EFFECTS OF GRAVITY-GRADIENT TORQUE ON THE ROTATIONAL MOTION OF A TRIAXIAL SATELLITE IN A PRECESSING ELLIPTIC ORBIT

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Abstract. A method of general perturbations, based on the use of Lie series to generate approximate canonical transformations, is applied to study the effects of gravity-gradient torque on the rotational motion of a triaxial, rigid satellite. The center of mass of the satellite is constrained to move in an elliptic orbit about an attracting point mass. The orbit, which has a constant inclination, is free to precess and spin. The method of general perturbations is used to obtain the Hamiltonian for the nonresonant secular and long-period rotational motion of the satellite to second order in n/ω_0, where n is the orbital mean motion of the center of mass and ω_0 is a reference value of the magnitude of the satellite's rotational angular velocity. The differential equations derivable from the transformed Hamiltonian are integrable and the solution for the long-term motion may be expressed in terms of Jacobian elliptic functions and elliptic integrals. Geometrical aspects of the long-term rotational motion are discussed and a comparison of theoretical results with observations is made.

1. Introduction

Since the advent of artificial satellites, there has been renewed interest in obtaining analytical theories for the rotational motions of rigid bodies about their centers of mass when the centers of mass are constrained to orbit attracting primary bodies. Earlier works, dealing with natural bodies, include Laplace's (1829) and Tisserand's (1891) investigations of the rotational motions of the earth and the moon.

Among the many recent works on the subject are those of Colombo (1964), Beletskii (1965) and Holland and Sperling (1969). However, the studies most closely related to the one discussed here are those made by Crenshaw and Fitzpatrick (1968) and Hitzl and Breakwell (1969).

Crenshaw and Fitzpatrick developed a first-order, gravity-gradient theory for the complete rotational motion of a rapidly spinning, uniaxial, rigid body, the center of mass of which was required to move in a uniformly precessing, circular orbit about an attracting point mass. They used the theory of canonical transformations to obtain a solution to the unperturbed, free-Eulerian motion, derived a set of differential equations analogous to Lagrange's planetary equations, and integrated these to first order. Hitzl and Breakwell used canonical transformation theory to study the rotational motion of a rapidly tumbling triaxial, rigid satellite, the center of mass of which was constrained to move in a fixed elliptic orbit about an attracting point mass, by applying an averaging procedure to the perturbing Hamiltonian. They studied the nonresonant and internally near-resonant effects of the gravity-gradient perturbations.

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The mathematical model used in the current study extends that of Hitzl and Breakwell to include effects of orbital evolution. That is, the plane of the orbit of the satellite's center of mass is constrained to precess and spin (movement of apsidal line) at constant rates $\dot{\Omega}$ and $\dot{\omega}$, respectively. This problem will be called the 'triaxial problem' henceforth.

The triaxial problem is studied using a slight modification of a new theory of general perturbations introduced by Hori (1966).* The method of general perturbations is used to obtain the Hamiltonian for the nonresonant secular and long-period rotational motion of the triaxial satellite to second order in n/ω_0, where n is the orbital mean motion of the center of mass and ω_0 is a reference value of the satellite's rotational angular speed. The differential equations derivable from the transformed Hamiltonian are integrable in terms of Jacobian elliptic functions and elliptic integrals. Geometrical aspects of the long-term rotational motion are discussed and a comparison of theoretical results with observations is made.

2. Coordinate Systems

It is convenient at the outset to define certain coordinate systems which will be used in what follows. In Figure 1, five orthogonal coordinate systems with their common origin O at the center of mass of the triaxial satellite are shown. The $OXYZ$ coordinate system is a nonrotating system to which rotational motion about O is referred. The $Ox^oy^oz^o$ system is associated with the orbit of the attracting point mass P about O. It may be obtained from the $OXYZ$ system by positive rotations through the angles Ω and I, the longitude of the ascending node and the inclination, respectively, of the orbit of P. The z^o-axis is directed along the normal to the orbital plane. The $Oxyz$ system is associated with the rotational angular momentum $\mathbf{H}$ of the satellite about O. The z-axis of this system is directed in the sense of $\mathbf{H}$ and the x-axis lies along the line of intersection of the XY plane and the plane normal to $\mathbf{H}$ through O, as shown in Figure 1b. The $Oxyz$ system may be obtained from the $OXYZ$ system by rotations through the angles ψ^* and θ^*. The $Ox_Hy_Hz_H$ system is associated with both the vector $\mathbf{H}$ and the orbital system $Ox^oy^oz^o$. It may be obtained from the $Ox^oy^oz^o$ system by rotations through the angles ψ_H and θ_H. Finally, the $Ox'y'z'$ coordinate system (see Figure 1c) is such that its axes are principal axes for the triaxial satellite at its center of mass. It may be obtained from $OXYZ$, $Oxyz$, and $Ox_Hy_Hz_H$ systems by rotations through the angles (ψ, θ, ϕ), (ϕ^*, θ', ϕ'), and (ϕ_H, θ', ϕ') respectively.

3. The Hamiltonian for the Triaxial Problem

It is a straightforward matter to obtain the dynamical Hamiltonian H for the triaxial problem using the Euler angles ψ, θ, ϕ, as generalized coordinates. However, if this is

* A theory similar to Hori's has been set forth by Deprit (1970) and much controversy has arisen as to the exact connection between the two theories. See, for example, Campbell and Jefferys (1970), Mersman (1971) and Henrard (1971).

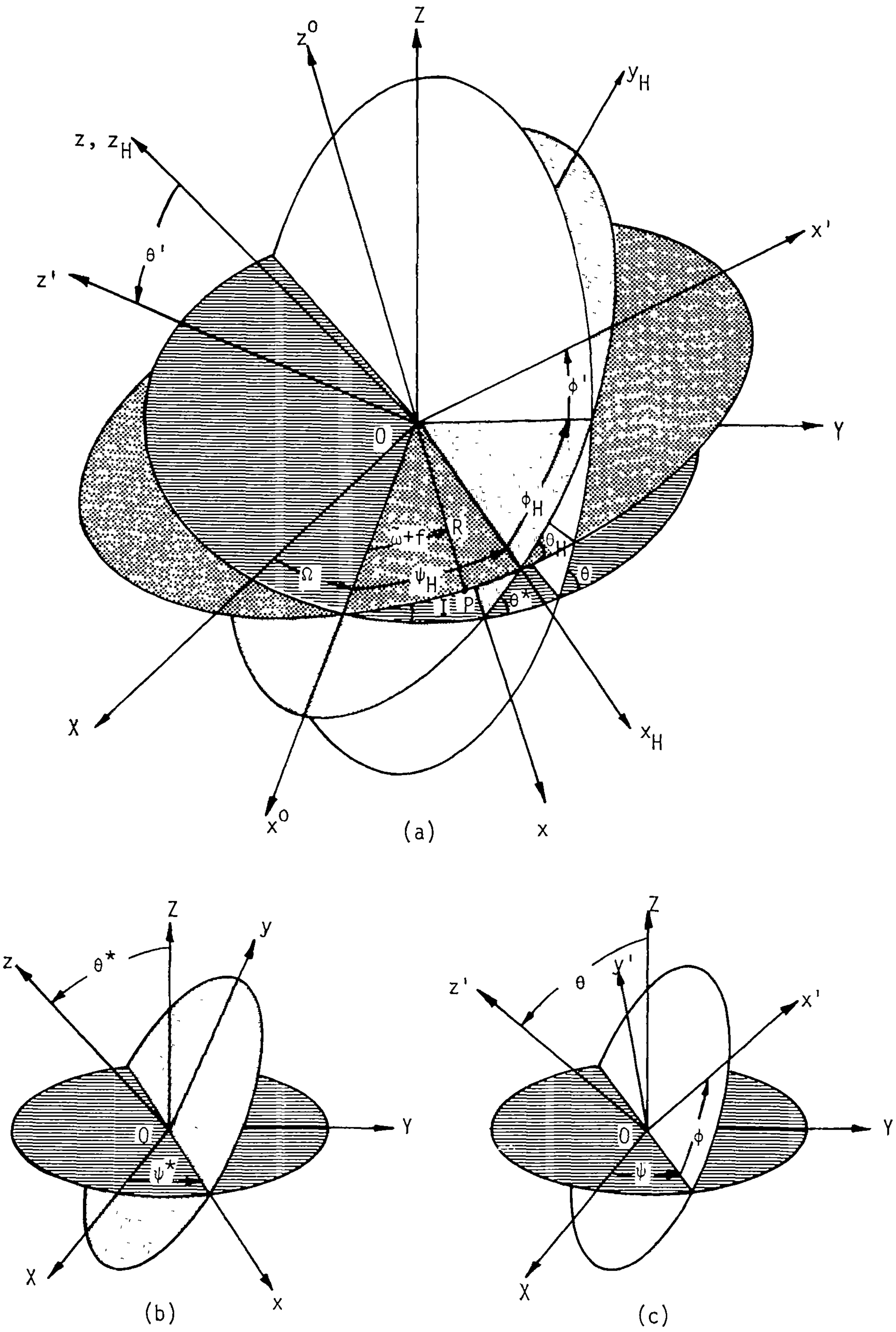

Fig. 1. Coordinate systems.

