU, i.e. $Z^\perp = U^i\big(\frac{\partial}{\partial x^i}\big)^H$. It is well known that for the geodesic flow $\nabla'_Z Z = 0$ and similarly, using (3.1), (3.2) and differentiating $G_{ij}U^iv^j = 0$, one can show that $\nabla'_{Z^\perp} Z^\perp = 0$.

Now

$$g'(Q'U, Z)_t = -\frac{1}{2}KU\mathbf{K}, \quad g'(Q'U, Z^\perp)_t = \frac{1}{2}t\mathbf{K}.$$

Continuing we have

$$g'(Q'Z, Z)_t = R'(Z, U, U, Z)_t + R'(Z, Z^\perp, Z^\perp, Z)_t$$

$$= -\frac{1}{4}G(\mathbf{R}(t, KU)\mathbf{R}(t, KU)t, t)$$

$$+G(\mathbf{R}(t, KU)KU + \frac{1}{4}\mathbf{R}(t, \mathbf{R}(t, KU)t)KU + \frac{1}{2}\mathbf{R}(t, \mathbf{R}(t, KU)t)KU, t)$$

$$= \frac{1}{4}\mathbf{K}^2 + \mathbf{K} - \frac{3}{4}\mathbf{K}^2 = \mathbf{K} - \frac{1}{2}\mathbf{K}^2.$$

Similarly we obtain

$$g'(Q'Z, Z^\perp)_t = 0, \quad g'(Q'Z^\perp, Z^\perp)_t = \mathbf{K} - \frac{1}{2}\mathbf{K}^2.$$

Thus the Ricci operator Q' is given by

$$Q'U = \frac{1}{2}(\mathbf{K}^2)U - \frac{1}{2}(KU\mathbf{K})Z + \frac{1}{2}(t\mathbf{K})Z^\perp,$$

$$Q'Z = -\frac{1}{2}(KU\mathbf{K})U + (\mathbf{K} - \frac{1}{2}\mathbf{K}^2)Z,$$

$$Q'Z^\perp = \frac{1}{2}(t\mathbf{K})U + (\mathbf{K} - \frac{1}{2}\mathbf{K}^2)Z^\perp, \tag{4.7}$$

and the scalar curvature is

$$R' = 2\mathbf{K} - \frac{1}{2}\mathbf{K}^2.$$

Since T_1M is conformally flat,

$$(\nabla'_Z Q')Z^\perp - (\nabla'_{Z^\perp} Q')Z = \frac{1}{4}((ZR')Z^\perp - (Z^\perp R')Z) \tag{4.8}$$

and our proof will be to expand this equation. For this we first need to compute $\nabla'_Z Z^\perp$, $\nabla'_{Z^\perp} Z$, $\nabla'_Z U$ and $\nabla'_{Z^\perp} U$. Since the second fundatmental form of T_1M in TM vanishes on horizontal vectors, we may use (3.2) to compute these. The results are

$$\nabla'_Z Z^\perp = \frac{\mathbf{K}}{2}U, \quad \nabla'_{Z^\perp} Z = -\frac{\mathbf{K}}{2}U, \quad \nabla'_Z U = -\frac{\mathbf{K}}{2}Z^\perp, \quad \nabla'_{Z^\perp} U = \frac{\mathbf{K}}{2}Z. \tag{4.9}$$

Now expanding (4.8) using (4.7) and (4.9) and taking the U-component we obtain

$$\frac{1}{2}tt\mathbf{K} + \mathbf{K}(\mathbf{K} - \frac{1}{2}\mathbf{K}^2) - \frac{1}{2}\mathbf{K}^3 + \frac{1}{2}KUKUK\mathbf{K} = 0. \tag{4.10}$$

Similarly both the Z and $Z^{\perp}$-components yield

$$(1 - 3\mathbf{K})t\mathbf{K} = 0.$$

From this we see that $\mathbf{K}$ must be a constant and then (4.10) simplifies to $\mathbf{K}^2 - \mathbf{K}^3 = 0$ giving $\mathbf{K} = 0$ or 1.

5 IMPLICATION OF $Q\phi = \phi Q$

On a Sasakian manifold the Ricci tensor satisfies $Q\phi = \phi Q$, but in general $Q\phi \neq \phi Q$ [7]. Some results concerning the Ricci and scalar curvatures of a conformally flat contact metric manifold have been obtained in [10,14]. Moreover in [3] it is shown that the critical point condition of the integral of the scalar curvature over a compact contact manifold considered as a functional on the space of all associated metrics is that Q commutes with ϕ when restricted to the contact subbundle $\{\eta = 0\}$. In this section we prove that every conformally flat contact metric manifold M^{2n+1} on which Q commutes with ϕ is of constant curvature.

