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On Electromagneto-Thermoelastic Plane Waves

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Summary — Zusammenfassung

On Electromagneto-Thermoelastic Plane Waves. The propagation of harmonically time-dependent electromagneto-thermoelastic plane waves of assigned frequency in an unbounded, homogeneous, isotropic, elastic, thermally and electrically conducting medium is considered. The theory of thermoelasticity recently proposed by Green and Lindsay is used to account for the interactions between the elastic and thermal fields. The results pertaining to phase velocity and attenuation coefficient of various types of waves are compared with those of Nayfeh and Nemat-Nasser who have dealt with a theory of thermoelasticity having a thermal relaxation time.

Elektro-magnetothermoelastische ebene Wellen. Die Fortpflanzung von harmonischen, zeitabhängigen, elektro-magnetothermoelastischen ebenen Wellen von gegebener Frequenz in einem unbegrenzten, homogenen, isotropischen, elastischen, wärme- und elektrisch leitendem Material wird behandelt. Die Wechselwirkung zwischen den elektrischen und thermischen Feldern wird durch die kürzlich vorgeschlagene Thermoelastizitätstheorie von Green und Lindsay beschrieben. Die Dämpfungskoeffizienten der verschiedenen Wellentypen werden mit denen von Nayfeh und Nemat-Nasser verglichen, welche schon früher eine Thermoelastizitätstheorie mit thermischer Relaxationszeit behandelt hatten.

1. Introduction

The classical theory of thermoelasticity is based on Fourier's law of heat conduction, and, consequently, permits disturbances to propagate at infinite speed. This situation is physically unacceptable and many new theories have been formulated to remedy it (see [1]). Paria [2], [3], Willson [4], and Purushothama [5] have used the classical theory of thermoelasticity along with electromagnetic theory when considering the propagation of harmonically time-dependent plane waves of assigned frequency in a homogeneous, isotropic, and unbounded solid. Nayfeh and Nemat-Nasser [6] used a theory of thermoelasticity having a thermal relaxation time [7], [8], and which in contrast to [2], [3], [4], [5] includes the electric displacement current in the electromagnetic field equations. Perturbation techniques are used in [6] to study the effect that small thermoelastic and magnetoelastic coupling parameters have on the phase velocity and attenuation coefficient of plane electromagneto-thermoelastic waves. Explicit solutions are obtained in [6] for two cases which are shown to include the results obtained by the other authors [2], [3], [4], [5].

In this paper we consider the problem of the propagation of electromagneto-

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thermoelastic plane waves within the context of a theory of thermoelasticity[1] proposed recently by Green and Lindsay [9]. The latter theory has certain important features that contrast with the theory having a thermal relaxation time:

The Fourier law of heat conduction remains unchanged while the classical energy equation and the stress-strain-temperature (Hooke-Duhamel-Neumann) relations are modified.

Two constitutive constants α and α^*, which also have the dimension of time, now appear in the governing equations instead of one relaxation time τ.

In the absence of electromagnetic effects, the writer has shown [1] that all the results pertaining to phase velocity, attenuation coefficient, behaviour of amplitude ratios, and the stability of thermoelastic waves for the theory of Lord and Shulman [11] are recovered as a special case of the results of Green and Lindsay's theory, and the structure of the two theories remains distinct.

Moreover, in the theory of Green and Lindsay the specific loss and the attenuation coefficient for the quasi-elastic waves are both increased at low frequencies. It thus also yields results which are qualitatively different [1].

Ignaczak [12] has proved a domain of influence theorem which asserts that thermoelastic disturbances produced by the data of bounded support propagate with a finite velocity; a uniqueness theorem was established earlier by Green [10].

In Section 2 the basic equations governing the electromagnetic, thermal, and elastic fields and the interactions between them are recalled. The propagation of time-harmonic plane electromagneto-thermoelastic waves of assigned frequency is considered in Section 3. The solutions for the phase velocity and the attenuation coefficient are obtained for small thermoelastic and magnetoelastic couplings by perturbation technique for two important cases similar to those in [6]. The results are compared with those of Nayfeh and Nemat-Nasser, and, in order to facilitate this comparison, the notation of Nayfeh and Nemat-Nasser is retained as far is possible. It is shown that many of the results in [6] form a special case (where $\alpha = \alpha^* = \tau$) of the present results. Some results obtained here, however, do not agree with the results of Nayfeh and Nemat-Nasser. This appears to be due to certain errors in the calculations and conclusions of [6].

2. Basic Equations

The electromagnetic equations are taken as

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}, \qquad \operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{B} = 0, \qquad \operatorname{div} \boldsymbol{D} = \varrho_e, \tag{2.2}$$

$$\boldsymbol{B} = \mu_e \boldsymbol{H}, \qquad \boldsymbol{D} = \varepsilon \boldsymbol{E}, \tag{2.3}$$

and the modified Ohm's law

$$\boldsymbol{j} = \sigma \left[\boldsymbol{E} + \frac{\partial \boldsymbol{u}}{\partial t} \wedge \boldsymbol{B}\right] + \varrho_e \frac{\partial \boldsymbol{u}}{\partial t} - k_0 \nabla \theta. \tag{2.4}$$

[1] We consider only the linearised form of this nonlinear theory (see Green [10]).