done (see Fitzpatrick, 1970), the form of H_0, the part of the Hamiltonian consisting of the rotational kinetic energy of the satellite about O, is not in a very simple form. Deprit (1967), following Tisserand to some extent, obtained a simpler form for H_0 using cononical transformations. The same technique may be applied to the triaxial Hamiltonian H expressed in terms of ψ, θ, ϕ, and their conjugate moments, to obtain the transformed Hamiltonian,

$$H^* = \frac{1}{2}\left[\frac{\sin^2\phi'}{A} + \frac{\cos^2\phi'}{B}\right](P_{\phi*}^2 - P_{\phi'}^2) + \frac{1}{2C}P_{\phi'}^2 + V,$$

where A, B, and C denote the principal moments of inertia of the satellite about the x'-, y'-, and z'-axes, respectively, and $P_{\phi*} = h \equiv |\mathbf{H}|$, $P_{\phi'} = h\cos\theta^*$, and $P_{\psi*} = h\cos\theta^*$ are the momenta conjugate to ϕ^*, ϕ', and ψ^*, respectively. The term V which represents that part of the potential energy due to the gravity-gradient torque which depends explicitly on the orientation of the satellite is given by

$$V = \frac{3}{2}\frac{GM_P}{R^3}\left[(A - B)\cos^2\gamma + (C - B)\cos^2\chi\right]. \tag{1}$$

In Equation (1), G is the universal gravitational constant, M_P is the mass of P, R is the distance form O to P, and γ and χ are the angles between the line segment OP and the x'- and z'-axes, respectively. Also, for an elliptic orbit,

$$R = a(1 - e^2)/(1 + e\cos f),$$

where a, e, and f denote the semi-major axis, eccentricity, and true anomaly, respectively.

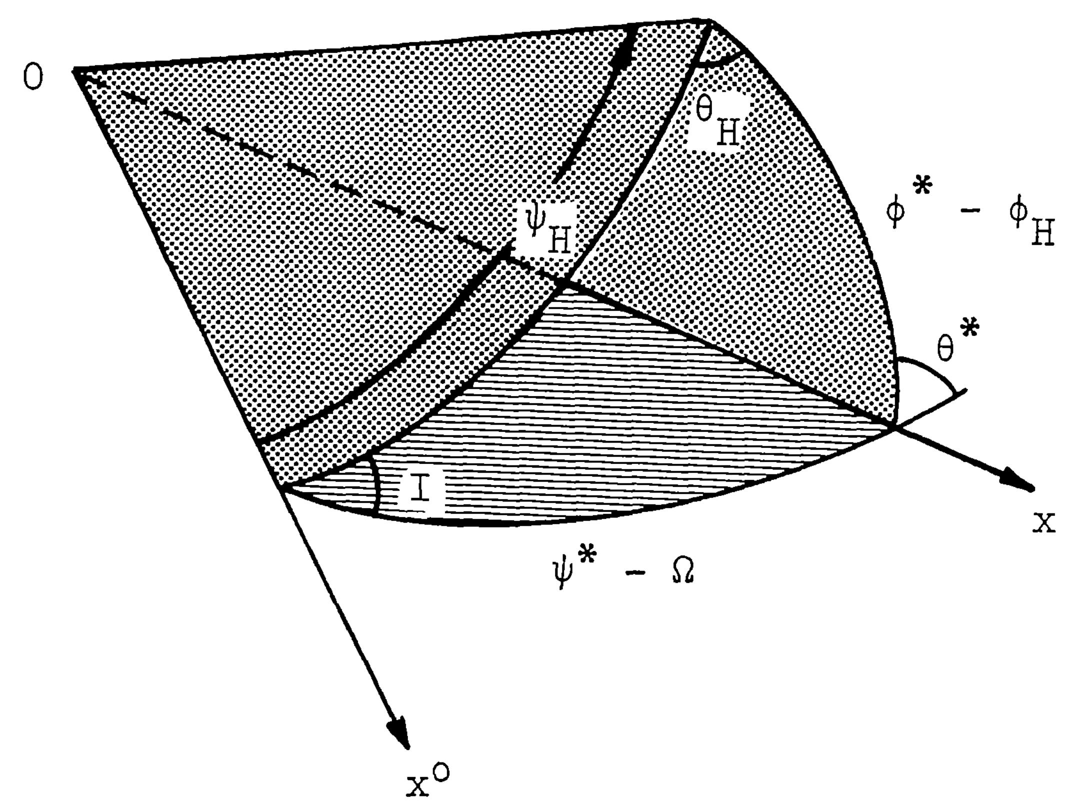

Fig. 2. Spherical triangle for the transformation $(\phi', \phi^*, \psi^*, P_{\phi'}, P_{\phi^*}, P_{\psi^*}) \to (\phi', \phi_H, \psi_H, P_{\phi'}, P_{\phi_H}, P_{\psi_H})$.

In this paper the orbit of P is assumed to precess at a constant rate $\dot{\Omega}$. For earth satellites, $\dot{\Omega}$ is $\mathcal{O}(10^{-3})$ compared to n, so that it is convenient to treat the precession as well as the gravity-gradient torque as a perturbation. This may be done by referring the rotational motion to the $OXYZ$ system using the angles Ω, I, ψ_H, θ_H, ϕ_H, θ' and ϕ'. The angles Ω, I, ψ_H, θ_H, and ϕ_H can be introduced through a canonical transformation.

By referring to the spherical triangle in Figure 2, $\sin\phi^*$, $\cos\phi^*$, $\sin\psi^*$ and $\cos\psi^*$ may be replaced by functions of Ω, I, ψ_H, θ_H and ϕ_H as follows:

$$\begin{aligned}
\sin\phi^* &= a_1 \sin\phi_H + b_1 \cos\phi_H \\
\cos\phi^* &= a_1 \cos\phi_H - b_1 \sin\phi_H \\
\sin\psi^* &= c_1 \sin\Omega + d_1 \cos\Omega \\
\cos\psi^* &= c_1 \cos\Omega - d_1 \sin\Omega,
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= (\cos I - \cos\theta^* \cos\theta_H)/\sin\theta^* \sin\theta_H \\
b_1 &= \sin I \sin\psi_H/\sin\theta^*
\end{aligned}$$

$$c_1 = (\cos\theta_H - \cos\theta^* \cos I)/\sin\theta^* \sin I$$
$$d_1 = \sin\theta_H \sin\psi_H/\sin\theta^*$$
$$\cos\theta^* = \cos\theta_H \cos I - \sin\theta_H \sin I \cos\psi_H .$$

Furthermore, by using the differential identity,

$$\mathrm{d}\phi^* = \mathrm{d}\phi_H - \cos\theta^* (\mathrm{d}\psi^* - \mathrm{d}\Omega) - \cos\theta_H \, \mathrm{d}\psi_H ,$$

obtained from the elementary spherical trigonometry of Figure 2, we find that the differential condition,

$$P_{\psi*} \, \mathrm{d}\psi^* + P_{\phi*} \, \mathrm{d}\phi^* - H^* \, \mathrm{d}t = P_{\psi_H} \, \mathrm{d}\psi_H + P_{\phi_H} \, \mathrm{d}\phi_H - H_H \, \mathrm{d}t ,$$

which is sufficient for a canonical transformation, is satisfied by

$$\begin{aligned} P_{\psi_H} &= P_{\phi*} \cos\theta_H \\ P_{\phi_H} &= P_{\phi*} \\ H_H &= H^* - \dot{\Omega} \left(P_{\psi_H} \cos I - \sqrt{P_{\phi_H}^2 - P_{\psi_H}^2} \sin I \cos\psi_H\right). \end{aligned} \tag{2}$$

In the last of Equations (2), the new Hamiltonian H_H is to be formed as indicated, after expressing H^* in terms of the variables $(\phi', \phi_H, \psi_H, P_{\phi'}, P_{\phi_H}, P_{\psi_H})$ and the time t. It turns out that the angle Ω does not appear in H_H, and since $\dot{\Omega}$, the constant rate of orbital precession, is small, the second term H_H will be treated as a perturbing Hamiltonian along with $V(\phi', \phi_H, \psi_H, P_{\phi'}, P_{\phi_H}, P_{\psi_H}, t)$.