First from (2.2) and (2.7) we have

$$lX = \frac{1}{2n-1}(QX - g(X,\xi)Q\xi + g(Q\xi,\xi)X - g(QX,\xi)\xi) - \frac{R}{2n(2n-1)}(X - g(X,\xi)\xi). \tag{5.1}$$

Since $g(\phi X, Y) = -g(X, \phi Y)$, $\phi\xi = 0$ and $Q\phi = \phi Q$ we obtain from (5.1) $\phi l = l\phi$. This together with equations (2.4) and (2.5) yields $-l = \phi^2 + h^2$, and $\nabla_\xi h = 0$ Differentiating $-l = \phi^2 + h^2$ with respect to ξ and using (2.6) we have

$$\nabla_\xi l = 0 \ \ (\text{and so } \xi\, Trl = 0). \tag{5.2}$$

Since $\phi^2 Q\xi = -Q\xi + g(Q\xi,\xi)\xi$ and $\phi^2 Q\xi = \phi Q\phi\xi = 0$ we get $Q\xi = g(Q\xi,\xi)\xi$ giving

$$Q\xi = (Trl)\xi. \tag{5.3}$$

Now differentiating (5.3) and using (2.3) we have

$$(\nabla_X Q)\xi - Q(\phi X + \phi hX) = (X\,Trl)\xi - (Trl)(\phi X + \phi hX) \tag{5.4}$$

from which using the commutativity and (2.1), we have

$$g((\nabla_X Q)X,\xi) + g(QhX,\phi X) = (Trl)g(X, h\phi X). \tag{5.5}$$

Now let X_i, ϕX_i, ξ, $(i = 1, \ldots, n)$ be a ϕ-basis. Since $Trh\phi = 0$ and $g(Qh\phi X, \phi^2 X) = -g(QhX, \phi X)$ we get from (5.5)

$$\sum_{i=1}^{n} g((\nabla_{X_i} Q)X_i + (\nabla_{\phi X_i} Q)\phi X_i, \xi) = 0. \tag{5.6}$$

Now from (5.6) and the contraction of the second Bianchi identity, we have $g((\nabla_\xi Q)\xi, \xi) = \frac{1}{2}\xi R$. But differentiating (5.3) with respect to ξ and using $\nabla_\xi \xi = 0$ and $\xi Trl = 0$, (5.2), we have $(\nabla_\xi Q)\xi = 0$ and hence

$$\xi R = 0. \tag{5.7}$$

For X orthogonal to ξ, (5.1) becomes

$$lX = \frac{1}{2n-1}(QX + (Trl)X) - \frac{R}{2n(2n-1)}X. \tag{5.8}$$

Differentiating (5.8) and using (5.2) and (5.7) we have

$$l\nabla_\xi X = \frac{1}{2n-1}(Q\nabla_\xi X + (Trl)\nabla_\xi X) - \frac{R}{2n(2n-1)}\nabla_\xi X.$$

Thus $(\nabla_\xi l)X = \frac{1}{2n-1}(\nabla_\xi Q)X$. So from (5.2) and $(\nabla_\xi Q)\xi = 0$ we get

$$\nabla_\xi Q = 0. \tag{5.9}$$

LEMMA: On any conformally flat contact metric manifold M^{2n+1} with $Q\phi = \phi Q$ we have $(\nabla_\xi R)(X, Y, Z) = 0$ and

$$\begin{aligned} &2n\big[g(\xi, Z)((\nabla_X Q)Y - (\nabla_Y Q)X) - g(Y, Z)(\nabla_X Q)\xi + g(X, Z)(\nabla_Y Q)\xi \\ &+g((\nabla_X Q)\xi, Z)Y - g((\nabla_Y Q)\xi, Z)X - g((\nabla_X Q)Y - (\nabla_Y Q)X, Z)\xi\big] \\ &\quad = (XR)\big[g(\xi, Z)Y - g(Y, Z)\xi\big] - (YR)\big[g(\xi, Z)X - g(X, Z)\xi\big]. \end{aligned} \tag{5.10}$$

PROOF: The proof is a straightforward computation, differentiating (2.7) with respect to ξ using (5.7), applying (2.7) to $R(\nabla_\xi X, Y)Z$, etc., and then combining using (5.9). Now from the second Bianchi identity we have

$$(\nabla_X R)(Y, \xi, Z) = (\nabla_Y R)(X, \xi, Z). \tag{5.11}$$

Calculating the terms $(\nabla_X R)(Y, \xi, Z)$ and$(\nabla_Y R)(X, \xi, Z)$ and substituting into (5.11) we obtain (5.10).