The equations of motion are

$$t_{ij,j} + (\boldsymbol{j} \wedge \boldsymbol{B})_i + \varrho_e E_i = \varrho \ddot{u}_i. \tag{2.5}$$

The equations of thermoelasticity are [10]

$$e_{ij} = u_{i,j} + u_{j,i}, \tag{2.6}$$

$$-q_{i,i} = \varrho c_V(\dot{\theta} + \alpha^* \ddot{\theta}) + \gamma \theta_0 \dot{e}, \tag{2.7}$$

$$t_{ik} = \lambda e \delta_{ik} + 2\mu e_{ik} - \gamma(\theta + \alpha \dot{\theta})\, \delta_{ik}, \tag{2.8}$$

$$q_i = -k\theta_{,i} + \pi_0 j_i, \tag{2.9}$$

$$e = u_{i,i}, \tag{2.10}$$

$$\gamma = (3\lambda + 2\mu)\, \frac{\beta}{3}, \tag{2.11}$$

where all the terms have their usual meaning (see, for example, [2]—[6]), a superposed dot denotes $\partial/\partial t$, $(\ )_{,k} = \partial(\ )/\partial x_k$; $i, j, k = 1, 2, 3$, and the summation convention is used for the repeated subscripts.

Combining (2.7) and (2.9) we obtain

$$\varrho c_V(\dot{\theta} + \alpha^* \ddot{\theta}) + \gamma \theta_0 \dot{e} = k\theta_{,ii} - \pi_0 j_{i,i}. \tag{2.12}$$

If $\boldsymbol{D}$, $\boldsymbol{B}$, ϱ_e, $\boldsymbol{j}$ are eliminated from the above equations and the resulting equations are then linearised by taking

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{h},$$

where $\boldsymbol{H}_0$ is the primary magnetic field (constant in space and time) and $\boldsymbol{h}$ is a small perturbation, we obtain

$$\varrho c_V(\dot{\theta} + \alpha^* \ddot{\theta}) + \gamma \theta_0 \dot{e} = k\theta_{,ii} + \pi_0 \varepsilon (\nabla \cdot \dot{\boldsymbol{E}}), \tag{2.13}$$

$$\varrho \ddot{\boldsymbol{u}} = \nabla \boldsymbol{T} + \mu_e \sigma (\boldsymbol{E} \wedge \boldsymbol{H}_0) + \mu_e^2 \sigma (\dot{\boldsymbol{u}} \wedge \boldsymbol{H}_0) \wedge \boldsymbol{H}_0 - \mu_e k_0 (\nabla \theta \wedge \boldsymbol{H}_0), \tag{2.14}$$

$$\nabla^2 \boldsymbol{E} - \nabla(\nabla \cdot \boldsymbol{E}) = \mu_e \sigma [\dot{\boldsymbol{E}} + \mu_e (\ddot{\boldsymbol{u}} \wedge \boldsymbol{H}_0)] - \mu_e k_0 \nabla \dot{\theta} + \mu_e \varepsilon \ddot{\boldsymbol{E}}. \tag{2.15}$$

In order to nondimensionalise (2.13)—(2.15) we follow Nayfeh and Nemat-Nasser in defining

$$\omega^* = \frac{\varrho c c_1^2}{k}, \qquad c_1^2 = \frac{\lambda + 2\mu}{\varrho}, \qquad g = \frac{\gamma}{\varrho c},$$

$$\Lambda^2 = \frac{\lambda + 2\mu}{\mu}, \qquad s = c_1^2 \varepsilon \mu_e, \qquad \nu = \frac{1}{\sigma \mu_e}$$

$$\varepsilon_\theta = \frac{bg}{\Lambda^2}, \qquad \varepsilon_E = \frac{\mu_e \bar{H}_0^2}{\varrho c_1^2}, \qquad \bar{\nu} = \frac{\nu \omega^*}{c_1^2},$$

$$\pi = \frac{\pi_0 \mu_e \varepsilon \omega^* \bar{H}_0}{g \varrho c \theta_0}, \qquad \bar{k} = \frac{g k_0 \theta_0}{\bar{H}_0^2},$$

$$\boldsymbol{H}_0 = \bar{H}_0 \boldsymbol{n}, \qquad \boldsymbol{n} = (n_1, n_2, n_3), \qquad \text{a unit vector}, \tag{2.16}$$

and in keeping the same symbols for nondimensional quantities. Thus in what follows, u_i, t, θ, x_i, E_i, α, α^* will represent $u_i/(c_1/g\omega^*)$, $t\omega^*$, θ/θ_0, $x_i/(c_1/w^*)$, $E_i/(\bar{H}_0 \mu_e c_1/g)$, $\alpha\omega^*$, $\alpha^*\omega^*$, respectively.

Assuming that all the field variables are functions of x $(= x_1)$ and t only, it is a straight-forward matter to obtain from (2.13)—(2.15) the following cases, referred to in [6] as Systems I_a, II_a, I_b and II_b:

Case (a): $n_2 = n_3 = 0$, $n_1 = 1$.