For sufficiently small values of the eccentricity, the functions of f which appear in V may readily be expressed as functions of the mean anomaly, $M = n(t - t_0)$. The resulting Hamiltonian will, because of M and $\tilde{\omega}$, be non-autonomous, but by treating M and $\tilde{\omega}$ as additional coordinates and thereby artificially increasing the order of our system, the explicit dependence of H_H on t may be removed. The reason for doing this is that we desire an autonomous Hamiltonian as a starting point for application of the perturbation scheme which we shall employ here.

In addition to eliminating t, we shall introduce the dimensionless variables $P_1 \equiv P_{\phi'}/I_0\omega_0$, $P_2 \equiv P_{\phi_H}/I_0\omega_0$, and $P_3 \equiv P_{\psi_H}/I_0\omega_0$, where I_0 has the units of a moment of inertia and ω_0 is an angular speed such that, at $t = t_0$, $h = I_0\omega_0$ and the rotational kinetic energy, $T = \frac{1}{2} I_0 \omega_0^2$. We also introduce the notation, $\bar{A} \equiv A/I_0$, $\bar{B} \equiv B/I_0$, $\bar{C} \equiv C/I_0$, $Q_1 \equiv \phi'$, $Q_2 \equiv \phi_H$, $Q_3 \equiv \psi_H$, $Q_4 \equiv M$, and $Q_5 \equiv \tilde{\omega}$. Using these definitions, removing the explicit time dependence from H_H, dividing by $I_0\omega_0^2$, and letting F denote the negative of the result, we have

$$F = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 ,$$

where

$$F_0 = -\frac{1}{2}\left[\left(\frac{\sin^2 Q_1}{\bar{A}} + \frac{\cos^2 Q_1}{\bar{B}}\right)(P_2^2 - P_1^2) + P_1^2/\bar{C}\right]$$
$$F_1 = -P_4$$
$$F_2 = -\frac{3}{2}\left(\frac{a}{R}\right)^3 [(\bar{A} - \bar{B}) \cos^2\gamma + (\bar{C} - \bar{B}) \cos^2\chi] +$$

$$+\left(\frac{\dot{\Omega}}{n}\right)\left(\frac{\omega_0}{n}\right)[P_3 \cos I - \sqrt{P_2^2 - P_3^2}\sin I \cos Q_3] -$$

$$-\left(\frac{\tilde{\omega}}{n}\right)\left(\frac{\omega_0}{n}\right)P_5 .$$

and $(a/R)^3 \cos^2\gamma$ and $(a/R)^3 \cos^2\chi$ are functions of the Q_j, $j=1, 2, 3, 4, 5$, and P_j $j=1, 2, 3$, and do not contain $t^* \equiv t_0\omega_0$, the new independent variable, explicitly. The small parameter $\varepsilon \equiv n/\omega_0$ has appeared naturally during the introduction of dimensionless variables and inertia parameters.

The problem embodied by F will be approached as a standard, general perturbation problem treating F_1 and F_2 as perturbing Hamiltonians, and using the method of general perturbations which will now be described.

4. Method of General Perturbations

We will study the triaxial problem using a method of general perturbations which is based on the use of Lie series to generate approximate, direct canonical transformations. The procedure we will follow is essentially that developed by Hori (1966).

A convenient way to introduce the method is by considering the following autonomous, canonical, differential system:

$$\begin{aligned} \mathrm{d}P_j/\mathrm{d}\tau &= \partial S/\partial Q_j\,; & P_j(0) &= x_j \\ \mathrm{d}Q_j/\mathrm{d}\tau &= -\,\partial S/\partial P_j\,; & Q_j(0) &= y_j \quad (j = 1, 2, 3, \ldots, n), \end{aligned} \tag{3}$$

where

$$S = S(P_1, P_2, \ldots, P_n, Q_1, Q_2, \ldots, Q_n; \varepsilon)$$

is an arbitrary function except that it must be such that a solution to the system (3) exists in a domain, $0 \le |\tau| \le \delta$, $0 \le \varepsilon \le \delta$, and should have a convergent Taylor series expansion about $\varepsilon = 0$.† In Equations (3) ε is not a true variable, but a small positive constant and τ, the independent variable, is not necessarily the time.

For convenience, we let

$$\begin{aligned} \mathbf{x} &\equiv (x_1\, x_2\, x_3 \ldots x_n)^T, & \mathbf{y} &\equiv (y_1\, y_2\, y_3 \ldots y_n)^T \\ \mathbf{P} &\equiv (P_1\, P_2\, P_3 \ldots P_n)^T, & \mathbf{Q} &\equiv (Q_1\, Q_2\, Q_3 \ldots Q_n)^T \end{aligned}$$

where the superscript T denotes the transpose. Furthermore, a function $f(P_1, P_2, \ldots, P_n, Q_1, Q_2, Q_3, \ldots, Q_n; \varepsilon)$ will be denoted by $f(\mathbf{P}, \mathbf{Q}; \varepsilon)$ and the notation

$$\frac{\partial S}{\partial \mathbf{P}} = \left(\frac{\partial S}{\partial P_1}\,\frac{\partial S}{\partial P_2} \cdots \frac{\partial S}{\partial P_n}\right)^T$$

will be used.

† Since we are more concerned with obtaining a 'formal' solution, no attempt to prove a particular function S has such properties will be made.

The powers series solution to (3) is of particular interest. To implement obtaining such a solution the additional notation

$$\begin{aligned} D_s^0 v &\equiv v \\ D_s^1 v &\equiv \{v, S\} \\ D_s^2 v &\equiv \{\{v, S\}, S\} \\ &\vdots \\ D_s^k &\equiv \{D_s^{k-1} v, S\}, \end{aligned}$$

where $\{v, S\}$ denotes the Poisson bracket of v, a differentiable function of the P_j and Q_j, and of S, is adopted. Then, the power series solution to (3) may be written as

$$\begin{aligned} \mathbf{P} &= \mathbf{x} + \sum_{k=1}^{\infty} \frac{\tau^k}{k!} D_s^k \mathbf{x} \\ \mathbf{Q} &= \mathbf{y} + \sum_{k=1}^{\infty} \frac{\tau^k}{k!} D_s^k \mathbf{y}. \end{aligned} \tag{4}$$

In Equations (4), $S = S(\mathbf{x}, \mathbf{y}; \varepsilon)$.

It may also be easily shown that an indefinitely differentiable function, $f(\mathbf{P}, \mathbf{Q}) = f(P_1, P_2, \ldots, P_n, Q_1, Q_2, \ldots Q_n)$, has the series representation

$$f(\mathbf{P}, \mathbf{Q}) = f(\mathbf{x}, \mathbf{y}) + \sum_{k=1}^{\infty} \frac{\tau^k}{k!} D_s^k f. \tag{5}$$

Furthermore, because the system (3) is autonomous, we have

$$\begin{aligned} \mathbf{x} &= \mathbf{P} + \sum_{k=1}^{\infty} \frac{(-\tau)^k}{k!} D_s^k \mathbf{P} \\ \mathbf{y} &= \mathbf{Q} + \sum_{k=1}^{\infty} \frac{(-\tau)^k}{k!} D_s^k \mathbf{Q} \end{aligned} \tag{6}$$

and

$$f(\mathbf{x}, \mathbf{y}) = f(\mathbf{P}, \mathbf{Q}) + \sum_{k=1}^{\infty} \frac{(-\tau)^k}{k!} D_s^k f(\mathbf{P}, \mathbf{Q}), \tag{7}$$

where $S = S(\mathbf{P}, \mathbf{Q}; \varepsilon)$.