THEOREM 5.1: Let M^{2n+1} be a conformally flat contact metric manifold. If $Q\phi = \phi Q$, then M^{2n+1} is of constant curvature $+1$ if $n > 1$ and 0 or $+1$ if $n = 1$.

PROOF: Case 1: $n \geq 2$. For X,Y,Z mutually orthogonal and normal to ξ, (5.10) gives

$$g((\nabla_X Q)Z, \xi) = 0. \tag{5.12}$$

Taking the inner product of (5.4) with X and using (5.12) we have

$$g(Q(X + hX), X) = (Trl)g(X + hX, X). \tag{5.13}$$

Using ϕX instead of X and simplifying we also have

$$g(Q(X - hX), X) = (Trl)g(X - hX, X). \tag{5.14}$$

Adding (5.13) and (5.14) we have

$$g(QX, X) = (Trl)g(X, X).$$

Linearization yields for X normal to ξ, $QX = (Trl)X$ which with (5.3) shows that M^{2n+1} is a conformally flat Einstein space and hence of constant curvature; but a contact metric manifold of constant curvature and dimension ≥ 5 is of constant curvature $+1$ [10].

Case 2: $n = 1$. From $l\xi = 0$ and $\phi l = l\phi$ we have for X orthogonal to ξ, $g(lX, \xi) = 0$ and $g(lX, \phi X) = 0$. Thus lX is parallel to X for any X orthogonal to ξ. Let $lX = \alpha X$ for such X. Using (5.8) for $n = 1$ we get

$$QX + (Trl - \frac{R}{2} - \alpha)X = 0. \tag{5.15}$$

Computing the scalar curvature both directly and using (5.15) yields $\alpha = \frac{1}{2} Trl$. Therefore

$$lX = \frac{1}{2}(Trl)(X - g(X, \xi)\xi). \tag{5.16}$$

Moreover by virtue of (5.15) and (5.3) we have

$$QX = \frac{1}{2}(R - Trl)(X - g(X, \xi)\xi) + (Trl)g(X, \xi)\xi. \tag{5.17}$$

Now using (5.16), (2.7) and (5.3) we calculate for X, Y orthogonal to ξ,

$$\nabla_X R(Y, \xi)\xi = \frac{1}{2}((X\,Trl)Y + (Trl)\nabla_X Y), \; R(\nabla_X Y, \xi)\xi = \frac{1}{2}(Trl)(\nabla_X Y - g(\nabla_X Y, \xi)\xi)$$

and $R(Y, \nabla_X \xi)\xi = 0$ since ξ is unit. Also using (2.3) we have

$$R(Y, \xi)\nabla_X \xi = (Trl)g(Y, \phi X)\xi + g(QY, \phi X)\xi - \frac{R}{2}g(Y, \phi X)\xi$$

$$+(Trl)g(Y, \phi hX)\xi + g(QY, \phi hX)\xi - \frac{R}{2}g(Y, \phi hX)\xi.$$

Thus

$$(\nabla_X R)(Y,\xi,\xi) = \frac{1}{2}(X\,Trl)Y + \big[(Trl)(\frac{1}{2}g(\nabla_X Y,\xi) - g(Y,\phi X + \phi hX))$$
$$+\frac{R}{2}g(Y,\phi X + \phi hX) - g(QY,\phi X + \phi hX)\big]\xi. \tag{5.18}$$

From (5.18) and (5.11) with $Z = \xi$ we get $X\,Trl = 0$, but $\xi\,Trl = 0$ by (5.2) and so

$$Trl = \text{constant}. \tag{5.19}$$

Since M^3 is conformally flat the tensor field $P = -Q + \frac{R}{4}Id$ satisfies (2.8). Using (5.17) and (5.19) and differentiating we get for X, Y orthogonal to ξ,

$$(\nabla_X P)Y = -\frac{1}{4}(XR)Y + \frac{1}{2}(R - 3\,Trl)g(Y,\nabla_X\xi)\xi.$$

So by (2.8) we get $(R - 3\,Trl)(g(Y,\nabla_X\xi) - g(X,\nabla_Y\xi)) = 0$, and hence

$$(R - 3\,Trl)d\eta(X,Y) = 0.$$

Thus $R = 3\,Trl$ and by (5.17), $QX = (Trl)X$ i.e. M^3 is Einstein and hence of constant curvature. However a 3-dimensional contact metric manifold of constant curvature is of constant curvature 0 or +1 [4], completing the proof.

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