I_a:
$$\begin{aligned} \dot\theta + \alpha^*\ddot\theta - \theta'' + \dot u' &= \pi \dot E_1{}', \\ \bar\nu s \ddot E_1 + \dot E_1 - \bar\nu \bar k \dot\theta' &= 0, \\ \ddot u &= u'' - \varepsilon_\theta \theta'; \end{aligned} \tag{2.17}$$

II_a:
$$\begin{aligned} \bar\nu s \ddot F + \dot F + \ddot G &= \bar\nu F'', \\ \bar\nu \beta^2 \ddot G &= \bar\nu G'' - \beta^2 \varepsilon_E F - \beta^2 \varepsilon_E \dot G, \end{aligned} \tag{2.18}$$

where

$$F = E_2 - E_3, \qquad G = w + v,$$

u, v, w are the displacements and E_1, E_2, E_3 are components of the electric field in the x_1, x_2, and x_3-directions of the Cartesian coordinate system. The prime indicates differentiation with respect to x $(= x_1)$.

Case (b): $\varrho_e = 0, \qquad \varepsilon \neq 0, \qquad \bar\nu \bar k \neq 0.$

I_b:
$$\begin{aligned} \bar\nu \beta^2 \ddot v &= \bar\nu v'' + \beta^2 \varepsilon_E [n_1 E_3 - (1 + n_3^2)\, \dot v], \\ \bar\nu s \ddot E_3 + \dot E_3 - n_1 \ddot v &= \bar\nu E_3{}''; \end{aligned} \tag{2.19}$$

II_b:
$$\begin{aligned} \dot\theta + \alpha^* \ddot\theta - \theta'' + \dot u' &= 0, \\ \bar\nu s \ddot E_2 + \dot E_2 + n_1 \ddot w - n_3 \ddot u &= \bar\nu E_2{}'', \\ \bar\nu \ddot u = \bar\nu u'' - \bar\nu \varepsilon_\theta (\theta' + \alpha \dot\theta') &+ \varepsilon_E n_3 (E_2 + n_1 \dot w - n_3 \dot u) \\ \bar\nu \beta^2 \ddot w &= \bar\nu w'' - \varepsilon_E n_1 \beta^2 (E_2 + n_1 \dot w - n_3 \dot u). \end{aligned} \tag{2.20}$$

Two further special cases are also needed and are recorded below.

I_c (special case of II_b): $\alpha = \alpha^* = s = \pi = \bar k = 0.$

$$\begin{aligned} \dot\theta - \theta'' + \dot u' &= 0, \\ \bar\nu \ddot u &= \bar\nu u'' - \bar\nu \varepsilon_\theta \theta' + \varepsilon_E n_3 (E_2 + n_1 \dot w - n_3 \dot u), \\ \bar\nu \beta^2 \ddot w &= \bar\nu w'' - n_1 \beta^2 \varepsilon_E (E_2 + n_1 \dot w - n_3 \dot u), \\ \dot E_2 + n_1 \ddot w - n_3 \ddot u &= \bar\nu E_2{}''; \end{aligned} \tag{2.21}$$

II_c (special case of I_b): $s = 0, \quad n_3 = 0.$

$$\begin{aligned} \beta^2 \ddot v &= \dot v'' + \beta^2 \varepsilon_E n_1 E_3{}'', \\ \dot E_3 - n_1 \ddot v &= \bar\nu E_3{}''. \end{aligned} \tag{2.22}$$

In the present formulation II_a, I_b, I_c, II_c, are exactly the same as (28a, b)[2], (33a, b), (43a—d), (44a, b), respectively of [6], but I_a and II_b differ from (27a, b, c)[2] and (34a, b).

[2] (27c) and (28a) have some minor misprints.

3. Plane Waves

We assume that each of the field variables in (2.17)—(2.22) is of the form a exp $[i(qx + \omega t)]$, where ω, the prescribed frequency, is the same real number for all variables; q, the wavenumber, possibly complex, is to be determined and a, also an unknown, is the amplitude associated with each variable. Then we obtain the characteristic equation and analyse it for low and high frequencies for each case.

I_a (Eq. (2.17)): The characteristic equation is

$$\det \begin{bmatrix} -q\omega & (i\omega - \alpha^*\omega^2 + q^2) & \pi q\omega \\ 0 & \bar{\nu}\bar{k}q\omega & (-\bar{\nu}s\omega^2 + i\omega) \\ (q^2 - \omega^2) & \varepsilon_\theta q(i - \alpha\omega) & 0 \end{bmatrix} = 0. \tag{3.1}$$

We note that if

$$\bar{\nu} = 0, \qquad \text{or} \qquad \pi = 0, \qquad \text{or} \qquad \bar{k} = 0 \tag{3.2}$$

in (3.1), the characteristic equation becomes

$$(q^2 - \omega^2)(i\omega - \alpha^*\omega^2 + q^2) + \varepsilon_\theta q^2\omega(i - \alpha\omega) = 0, \tag{3.3}$$

which is the same as in generalized thermoelasticity [1]. Therefore, all the results of [1] must be recovered for I_a whenever (3.2) holds. Now, for $\bar{\nu} \neq 0$, we define

$$\alpha_1^* = \alpha - \frac{i}{\omega}, \qquad \alpha_2^* = \alpha^* - \frac{i}{\omega}, \qquad s^* = s - \frac{i}{\bar{\nu}\omega} \qquad (\bar{\nu} \neq 0). \tag{3.4}$$

The case of $\bar{\nu} = 0$ is treated separately in what follows.