Equations (4) represent infinitely many canonical transformations and equations (6) represent the corresponding inverse transformations. The particular transformation we will use is that obtained from (4) by letting $\tau = \varepsilon$,

$$\mathbf{P}(\varepsilon) = \mathbf{x} + \sum_{k=1}^{\infty} \frac{\varepsilon^k}{k!} D_s^k \mathbf{x}$$
$$\mathbf{Q}(\varepsilon) = \mathbf{y} + \sum_{k=1}^{\infty} \frac{\varepsilon^k}{k!} D_s^k \mathbf{y}. \tag{8}$$

Equations (8), along with the corresponding transformation equation,

$$f(\mathbf{P}, \mathbf{Q}) = f(\mathbf{x}, \mathbf{y}) + \sum_{k=1}^{\infty} \frac{\varepsilon^k}{k!} D_s^k f(\mathbf{x}, \mathbf{y}), \tag{9}$$

for an arbitrary function $f(\mathbf{P}, \mathbf{Q})$, form the basis of our perturbation method.

Let S be expressed in the form $S = S_1(\mathbf{P}, \mathbf{Q}) + \varepsilon S_2(\mathbf{P}, \mathbf{Q}) + \varepsilon^2 S_3(\mathbf{P}, \mathbf{Q}) + \ldots$ and let $F(\mathbf{P}, \mathbf{Q}; \varepsilon) = F_0(\mathbf{F}, \mathbf{Q}) + \varepsilon F_1(\mathbf{P}, \mathbf{Q}) + \varepsilon^2 F_2(\mathbf{P}, \mathbf{Q}) + \ldots$ denote the Hamiltonian for an autonomous dynamical system which has a solution when $\varepsilon = 0$. Then, if we transform from the canonical set $(\mathbf{P}, \mathbf{Q})$ to the set $(\mathbf{x}, \mathbf{y})$ using the autonomous transformation Equations (8), we have $F^*(\mathbf{x}, \mathbf{y}; \varepsilon) = F(\mathbf{P}(\mathbf{x}, \mathbf{y}; \varepsilon), Q(\mathbf{x}, \mathbf{y}; \varepsilon); \varepsilon)$ as the new Hamiltonian. It then follows, from (9) and the expressions for S and $F(\mathbf{P}, \mathbf{Q}; \varepsilon)$, that

$$F^*(\mathbf{x}, \mathbf{y}; \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k F_k^* \tag{10}$$

where

$$\begin{aligned} F_0^* &= F_0(\mathbf{x}, \mathbf{y}) \\ F_1^* &= F_1(\mathbf{x}, \mathbf{y}) + \{F_0, S_1\} \\ F_2^* &= F_2(\mathbf{x}, \mathbf{y}) + \tfrac{1}{2}\{F_1 + F_1^*, S_1\} + \{F_0, S_2\} \\ &\vdots \end{aligned} \tag{11}$$

We have given only the first three terms of F^* explicitly, since only these will be used here.

For a particular $F(\mathbf{P}, \mathbf{Q}; \varepsilon)$, it is our aim to choose the functions S_k in such a manner that the differential equations associated with F^* are more tractable than the original canonical equations. If F_1 contains both momenta and coordinates, the partial differential equations which are obtained by expanding the Poisson brackets in Equations (11) may be complicated, thus making the choices of the S_k difficult. To simplify the brackets $\{F_0, S_k\}$ and implement the choices of the S_k, we shall introduce an auxiliary transformation into Equations (11). This transformation is defined by a complete integral of the Hamilton-Jacobi equation,

$$F_0\left(-\frac{\partial \bar{S}}{\partial \mathbf{y}}, \mathbf{y}\right) + \frac{\partial \bar{S}}{\partial t} = 0. \tag{12}$$

The complete integral $\bar{S}=\bar{\alpha}_1 t+\bar{S}_1(\bar{\boldsymbol{\alpha}}, \mathbf{y})$, where $\bar{\boldsymbol{\alpha}}=(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n)^T$ and the $\bar{\alpha}_j$ are new canonical variables, defines the transformation

$$\begin{aligned} \mathbf{x} &= -\ \partial \bar{S}/\partial \mathbf{y} \\ t+\bar{\beta}_1 &= -\ \partial \bar{S}_1/\partial \bar{\alpha}_1 \\ \bar{\beta}_j &= -\ \partial \bar{S}_1/\partial \bar{\alpha}_j \quad (j=2,3,\ldots,n), \end{aligned} \tag{13}$$

and the new Hamiltonian, $K^*(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t;\varepsilon)$ is given by

$$K^*(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t;\varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k K_k^*, \tag{14}$$

where

$$K_k^* = F_k^*(\mathbf{x}(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t), \mathbf{y}(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t)).$$

Since a Poisson bracket is invariant under a canonical transformation, by using Equations (13) and the fact that $F_0=-\bar{\alpha}_1$, we find that Equations (11) become

$$\begin{aligned} K_1^* &= F_1(\mathbf{x}(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t), \mathbf{y}(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t)) - \partial \bar{S}/\partial \bar{\beta}_1 \\ K_2^* &= F_2(\mathbf{x}(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t), \mathbf{y}(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t)) + \tfrac{1}{2}\{F_1+F_1^*, S_1\} - \partial S_2/\partial \bar{\beta}_1 . \\ &\vdots \end{aligned} \tag{15}$$

We now try to choose the S_k successively, in such a manner that our transformed dynamical system is easier to solve than the original system. If the Hamiltonian $F(\mathbf{P}, \mathbf{Q};\varepsilon)$ is periodic in the Q_j with period 2π, an acceptable choice of S_1 is

$$S_1 = \int F_{1_p}\, \mathrm{d}\bar{\beta}_1, \tag{16}$$

where F_{1_p} is the part of $F_1(\mathbf{x}(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t), \mathbf{y}(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t))$ which contains $\bar{\beta}_1$. In general, we will let the subscript s denote the part of a function of $\bar{\boldsymbol{\alpha}}$, $\bar{\boldsymbol{\beta}}$, and t which does not contain $\bar{\beta}_1$ explicitly, while the subscript p will denote the remainder of that same function. Then, along with (16), we have, through second order in ε,

$$\begin{aligned} K_1^* &= F_{1s} \\ S_2 &= \int [F_2 + \tfrac{1}{2}\{F_1+F_1^*, S_1\}]_p\, \mathrm{d}\bar{\beta}_1 \\ K_2^* &= F_{2s} + \tfrac{1}{2}\{F_1+F_1^*, S_1\}_s . \end{aligned} \tag{17}$$

If K_1^* and K_2^* are chosen according to (17) and F is periodic in the Q_j with period 2π, they will not contain t or $\bar{\beta}_1$ explicitly. Thus, through second order in ε, we have

$$K^*(\bar{\boldsymbol{\alpha}}, -, \bar{\beta}_2, \bar{\beta}_3, \ldots, \bar{\beta}_n, -) = \text{constant}, \tag{18}$$

where a dash is used in place of a variable to emphasize its explicit absence from K^*, and since $\bar{\beta}_1$ is ignorable, we also have

$$\dot{\bar{\alpha}}_1 = \partial K^*/\partial \bar{\beta}_1 = 0,$$

or

$$\bar{\alpha}_1 = \text{constant}, \tag{19}$$

which is a new integral of the system. The remaining canonical equations are

$$\begin{aligned}\dot{\bar{\alpha}}_j &= \partial K^*/\partial \bar{\beta}_j \quad (j = 2, 3, \ldots, n) \\ \dot{\bar{\beta}}_j &= -\,\partial K^*/\partial \bar{\alpha}_j \quad (j = 1, 2, \ldots, n).\end{aligned} \tag{20}$$

If Equations (20) are integrable, the problem is solved through second order in ε. If they are not integrable, but the equations,

$$\begin{aligned}\dot{\bar{\boldsymbol{\alpha}}} &= \varepsilon\,(\partial K_1^*/\partial \bar{\boldsymbol{\beta}}) \\ \dot{\bar{\boldsymbol{\beta}}} &= -\,\varepsilon\,(\partial K_1^*/\partial \bar{\boldsymbol{\alpha}}),\end{aligned} \tag{21}$$

are integrable, then the procedure just described may be applied to the Hamiltonian $K^*(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}; \varepsilon)$ in another effort to obtain additional integrals and/or integrable equations.

It may be seen that each time the method we have outlined is applied, two canonical transformations, one approximate transformation via Lie series and one exact, auxiliary transformation using the solution to the Hamilton-Jacobi equation for the 'unperturbed' problem are made. The method given here differs from Hori's method (Hori, 1966) in the use of the auxiliary Hamilton-Jacobi equation and the invariance of the Poisson bracket under a canonical transformation to obtain the simplified Equations (15).

It should be noted that the auxiliary transformation may not always be needed. If F_0 is a function of only the P_j, then the brackets $\{F_0, S_k\}$ will not be very complicated and the S_k can easily be chosen so that one or more of the y_j are eliminated from F^*.

5. Long-Term Motion in the Triaxial Problem

5.1. Transformations

In applying the perturbation method just described to the triaxial problem, we first make a second-order canonical transformation defined by

$$\begin{aligned}\mathbf{P} &= \mathbf{x} + \varepsilon\{\mathbf{x}, S_1\} + \varepsilon^2\left[\{\mathbf{x}, S_2\} + \tfrac{1}{2}\{\{\mathbf{x}, S_1\}, S_1\}\right] \\ \mathbf{Q} &= \mathbf{y} + \varepsilon\{\mathbf{y}, S_1\} + \varepsilon^2\left[\{\mathbf{y}, S_2\} + \tfrac{1}{2}\{\{\mathbf{y}, S_1\}, S_1\}\right].\end{aligned} \tag{22}$$

This gives us the new Hamiltonian

$$F^*(\mathbf{x}, \mathbf{y}; \varepsilon) = F_0^* + \varepsilon F_1^* + \varepsilon^2 F_2^*, \tag{23}$$

where

$$\begin{aligned}F_0^* &= -\tfrac{1}{2}\left[\left(\frac{\sin^2 y_1}{\bar{A}} + \frac{\cos^2 y_1}{\bar{B}}\right)(x_2^2 - x_1^2) + x_1^2/\bar{C}\right] \\ F_1^* &= -\,x_4 + \{F_0, S_1\} \\ F_2^* &= -\frac{3}{2}\left(\frac{a}{R}\right)^3 \left[(\bar{A} - \bar{B})\cos^2\gamma + (\bar{C} - \bar{B})\cos^2\chi\right]\Big|_{\substack{P=x\\Q=y}} + \\ &\quad + \left(\frac{\dot{\Omega}}{n}\right)\left(\frac{\omega_0}{n}\right)\left[x_3\cos I - \sqrt{x_2^2 - x_3^2}\,\sin I\cos y_3\right] - \\ &\quad - \left(\frac{\tilde{\omega}}{n}\right)\left(\frac{\omega_0}{n}\right)x_4 + \tfrac{1}{2}\{F_1 + F_1^*, S_1\} + \{F_0, S_2\}.\end{aligned} \tag{24}$$

The second and third of Equations (24) are to be simplified by choosing S_1 and S_2, respectively. Clearly, the second equation is simple if we choose $S_1 \equiv 0$, and this choice simplifies the third equation. The choice of S_2 is more difficult.

Since F_0 contains x_1 *and* y_1, the expanded form of the Poisson bracket $\{F_0, S_2\}$ in terms of x_j and the y_j is complicated. To simplify this form, we introduce an auxiliary canonical transformation which is defined by the complete integral,

$$\bar{S} = \bar{\alpha}_1 t^* - \sum_{j=2}^{5} \bar{\alpha}_j y_j - \int_{y_{10}}^{y_1} x_1 \, \mathrm{d}y_1 , \tag{25}$$

of the Hamilton-Jacobi equation,

$$-\left[\frac{\sin^2 y_1}{2\bar{A}} + \frac{\cos^2 y_1}{2\bar{B}}\right]\left[\left(\frac{\partial \bar{S}}{\partial y_2}\right)^2 - \left(\frac{\partial \bar{S}}{\partial y_1}\right)^2\right] - \left(\frac{\partial \bar{S}}{\partial y_1}\right)^2 /2\bar{C} + \frac{\partial \bar{S}}{\partial t^*} = 0 . \tag{26}$$

In (25), $\bar{\alpha}_1 = -F_0(\mathbf{x}, \mathbf{y})$ and $\bar{\alpha}_j, j=2, 3, 4, 5$, are new canonical variables, while

$$x_1 = \sqrt{\bar{C}\,\frac{(a' + b' \sin^2 y_1)}{(c' + d' \sin y_1)}}, \tag{27}$$

where

$$\begin{aligned} a' &= \bar{A}\,(2\bar{B}\bar{\alpha}_1 - \bar{\alpha}_2^2) \\ b' &= \bar{\alpha}_2^2\,(\bar{A} - \bar{B}) \\ c' &= \bar{A}\,(\bar{B} - \bar{C}) \\ d' &= \bar{C}\,(\bar{A} - \bar{B}) \end{aligned}$$

and $y_{1\,0}$will be specified in what follows. The plus sign is taken on the radical in (27), since by proper labeling of the principal axes we may make $0 \leq \theta' < \pi/2$ and since $\theta' \equiv \equiv \cos^{-1}(x_1/x_2)$ should also be in this range for small perturbations in P_1 and P_2.†

The transformation equations derived from $\bar{S}(\bar{\boldsymbol{\alpha}}, \mathbf{y}, t)$, according to (13), in addition to (27) are

$$\begin{aligned} x_j &= \bar{\alpha}_j \quad (j = 2, 3, 4, 5) \\ t^* + \bar{\beta}_1 &= I_1 \\ y_2 - \bar{\beta}_2 &= I_2 \\ y_j &= \bar{\beta}_j \quad (j = 3, 4, 5), \end{aligned} \tag{28}$$

where

$$\begin{aligned} I_1 &= \int_{y_{10}}^{y_1} \frac{\bar{A}\bar{B}\sqrt{\bar{C}}\,\mathrm{d}y_1}{\sqrt{(a' + b' \sin^2 y_1)\,(c' + d' \sin^2 y_1)}} \\ I_2 &= \int_{y_{10}}^{y_1} \frac{\sqrt{\bar{C}}\,[\bar{B} + (\bar{A} - \bar{B}) \cos y_1]\,\mathrm{d}y_1}{\sqrt{(a' + b' \sin^2 y_1)\,(c' + d' \sin^2 y_1)}} . \end{aligned} \tag{29}$$

† Only motion which does not correspond to a separatrix polhode will be considered here. See, for example, MacMillan (1960).

The integral I_1 may be simplified by making a change of variable defined by $U \tan y_1 = \cot\zeta$, where $U = [\bar{A}(\bar{B}-\bar{C})/\bar{B}(\bar{A}-\bar{C})]^{\frac{1}{2}}$. Setting $y_{10} = -\pi/2$ so that $\zeta_0 = 0$, from (29), we get

$$u = \int_0^{\zeta} \frac{d\zeta}{\sqrt{1 - k^2 \sin^2 \zeta}}, \tag{30}$$

where

$$u = \lambda(t^* + \bar{\beta}_1)$$
$$\lambda = \sqrt{\frac{(\bar{B} - \bar{C})(2\bar{B}\bar{\alpha}_1 - \bar{\alpha}_2^2)}{\bar{A}\bar{B}\bar{C}}}$$
$$k^2 = \frac{(\bar{A} - \bar{B})(\bar{\alpha}_2^2 - 2C\bar{\alpha}_1)}{(\bar{B} - \bar{C})(2A\bar{\alpha}_1 - \bar{\alpha}_2^2)}.$$

From (30) and the theory of elliptic functions, it follows that $\sin\zeta = \mathrm{sn}\,u$. We note that $-\bar{\beta}_1$ is the value of t^* when $y_{10} = -\pi/2$.

Using the above results and formulae in Whittaker and Watson (1963), we find that, if $\alpha^2 \equiv C(A-B)/A(B-C)$ and $k^2\,\mathrm{sn}\,ia \equiv -\alpha^2$, where $i = \sqrt{-1}$, then

$$I_2 = \left[\frac{\bar{\alpha}_2}{\bar{A}} - iZ(ia)\,\lambda\right](t^* + \bar{\beta}_1) + (i/2)\ln\left[\frac{\Theta(u + ia)}{\Theta(u - ia)}\right]. \tag{31}$$

In Equation (31), $Z(ia)$ is Jacobi's Zeta-function and $\Theta(u \pm ia)$ are Theta-functions. The coefficient of $(t^* + \bar{\beta}_1)$ in Equation (31) is the mean rate of precession of the z'-axis about the z-axis if $\varepsilon = 0$ and $\dot{\Omega} = 0$.