The use of (3.4) in (3.1) yields

$$(\omega^2 - q^2)\left[(q^2 - \alpha_2^*\omega^2)s^* + \pi\bar{k}q^2\right] + q^2\omega^2 s^*\alpha_1^*\varepsilon_\theta = 0, \tag{3.5}$$

which upon setting $\alpha_1^* = \alpha_2^* = \tau^*$ becomes[3] Eq. (30) of [6]. Eq. (3.5) can be put in the form

$$q^4 - (A - iB)q^2 + \omega(C - iD) = 0, \tag{3.6}$$

where

$$\begin{aligned} A &= \{\bar{\nu}(s + \pi\bar{k})[s\bar{\nu}(1 + \alpha^* + \alpha\varepsilon_\theta) + \pi\bar{\nu}\bar{k}]\,\omega^4 \\ &\quad + [(1 + \alpha^* + \alpha\varepsilon_\theta) - \pi\bar{\nu}\bar{k}(1 + \varepsilon_\theta)]\,\omega^2\}/\text{Den.} \\ B &= \{\bar{\nu}(s + \pi\bar{k})[(1 + \alpha^* + \alpha\varepsilon_\theta) + s\bar{\nu}(1 + \varepsilon_\theta)]\,\omega^3 \\ &\quad - [s\bar{\nu}(1 + \alpha^* + \alpha\varepsilon_\theta) + \pi\bar{\nu}\bar{k}]\,\omega^3 + (1 + \varepsilon_\theta)\,\omega\}/\text{Den.} \\ C &= \{s\bar{\nu}^2(s + \pi\bar{k})\,\alpha^*\omega^5 + (\alpha^* - \pi\bar{\nu}\bar{k})\,\omega^3\}/\text{Den.} \\ D &= \{[\bar{\nu}^2 s^2 + \pi\bar{\nu}\bar{k}(s\bar{\nu} + \alpha^*)]\,\omega^4 + \omega^2\}/\text{Den.} \\ \text{Den.} &= \bar{\nu}^2\omega^2(s + \pi\bar{k})^2 + 1. \end{aligned} \tag{3.7}$$

[3] $\bar{\nu}\bar{k}\pi$ in Eq. (30) of [6] should read $\bar{k}\pi$. This error has been carried along throughout the paper, especially in (30a, c), (30f, g), (37a, b), and (41a, b).

It is easy to verify that (3.6) has the solutions

$$\begin{aligned} 2q_{1,2} = &\pm\{A + (2\omega)^{1/2}\,\{(C^2 + D)^{1/2} + C\}^{1/2} \\ &- i[B + (2\omega)^{1/2}\,\{(C^2 + D)^{1/2} - C\}^{1/2}]\}^{1/2} \\ &\pm\{A - (2\omega)^{1/2}\,\{(C^2 + D)^{1/2} + C\}^{1/2} \\ &+ i[-B + (2\omega)^{1/2}\,\{(C^2 + D)^{1/2} - C\}^{1/2}]\}^{1/2}. \end{aligned} \tag{3.8}$$

One could now follow [1] and proceed to examine the exact solution (3.8) in the limits of low and high frequencies, and small values of thermoelastic coupling parameter ε_θ. However, it is more convenient here to use the perturbation technique. The solutions of (3.5) for $\varepsilon_\theta = 0$ are

$$q^2 = \omega^2, \qquad q^2 = \frac{\alpha_2^*}{1 + k_1^*}\,\omega^2, \tag{3.9}$$

where

$$k_1^* = \frac{\pi\bar{k}}{s^*} \qquad (s^* \neq 0).$$

For small values of ε_θ, the above solutions are modified to

$$q_u^2 = \omega^2\left[1 + \frac{\alpha_1^*}{1 + k_1^* - \alpha_2^*}\,\varepsilon_\theta\right] + 0(\varepsilon_\theta^2), \tag{3.10}$$

$$q_\theta^2 = \frac{\alpha_2^*\omega^2}{1 + k_1^*}\left[1 + \frac{\alpha_1^*}{\alpha_2^* - (1 + k_1^*)}\,\varepsilon_\theta\right] + 0(\varepsilon_\theta^2), \tag{3.11}$$

which correspond to the modified quasi-elastic and the modified quasi-thermal waves, respectively. The expressions (3.9), (3.10), and (3.11) coincide[4] with (30b, c) and (30f, g) of [6] for $\alpha_1^* = \alpha_2^* = \tau^*$. We now separate the real and imaginary parts of (3.10) and obtain

$$q_u = \operatorname{Re}(q_u) + i\operatorname{Im}(q_u) = \omega\left[1 + \frac{1}{2}\,\varepsilon_\theta\,\frac{N_r + iN_i}{D_r}\right], \tag{3.12}$$

where

$$\begin{aligned} N_r &= (\alpha s\bar{\nu}\omega^2 - 1)\,[\{s\bar{\nu}(1 - \alpha^*) + \pi\bar{\nu}\bar{k}\}\,\omega^2 + 1] - (\alpha + s\bar{\nu})\,\omega^2[s\bar{\nu} - (1 - \alpha^*)], \\ N_i &= -(\alpha + s\bar{\nu})\,\omega[\{s\bar{\nu}(1 - \alpha^*) + \pi\bar{\nu}\bar{k}\}\,\omega^2 + 1] - (\alpha s\bar{\nu}\omega^2 - 1)\,\omega[s\bar{\nu} - (1 - \alpha^*)], \\ D_r &= [\{s\bar{\nu}(1 - \alpha^*) + \pi\bar{\nu}\bar{k}\}\,\omega^2 + 1]^2 + [s\bar{\nu} - (1 - \alpha^*)]^2\,\omega^2. \end{aligned} \tag{3.13}$$

As $\omega \to \infty$, (3.12) and (3.13) yield the following expressions for the phase velocity and the attenuation coefficient:

$$C_u = \frac{\omega}{\operatorname{Re}(q_u)} = \left[1 - \frac{1}{2}\,\varepsilon_\theta\,\frac{\alpha}{(1 - \alpha^*) + k_1}\right], \tag{3.14}$$

$$S_\theta = -\operatorname{Im}(q_u) = \frac{1}{2}\,\varepsilon_\theta\,\frac{\left(1 + \dfrac{\alpha}{s\bar{\nu}}\right)k_1 + (1 + \alpha - \alpha^*)}{(1 - \alpha^* + k_1)^2}, \tag{3.15}$$

[4] It should be noted that (30g) of [6] is incorrect even after accounting for an extra $\bar{\nu}$. This error causes (37b) and (41b) to be incorrect in the same manner. After the corrections are made (30g) agrees with (3.11) for $a_1^* = \alpha_2^* = \tau^*$.

where

$$k_1 = \frac{\pi \bar{k}}{s} \qquad (s \neq 0).$$

Eq. (3.14) is the same as (41a) of [6] for $\alpha = \alpha^* = \tau$. The substitution of $\bar{k} = 0$ or $\pi = 0$ in (3.14) and (3.15) gives

$$C_u = 1 - \frac{1}{2}\,\varepsilon_\theta \alpha, \tag{3.16}$$

$$S_u = \frac{1}{2}\,\varepsilon_\theta (1 + \alpha + \alpha^*), \tag{3.17}$$

which are consistent with the results in [1] and [13]. Nayfeh and Nemat-Nasser neglected to calculate $\mathrm{Im}\,(q_u)$, and thus mistakenly concluded that the modified thermoelastic waves[5] propagate unattenuated. This conclusion is clearly inconsistent with their own work [13] as well as the work of the writer [1].

If $\omega \to 0$, the phase velocity and the attenuation coefficient are obtained from (3.12) and (3.13) as

$$C_u = \left(1 + \frac{1}{2}\,\varepsilon_\theta\right), \tag{3.18}$$

$$S_u \doteq \frac{1}{2}\,\varepsilon_\theta \omega^2 (1 + \alpha - \alpha^*), \tag{3.19}$$

which are the same as in generalized thermoelasticity [1]. The quasi-elastic waves are not affected by the electromagnetic quantities $\bar{k}$, $\bar{\nu}$, and π at low frequencies.

A calculation similar to the above for the quasi-thermal waves corresponding to (3.11) results in

$$\mathrm{Re}\,(q_\theta) = \omega \left(\frac{\alpha^*}{1 + k_1}\right)^{1/2} \left[1 + \frac{1}{2}\,\varepsilon_\theta\,\frac{\alpha}{\alpha^* - (1 + k_1)}\right], \tag{3.20}$$

$$-\mathrm{Im}\,(q_\theta) = \frac{1}{2}\,\frac{(1 + \alpha^*)\,k_1 + 1}{\sqrt{\alpha^*}\,(1 + k_1)^{3/2}}, \tag{3.21}$$

as $\omega \to \infty$. Eq. (3.20) is in agreement[6] with (41b) of [6] for $\alpha = \alpha^* = \tau$. Nayfeh and Nemat-Nasser do not have $\mathrm{Im}\,(q_\theta)$. For $\bar{k} = 0$ or $\pi = 0$, (3.20) and (3.21) become

$$\mathrm{Re}\,(q_\theta) = \omega \sqrt{\alpha^*}\left(1 - \frac{1}{2}\,\varepsilon_\theta \alpha\right), \tag{3.22}$$

$$-\mathrm{Im}\,(q_\theta) = \frac{1}{2\sqrt{\alpha^*}}, \tag{3.23}$$

which are consistent with [1]. When $\omega \to 0$ we obtain

$$\mathrm{Re}\,(q_\theta) = \sqrt{\omega/2}\left(1 + \frac{1}{2}\,\varepsilon_\theta\right), \tag{3.24}$$

$$-\mathrm{Im}\,(q_\theta) = \mathrm{Re}\,(q_\theta), \tag{3.25}$$

[5] The term thermoelastic waves is used for both the quasi-elastic and quasi-thermal waves [1].

[6] Only after (41b) has been corrected.

results which are also obtained in generalized thermoelasticity [1]. The quasi-thermal are also not affected by $\bar{k}$, π, and $\bar{\nu}$ at low frequencies.