Using the above results, and setting

$$\bar{R} = \sqrt{\frac{\bar{C}(2\bar{A}\bar{\alpha}_1 - \bar{\alpha}_2^2)}{\bar{\alpha}_2^2(\bar{A} - \bar{C})}},$$

we may write (27) as

$$x_1 = \bar{\alpha}_2 \bar{R}\,\mathrm{dn}\,u. \tag{32}$$

Also, since $\sin\zeta = \mathrm{sn}\,u$ and $\cos\zeta = \mathrm{cn}\,u$, we have

$$\begin{aligned} \sin y_1 &= -\,\mathrm{cn}\,u/\sqrt{1 + \alpha^2\,\mathrm{sn}^2 u} \\ \cos y_1 &= U^{-1}\,\mathrm{sn}\,u/\sqrt{1 + \alpha^2\,\mathrm{sn}^2 u}. \end{aligned} \tag{33}$$

The solution for the unperturbed problem, embodied in Equations (28), (31), (32), and (33), corresponds exactly to that given by Whittaker (1965) for a freely spinning, triaxial, rigid body. If $\varepsilon = 0$ and $\dot{\Omega} = 0$, $y_1 = \phi'$ is the angle of spin, $y_2 = \phi_H$ is angle of precession, and $\bar{\theta}' = \theta'$ is the nutation angle of the satellite.

When the equations for the transformation $(\mathbf{x}, \mathbf{y}) \to (\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}})$ are substituted into the last two of Equations (24), we get

$$\begin{aligned} F_1^* &= -\,\bar{\alpha}_4 - \partial S_1/\partial\bar{\beta}_1 \\ F_2^* &= F_2\big(x(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t^*), y(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t^*)\big) + \tfrac{1}{2}\{F_1 + F_1^*, S_1\} - \partial S_2/\partial\bar{\beta}_1. \end{aligned} \tag{34}$$

As pointed out earlier, the obvious choice for S_1 is $S_1 \equiv 0$; for, then we have, from (34),

$$F_1^* = -\bar{\alpha}_4$$
$$F_2^* = F_2\big(\mathbf{x}(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t^*), \mathbf{y}(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t^*)\big) - \partial S_2/\partial \bar{\beta}_1 .$$

In principle, the explicit substitution of the transformation equations (27) and (28) into F^* can be carried out using the expressions

$$\begin{aligned}\cos\gamma = {} & \cos\mu\,(\cos y_2 \cos y_1 - \sin y_2 \sin y_1 \cos\bar{\theta}') - \\ & - \cos\bar{\theta}_H \sin(\sin y_2 \cos y_1 + \cos y_2 \sin y_1 \cos\bar{\theta}') + \\ & + \sin\bar{\theta}_H \sin\mu \sin\bar{\theta}' \sin y_1\end{aligned}$$

and

$$\cos\chi = \cos\mu \sin y_2 \sin\bar{\theta}' + \cos\bar{\theta}_H \sin\mu \cos y_2 \sin\bar{\theta}' + \sin\bar{\theta}_H \sin\mu \cos\bar{\theta}' ,$$

where $\mu \equiv y_3 - (\tilde{\omega} + f)$, $\cos\bar{\theta}' \equiv x_1/x_2$, and $\cos\bar{\theta}_H \equiv x_3/x_2$. This will not be done here; however, it is fairly easy to show that $\cos^2\gamma$ and $\cos^2\chi$ have the forms

$$\begin{aligned}\cos^2\gamma = {} & \tfrac{1}{4}(1 - 3\cos^2\bar{\theta}_H)\sin^2\bar{\theta}' \sin^2 y_1 + \tfrac{1}{4}(1 - \sin^2\bar{\theta}' \sin^2 y_1) \times \\ & \times \sin^2\bar{\theta}_H \cos 2\mu + \sum_{i=-2}^{2}\sum_{j=-2}^{2}\sum_{k=-2}^{2} B_{ijk}\cos(iy_1 + ky_2)\end{aligned}$$

and

$$\cos^2\chi = \sum_{j=-2}^{2}\sum_{k=-2}^{2} C_{jk}\cos(j\mu + ky_2),$$

where $B_{000} = \frac{1}{4}(1 + \cos^2\bar{\theta}_H)$, $C_{00} = \frac{1}{4}[(1 + \cos^2\bar{\theta}_H) + (1 - 3\cos^2\bar{\theta}_H)\cos^2\bar{\theta}']$, $B_{i20} = 0$, $C_{20} = \frac{1}{4}\sin^2\bar{\theta}_H(1 - 3\cos^2\bar{\theta}')$, and B_{ij0} and C_{j0} are zero for $j \neq 2$.

Now, y_2 is monotonically increasing with t^* and, if the possibility of internal resonance (Hitzl and Breakwell, 1969) is not considered, the parts of $\cos^2\gamma$ and $\cos^2\chi$ which are free of $\bar{\beta}_1$ are those parts of the terms outside the summation signs which are free of $\bar{\beta}_1$, plus those parts of B_{000}, C_{00} and C_{20} free of $\bar{\beta}_1$. To determine the parts of these terms which are free of $\bar{\beta}_1$, we must consider only the functions $\cos^2\bar{\theta}'$ and $\sin^2\bar{\theta}' \sin^2 y_1$.

We have $\cos^2\bar{\theta}' = R^2 \,\mathrm{dn}^2 u$ and $\sin^2 y_1 = \mathrm{cn}^2 u/(1 + \alpha^2 \,\mathrm{sn}^2 u)$. Hence, by using $\cos^2\bar{\theta}' = 1 - \sin^2\bar{\theta}'$, $\mathrm{dn}^2 u = 1 - k^2 \,\mathrm{sn}^2 u$ and the identity†

$$\mathrm{dn}^2 u \equiv E/K + Z'(u),$$

where $Z'(u) = \mathrm{d}Z(u)/\mathrm{d}u$ is periodic in u, we find that the parts of $\cos^2\bar{\theta}'$ and $\sin^2\bar{\theta}'$ $\sin^2 y_1$ free of $\bar{\beta}_1$ are

$$\cos^2\bar{\theta}' = \bar{R}^2 E/K \tag{35}$$

and

$$\sin^2\bar{\theta}' \sin^2 y_1 = \bar{Q}^2(E - k'^2 K)/k^2, \tag{36}$$

respectively. In (36), $\bar{Q}^2 = \bar{B}(\bar{\alpha}_2^2 - 2\bar{C}\bar{\alpha}_1)/[\bar{\alpha}_2^2(\bar{B} - \bar{C})]$ and $k'^2 = 1 - k^2$.

† Here we have used the facts that $E(u) = \int \mathrm{dn}^2 u \,\mathrm{d}u$ and $E(u) = (E/K)u + Z(u)$, where $E(u)$ is the incomplete elliptic integral of the second kind and K and E are complete elliptic integrals of the first and second kinds, respectively.

Using (35) and (36) along with the expressions for $\cos^2\gamma$ and $\cos^2\chi$, we obtain

$$K^*(\bar{\boldsymbol{\alpha}}, \bar{\beta}_3, \bar{\beta}_4, \bar{\beta}_5; \varepsilon) = -\varepsilon\bar{\alpha}_4 - \varepsilon^2(\dot{\bar{\omega}}/n)(\omega_0/n)\,\bar{\alpha}_5 + \varepsilon^2[K_\Omega - \langle\!\langle V_G\rangle\!\rangle],$$

where

$$\begin{aligned} K_\Omega &= (\dot{\Omega}/n)(\omega_0/n)\left[\bar{\alpha}_3\cos I - \sqrt{\bar{\alpha}_2^2 - \bar{\alpha}_3^2}\sin I\cos\bar{\beta}_3\right] \\ \langle\!\langle V_G\rangle\!\rangle &= \tfrac{3}{8}(a/R)^3\{(\bar{A} + \bar{C} - 2\bar{B})(1 + \cos^2\bar{\theta}_H) + (1 - 3\cos^2\bar{\theta}_H)\,\Delta + \\ &\quad + (\bar{A} + \bar{C} - 2\bar{B})(1 - \cos^2\bar{\theta}_H)\cos 2\bar{\mu}\} \\ \Delta &= \bar{B}[(E/K)(\bar{B} - \bar{C})/\bar{B} - 1](2\bar{A}\bar{\alpha}_1/\bar{\alpha}_2^2 - 1) + \bar{A} - \bar{B} \end{aligned} \tag{37}$$

and

$$\bar{\mu} \equiv \bar{\beta}_3 - [\bar{\beta}_5 + f(\beta_4)].$$

The generating function S_2 is given by

$$S_2 = -\int[V_G(\mathbf{x}(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t^*), \mathbf{y}(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, t^*)) - \langle\!\langle V_G\rangle\!\rangle]\,\mathrm{d}\bar{\beta}_1, \tag{38}$$

where $V_G = (1/n^2)\,V$. Exact analytical evaluation of the integral in (38) appears to be a very formidable task and has not been done; however, an approximate analytical evaluation has been given by Cochran (1970).The numerical evaluation of the short-period perturbations derivable from S_2 involves only numerical quadratures, and is not considered in this paper. Only secular and long-period perturbations will be discussed.