II_a (Eq. (2.18)): Eq. (2.18) is the same as (28a, b) of [6]. The characteristic equation is

$$\det\begin{bmatrix}(-\bar{\nu}s\omega^2 + i\omega + \bar{\nu}q^2) & -\omega^2 \\ \beta^2\varepsilon_E & (-\bar{\nu}\beta^2\omega^2 + \bar{\nu}q^2 + i\beta^2\omega\varepsilon_E)\end{bmatrix} = 0. \qquad (3.26)$$

$\bar{\nu} = 0$ satisfies (3.26) identically. For $\bar{\nu} \neq 0$ we use (3.4) in (3.26) to obtain

$$\bar{\nu}^2(q^2 - s^*\omega^2)(q^2 - \beta^2\omega^2) + \beta^2\varepsilon_E\omega[\omega + i\bar{\nu}(q^2 - s^*\omega^2)] = 0, \qquad (3.27)$$

which is the same[7] as (31) of [6]. One can now obtain the exact solution (3.8) of (3.27) by recasting it in the form (3.6). However, we look for the perturbation solutions. For $\varepsilon_E = 0$ (3.27) gives

$$q^2 = s^*\omega^2, \qquad q^2 = \beta^2\omega^2, \qquad (3.28)$$

which are modified, for small values of ε_E, to

$$q_F{}^2 = s^*\omega^2\left[1 + \varepsilon_E\frac{\beta^2}{\bar{\nu}^2 s^*(\beta^2 - s^*)\,\omega^2}\right] + 0(\varepsilon_E{}^2) \qquad (s^* \neq \beta^2), \qquad (3.29)$$

$$= s\omega^2\left[1 + \varepsilon_E\frac{\beta^2(\beta^2 - s)}{s\{\bar{\nu}^2(\beta^2 - s)^2\,\omega^2 + 1\}}\right] - i\,\frac{\omega}{\bar{\nu}}\left[1 + \varepsilon_E\frac{\beta^2}{\bar{\nu}\{\bar{\nu}^2(\beta^2 - s)^2\,\omega^2 + 1\}}\right] \qquad (3.30)$$
$$+ 0(\varepsilon_E{}^2),$$

$$q_G{}^2 = \beta^2\omega^2\left[1 + \varepsilon_E\left\{\frac{1}{\bar{\nu}^2(s^* - \beta^2)\,\omega^2} - \frac{i}{\bar{\nu}\omega}\right\}\right] + 0(\varepsilon_E{}^2), \qquad (s^* \neq \beta^2) \qquad (3.31)$$

$$= \beta^2\omega^2\left[1 + \varepsilon_E\frac{(s - \beta^2)}{\bar{\nu}^2(s - \beta^2)^2\,\omega^2 + 1}\right] - i\varepsilon_E\frac{\beta^2\omega}{\bar{\nu}}\left[1 - \frac{1}{\bar{\nu}^2(s - \beta^2)^2\,\omega^2 + 1}\right] + 0(\varepsilon_E{}^2). \qquad (3.32)$$

Eqs. (3.29), (3.31), and (3.30), (3.32) coincide with (31a, b) and (38a, b) of [6]. For high frequencies, $\omega \to \infty$, we obtain the phase velocity, $C\ (= \omega/\mathrm{Re}\,(q))$, and the attenuation coefficient, $S\ (= -\mathrm{Im}\,(q))$, as

$$C_F = \frac{1}{\sqrt{s}}, \qquad S_F = \frac{1}{2\bar{\nu}\sqrt{s}}, \qquad (3.33)$$

$$C_G = \frac{1}{\beta}, \qquad S_G = \frac{\varepsilon_E\beta}{2\bar{\nu}}, \qquad (3.34)$$

and for low frequencies, $\omega \to 0$,

$$C_F = \frac{1}{\sqrt{s}}\left[1 - \frac{1}{2}\,\varepsilon_E\,\frac{\beta^2(\beta^2 - s)}{s}\right], \qquad S_F = \frac{1}{2\bar{\nu}\sqrt{s}}\left[1 + \varepsilon_E\beta^2\left\{\frac{1}{\bar{\nu}} - \frac{(\beta^2 - s)}{2s}\right\}\right], \qquad (3.35)$$

$$C_G = \frac{1}{\beta}\left[1 - \frac{1}{2}\,\varepsilon_E(s - \beta^2)\right], \qquad S_G = 0. \qquad (3.36)$$

[7] There is a minor misprint in (31) of [6].

I_b (Eq. (2.19)): I_b is the same as (33a, b) of [6]. The characteristic equation (for $\bar{\nu} \neq 0$) is

$$(q^2 - s^*\omega^2)(q^2 - \beta^2\omega^2) + \beta^2 \frac{\varepsilon_E n_1^2}{\bar{\nu}^2}\left[\omega^2 + \frac{i\bar{\nu}}{n_1^2}(1 + n_3^2)\,\omega(q^2 - s^*\omega^2)\right] = 0, \quad (3.37)$$

which becomes the same as (3.27) if we replace $\varepsilon_E n_1^2$ and $\bar{\nu}(1 + n_3^2)/n_1^2$ by ε_E and $\bar{\nu}$ respectively.