We note that K^* contains neither $\bar{\beta}_1$ nor $\bar{\beta}_2$, so that the new canonical equations are

$$\begin{aligned} \mathrm{d}\bar{\alpha}_j/\mathrm{d}t^* &= 0 && (j = 1, 2) \\ \mathrm{d}\bar{\alpha}_j/\mathrm{d}t^* &= \partial K^*/\partial\bar{\beta}_j && (j = 3, 4, 5) \\ \mathrm{d}\bar{\beta}_j/\mathrm{d}t^* &= -\partial K^*/\partial\bar{\alpha}_j && (j = 1, 2, 3, 4, 5). \end{aligned} \tag{39}$$

The integrals, $\bar{\alpha}_1 = \text{constant}$ and $\bar{\alpha}_2 = \text{constant}$, of Equations (39) express the facts that the average values of the rotational kinetic energy of the satellite and the magnitude of the satellite's rotational angular momentum, respectively, are constant.

Equations (39) do not appear to be completely integrable. To obtain equations which are integrable, we make the approximate transformation,

$$\begin{aligned} \bar{\boldsymbol{\alpha}} &= \boldsymbol{\xi} + \varepsilon(\partial S_1^*/\partial\boldsymbol{\eta}) \\ \bar{\boldsymbol{\beta}} &= \boldsymbol{\eta} - \varepsilon(\partial S_1^*/\partial\boldsymbol{\xi}). \end{aligned} \tag{40}$$

The function S_1^* is at this point arbitrary and the new Hamiltonian is

$$K^{**}(\boldsymbol{\xi}, \boldsymbol{\eta}; \varepsilon) = \varepsilon K_1^{**} + \varepsilon^2 K_2^{**}$$

where

$$\begin{aligned} K_1^{**} &= -\xi_4 \\ K_2^{**} &= K_2^*(\boldsymbol{\xi}, \boldsymbol{\eta}) - \partial S_1^*/\partial\eta_4. \end{aligned} \tag{41}$$

Since the second of Equations (41) is a simple partial differential equation, no auxiliary transformation will be made. Instead, we let $K_{2_p}^*$ denote the part of K_2^* which contains η_4 and choose

$$S_1^* = \int K_{2_p}^* \, \mathrm{d}\eta_4 . \tag{42}$$

To $\mathcal{O}(e^3)$, S_1^* is given by

$$\begin{aligned} S_1^* = {} & f_1 \left[\left(\tfrac{9}{8}\right) e \sin \eta_4 + \left(\tfrac{27}{32}\right) e^2 \sin 2\eta_4 + \left(\tfrac{3}{16}\right) e \sin (2\eta_3 - 2\eta_5 - \eta_4) \right] + \\ & + f_2 \left\{ \left[-\tfrac{3}{16} + \left(\tfrac{15}{32}\right) e^2 \right] \sin (2\eta_3 - 2\eta_5 - 2\eta_4) - \right. \\ & \left. - \left(\tfrac{7}{16}\right) e \sin (2\eta_3 - 2\eta_5 - 3\eta_4) - \left(\tfrac{51}{64}\right) e^2 \sin (2\eta_3 - 2\eta_5 - 4\eta_4) \right\}, \end{aligned}$$

where

$$\begin{aligned} f_1 &= (\bar{A} + \bar{C} - 2\bar{B}) (1 + \cos^2 \hat{\theta}_H) + (1 - 3 \cos^2 \hat{\theta}_H) \Delta \\ f_2 &= (\bar{A} + \bar{C} - 2\bar{B} - 3\Delta) (1 - \cos^2 \hat{\theta}_H) . \end{aligned}$$

In the expressions for $f_{1,2}$, Δ is to be obtained by replacing $\bar{\alpha}_1$ by ξ_1 and $\bar{\alpha}_2$ by ξ_2 in the last of Equations (37). Also, we have adopted the notation $\cos \hat{\theta}_H \equiv \xi_3/\xi_2$.

The perturbations in the variables $\bar{\alpha}_j$ and $\bar{\beta}_j$, which may be computed using Equations (40), may be termed 'quasi-long period perturbations' since they are perturbations with periods of order $2\pi/n$. We also note that the amplitudes of the seperturbations are of order ε.

Using the choice, (42), of S_1^*, we have, from the last of Equations (41),

$$K_2^{**} = K_\Omega^* (\boldsymbol{\xi}, \boldsymbol{\eta}) - (\dot{\tilde{\omega}}/n)(\omega_0/n)\, \xi_5 - \langle V_G \rangle ,$$

where

$$\begin{aligned} K_\Omega^* (\boldsymbol{\xi}, \boldsymbol{\eta}) &= (\dot{\Omega}/n)\, (\omega_0/n) \left[\xi_3 \cos I - \sqrt{\xi_2^2 - \xi_3^2} \sin I \cos \eta_3 \right] \\ \langle V_G \rangle &= \tfrac{3}{8} (1 - e^2)^{-3/2} \left[(\bar{A} + \bar{C} - 2\bar{B}) (1 + \cos^2 \hat{\theta}_H) + \right. \\ & \qquad \left. + (1 - 3 \cos^2 \hat{\theta}_H) \Delta \right] . \end{aligned} \tag{43}$$

In the second of Equations (43), $(1-e^2)^{-3/2}$ is the part of $(a/R)^3$ which does not contain $\eta_4 = M$. Note that the rotation of the apsidal line of the orbit of P does not affect the long-term rotational motion of the satellite.

The new Hamiltonian K^{**} does not contain η_1, η_2, η_4 or η_5, and the new canonical equations are

$$\begin{aligned} \mathrm{d}\xi_j/\mathrm{d}t^* &= 0 \quad (j = 1, 2, 4, 5) \\ \mathrm{d}\xi_3/\mathrm{d}t^* &= \partial K^{**}/\partial \eta_3 \\ \mathrm{d}\boldsymbol{\eta}/\mathrm{d}t^* &= - \partial K^{**}/\partial \boldsymbol{\xi} . \end{aligned} \tag{44}$$

Hence, ξ_1, ξ_2, ξ_4, and ξ_5 are constant. Also, from the last two of Equations (44),we immediately obtain

$$\begin{aligned} \eta_4 &= nt + \eta_{40} = M \\ \eta_5 &= \dot{\tilde{\omega}} t + \eta_{50} = \tilde{\omega} . \end{aligned} \tag{45}$$

These last results represent merely the recovery of our assumed variations for M and $\tilde{\omega}$.

The rest of Equations (44) can also be integrated. Their solution, which will now be considered, determines the long-term rotational motion of the satellite.

5.2. Long-term motion

Since a large portion of the previous work on problems of rotational motion dealt with uniaxial satellites, during the course of the work which led to this paper, a set of canonical variables which may be related to uniaxial variables was introduced for comparative purposes. The variables in this set are

$$\begin{aligned} L_1 &= \sqrt{\frac{\bar{B}\bar{C}(2\bar{A}\xi_1 - \xi_2^2)}{\bar{A}(\bar{B}-\bar{C})}} \\ L_j &= \xi_j \quad (j = 2, 3, 4, 5) \\ l_1 &= \hat{\lambda}(t^* + \eta_1) \\ l_2 &= (\xi_2/\bar{A})(t^* + \eta_1) + \eta_2 \\ l_j &= \eta_j \quad (j = 3, 4, 5), \end{aligned}$$

where

$$\hat{\lambda} = (\bar{B} - \bar{C})\, L_1/\bar{B}\bar{C}.$$

For the unperturbed, uniaxial ($\bar{A}=\bar{B}$) case, $l_1=\phi'$, $l_2=\phi_H$, $l_3=\psi_H$, $L_1=P_{\phi'}/I_0\omega_0$, $L_2=P_{\phi_H}/I_0\omega_0$, and $L_3=P_{\psi_H}/I_0\omega_0$.