II_b (Eq. (2.20)): The characteristic equation for this case is

$$\begin{aligned} &\bar{\nu}(q^2 - \beta^2\omega^2)(q^2 - s^*\omega^2)\left[(q^2 - \alpha_2^*\omega^2)(q^2 - \omega^2) - \varepsilon_\theta q^2\alpha_1^*\omega\right] \\ &\quad - i\beta^2 n_1^2\omega(q^2 - s\omega^2)\left[(q^2 - \alpha_2^*\omega^2)(q^2 - \omega^2) - \varepsilon_\theta q^2\alpha_1^*\omega\right]\varepsilon_E \\ &\quad + i n_3^2\omega(q^2 - \alpha_2^*\omega^2)(q^2 - \beta^2\omega^2)(q^2 - s\omega^2)\,\varepsilon_E = 0, \end{aligned} \quad (3.38)$$

which agrees with [6] for $\alpha_1^* = \alpha_2^* = \tau^*$. When $\varepsilon_E = 0$ we obtain the solutions

$$q^2 = \beta^2\omega^2, \qquad q^2 = s^*\omega^2, \quad (3.39)$$

$$(q^2 - \alpha_2^*\omega^2)(q^2 - \omega^2) - \varepsilon_\theta q^2\alpha_1^*\omega^2 = 0. \quad (3.40)$$

Eq. (3.40) is the characteristic equation obtained in generalized thermoelasticity [1]. For $\varepsilon_E \neq 0$ the solutions (3.39) and (3.40) are modified. Following Nayfeh and Nemat-Nasser we obtain, for small values of ε_E, the following solutions.

$$q_w^2 = \beta^2\omega^2 - i\omega\,\frac{n_1^2\beta^2(s - \beta^2)}{\bar{\nu}(s^* - \beta^2)}\,\varepsilon_E + 0(\varepsilon_E^2), \quad (3.41)$$

$$q_{E_2}^2 = s^*\omega^2 + \varepsilon_E\,\frac{i\beta^2(s^* - s)\,\omega}{\bar{\nu}} \cdot \left[\frac{n_1^2\{(s^* - \alpha_2^*)(s^* - 1) - \varepsilon_\theta s^*\alpha_1^*\} + n_3^2(s^* - \alpha_2^*)\,\dfrac{(s^* - \beta^2)}{\beta^2}}{(\beta^2 - s^*)\{(s^* - \alpha_2^*)(s^* - 1) - \varepsilon_\theta s^*\alpha_1^*\}}\right] + 0(\varepsilon_E^2) \quad (3.42)$$

$$q_\theta^2 = \alpha_2^*\omega^2\left[1 - \frac{\alpha_1^*}{1 - \alpha_2^*}\left\{1 + \frac{i n_3^2(\alpha_2^* - s)}{\bar{\nu}\omega(\alpha_2^* - s^*)(1 - \alpha_2^*)}\,\varepsilon_E\right\}\varepsilon_\theta\right] + 0(\varepsilon_E^2), \quad (3.43)$$

$$q_u^2 = \omega^2\left[1 + \frac{\alpha_1^*}{1 - \alpha_2^*}\,\varepsilon_\theta - \frac{i n_3^2(1 - s)}{\omega\bar{\nu}(1 - s^*)}\,\varepsilon_E\right] + 0(\varepsilon_E^2), \quad (3.44)$$

Eqs. (3.41)—(3.44) agree with ($36\,i - l$) of [6] for $\alpha_1^* = \alpha_2^* = \tau^*$. An interesting feature of the results (3.41)—(3.44) is that they help identify the roles of α_1^* and α_2^* in comparison with τ^*.

One can now separate the real and imaginary parts of the above expressions and determine the phase velocity and attenuation coefficient in each case, as has been done for I_a and II_a. We are interested in pursuing a special case of II_b.

The substitution of $\bar{\nu} = 0$ in the system of Eqs. (2.21) gives

$$\dot{\theta} - \theta'' + \dot{u}' = 0,$$

$$E_2 + n_1\dot{w} - n_3\dot{u} = 0,$$

which are not sufficient in number to solve for four unknowns θ, u, E_2, and w. However, Nayfeh and Nemat-Nasser obtain [6, p. 112] the dilatational wave speed

$$(1 + n_3^2 \varepsilon_E)^{1/2} \quad \text{as} \quad \bar{\nu} \to 0 \quad \text{and} \quad \omega \to \infty.$$

I_c (Eq. (2.21)): We recall from Section 2 that I_c is a special case of II_b, where $\alpha = \alpha^* = s = 0$ in (2.20). Since α and α^* now drop out, all the results deduced from (3.41)—(3.44) are the same as (45a—d) of [6]. We rewrite the expressions for wavenumbers of the displacement components after separating the real and imaginary parts. Thus

$$q_u^2 = \omega^2 \left[\left\{ 1 - \frac{\varepsilon_\theta}{1 + \omega^2} - \frac{n_3^2 \varepsilon_E}{1 + \bar{\nu}\omega^2} \right\} - i \left\{ \frac{\varepsilon_\theta \omega}{1 + \omega^2} + \frac{n_3^2 \bar{\nu} \omega \varepsilon_E}{1 + \bar{\nu}^2 \omega^2} \right\} \right], \tag{3.45}$$

$$q_w^2 = \beta^2 \omega^2 \left[\left\{ 1 - \frac{n_1^2 \beta^2 \varepsilon_E}{1 + \bar{\nu}^2 \omega^2 \beta^4} \right\} - i \, \frac{n_1^2 \beta^4 \bar{\nu} \omega \varepsilon_E}{1 + \bar{\nu}^2 \omega^2 \beta^4} \right], \tag{3.46}$$

and

$$q_\theta^2 = -i\omega \left[1 + \frac{1 + i\omega}{1 + \omega^2} \left\{ 1 - \frac{n_3^2 (1 + i\omega)}{(1 - \bar{\nu})(1 + \omega^2)} \varepsilon_E \right\} \varepsilon_\theta \right], \tag{3.47}$$

$$q_{E_2}^2 = -\frac{i\omega}{\bar{\nu}} \left[1 - \frac{\beta^2 n_1^2 \varepsilon_E}{(i\bar{\nu}\omega\beta^2 - 1)} + \frac{1 + i\bar{\nu}\omega}{1 + \bar{\nu}^2 \omega^2} \left\{ 1 + \frac{\bar{\nu}(1 + i\bar{\nu}\omega)}{(1 - \bar{\nu})(1 + \bar{\nu}^2 \omega^2)} \varepsilon_\theta \right\} n_3^2 \varepsilon_E \right]. \tag{3.48}$$

The results given below then follow for the limiting cases $\omega \to \infty$ and $\omega \to 0$.

High frequency ($\omega \to \infty$) *Low frequency* ($\omega \to 0$)

$$\left.\begin{aligned} &\operatorname{Re}(q_u) = \omega & &\operatorname{Re}(q_u) = \omega \left(1 - \frac{1}{2} \varepsilon_\theta - \frac{1}{2} n_3^2 \varepsilon_E \right) \\ &-\operatorname{Im}(q_u) = \frac{1}{2} \left(\varepsilon_\theta + \frac{n_3^2}{\bar{\nu}} \varepsilon_E \right) & &-\operatorname{Im}(q_u) = \frac{\omega^2}{2} (\varepsilon_\theta + \bar{\nu} n_3^2 \varepsilon_E) \end{aligned}\right\} \tag{3.49}$$

$$\left.\begin{aligned} &\operatorname{Re}(q_w) = \beta\omega & &\operatorname{Re}(q_w) = \beta\omega \left(1 - \frac{n_1^2 \beta^2 \varepsilon_E}{2} \right) \\ &-\operatorname{Im}(q_w) = \frac{1}{2} \frac{n_1^2 \beta \varepsilon_E}{\bar{\nu}} & &-\operatorname{Im}(q_w) = \frac{1}{2} n_1^2 \beta^5 \bar{\nu} \varepsilon_E \omega^2 \end{aligned}\right\} \tag{3.50}$$

$$\left.\begin{aligned} &\operatorname{Re}(q_\theta) = \sqrt{\frac{\omega}{2}} & &\operatorname{Re}(q_\theta) = \sqrt{\frac{\omega}{2}} \left[1 + \frac{1}{2} \left(1 - \frac{n_3^2}{1 - \bar{\nu}} \varepsilon_E \right) \varepsilon_\theta \right] \\ &-\operatorname{Im}(q_\theta) = \sqrt{\frac{\omega}{2}} & &-\operatorname{Im}(q_\theta) = \operatorname{Re}(q_\theta) \end{aligned}\right\} \tag{3.51}$$

$$\begin{aligned} &\operatorname{Re}(q_{E_2}) = \sqrt{\frac{\omega}{2\bar{\nu}}} & &\operatorname{Re}(q_{E_2}) = \sqrt{\frac{\omega}{2\bar{\nu}}} \left[1 + \frac{1}{2} \left\{ \beta^2 n_1^2 \right.\right. \\ & & &\qquad \left.\left. + \left(1 + \frac{\bar{\nu}}{1 - \bar{\nu}} \varepsilon_\theta \right) n_3^2 \right\} \varepsilon_E \right] \\ &-\operatorname{Im}(q_{E_2}) = \sqrt{\frac{\omega}{2\bar{\nu}}} & &-\operatorname{Im}(q_{E_2}) = \operatorname{Re}(q_{E_2}). \end{aligned} \tag{3.52}$$

Clearly for $\varepsilon_E = 0$, (3.49)—(3.52) agree with [13]. For $\varepsilon_E \neq 0$ they agree with the results of [6] except for the attenuation coefficients of the displacement components, given by $-\operatorname{Im}(q_u)$ and $-\operatorname{Im}(q_w)$. Nayfeh and Nemat-Nasser, however, state that the waves corresponding to the displacement components propagate unattenuated [6, p. 113].

II_c (Eq. (2.22)): As Nayfeh and Nemat-Nasser have noted, (2.22) follows directly from $(2.21)_{3,4}$ if we substitute in the latter v, E_3, $-n_1$ for w, E_2, n_1, and $n_3 = 0$. The expressions for q_v^2 and $q_{E_3}^2$ can then be obtained by making the same changes in notation of (3.46) and (3.48). They are recorded as (46a, b) in [6].

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