The transformation Equations (46) involve t^*, so that, when expressed in terms of the l_j and the L_j, the transformed Hamiltonian, $\hat{F}$, is given by

$$\hat{F} = -\tfrac{1}{2}(\bar{B}-\bar{C})\, L_1^2/\bar{B}\bar{C} - L_2^2/2\bar{A} + K^{**}(\boldsymbol{\xi}(\mathbf{L}), \boldsymbol{\eta}(\mathbf{L}, \mathbf{l}); \varepsilon),$$

and may be written explicitly in the form,

$$\begin{aligned} \hat{F} = &-\tfrac{1}{2}(\bar{B}-\bar{C})\, L_1^2/\bar{B}\bar{C} - L_2^2/2\bar{A} - \varepsilon L_4 - \varepsilon^2(\dot{\bar{\omega}}/n)(\omega_0/n)\, L_5 + \\ &+ \varepsilon^2\left[a_o L_3 + b_o\sqrt{L_2^2 - L_3^2}\cos l_3 + c_o L_3^2 + d_o\right], \end{aligned} \tag{47}$$

where

$$\begin{aligned} a_o &= (\dot{\Omega}/n)(\omega_0/n)\cos I \\ b_o &= (\dot{\Omega}/n)(\omega_0/n)\sin I \\ c_o &= -\tfrac{3}{8}(1-e^2)^{-3/2}(\bar{B}+\bar{C}-2\bar{A}+3\Delta_o)/L_2^2 \\ d_o &= -\tfrac{3}{8}(1-e^2)^{-3/2}\Delta_o \\ \Delta_o &= \Delta - \bar{A} + \bar{B}. \end{aligned} \tag{48}$$

Since t^* is absent form $\hat{F}$, we have the first integral, $\hat{F}=$constant, which may be expressed in the form

$$\hat{a}L_3^2 + \hat{b}L_3 + \hat{c} = -\sqrt{L_2^2 - L_3^2}\cos l_3, \tag{49}$$

where $\hat{a}=c_o/b_o$, $\hat{b}=-a_o/b_o$, and $\hat{c}$ is an arbitrary constant. Note that since l_1 and l_2 do not appear in $\hat{F}$, L_1 and L_2 are constant. Level curves for the long-term motion of a typical triaxial satellite (Pegasus A) are presented in Figure 3.

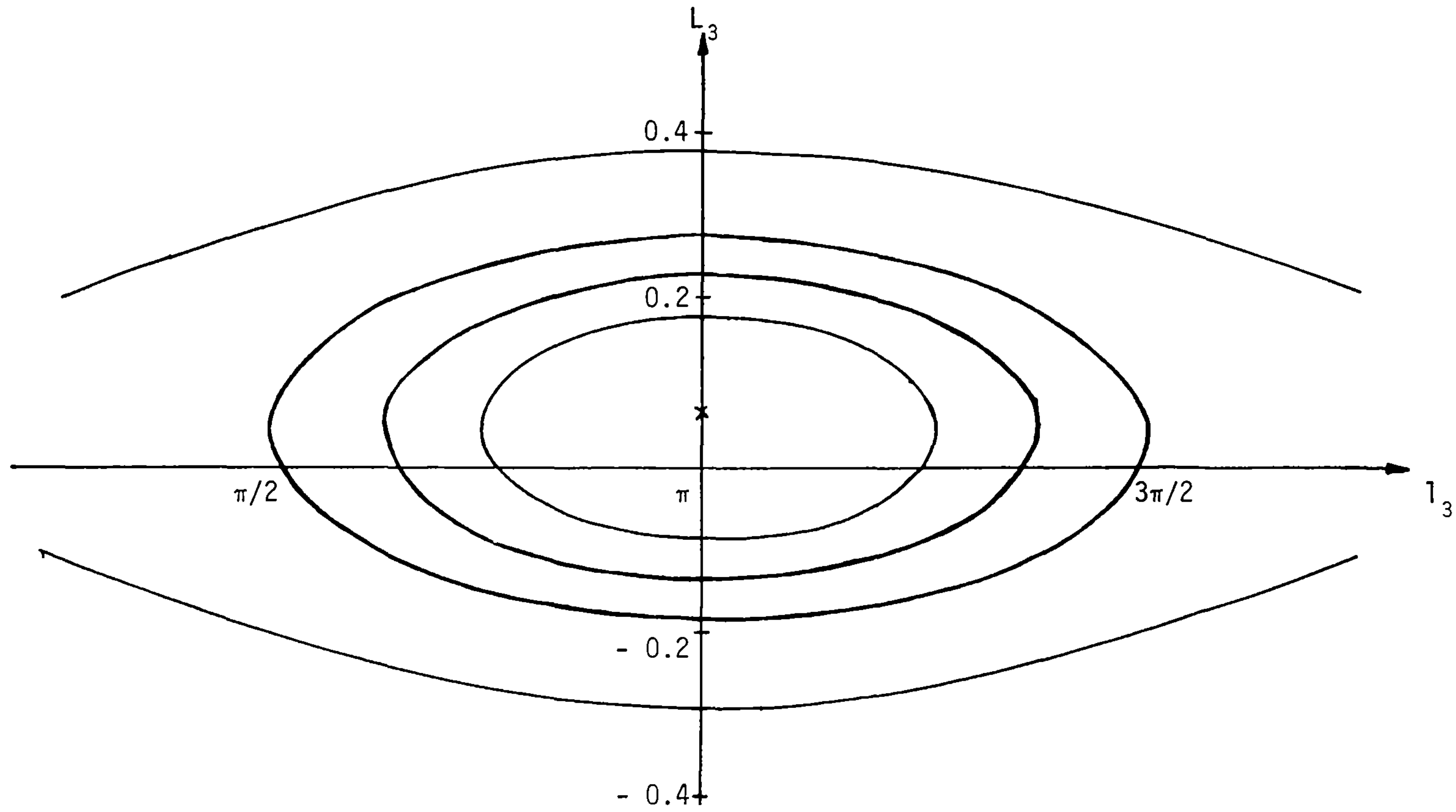

Fig. 3. Level curves, $\hat{F}(L_3, l_3) = \text{constant}$.

To obtain a geometrical interpretation of the integral (49), we let $\mathbf{i}^o$, $\mathbf{j}^o$, and $\mathbf{k}^o$ denote unit vectors directed along the positive x^o-, y^o-, and z^o-axes, respectively, and define a vector $\mathbf{L}$ by

$$\mathbf{L} \equiv L_{x^o}\mathbf{i}^o + L_{y^o}\mathbf{j}^o + L_{z^o}\mathbf{k}^o = \\ = \sqrt{L_2^2 - L_3^2}\sin l_3 \mathbf{i}^o - \sqrt{L_2^2 - L_3^2}\cos l_3 \mathbf{j}^o + L_3\mathbf{k}^o . \tag{50}$$

Then, (49) becomes

$$L_{y^o} = \hat{a}L_{z^o}^2 + \hat{b}L_{z^o} + \hat{c} . \tag{51}$$

Equation (51) represents a family of parabolic cylinders which open in either the positive or negative y^o-direction. Thus, the vector, $\mathbf{L}$, which may be construed as the averaged rotational angular momentum (nondimensionalized) of the satellite, must change in such a way that the projections of its terminus onto the $y^o z^o$-plane form a family of parabolas.

Since L_2 is constant and the magnitude of $\mathbf{L}$, we have

$$L_{x^o}^2 + L_{y^o}^2 + L_{z^o}^2 = L_2^2 = \text{constant} . \tag{52}$$

Equation (52) represents a family of spheres with origins at O.

For any particular problem, with given initial conditions which define the initial state of the rotational motion of the satellite, we have, according to (51) and (52), a unique parabolic cylinder and a unique sphere. The intersection(s) of these two surfaces is (are) the locus (possible loci) of the terminus of $\mathbf{L}$. A typical example is shown in Figure 4. If the parabolic cylinder penetrates the sphere as shown in Figure 5b, there are two possible loci. The 'occupied' locus may be determined from initial conditions.

To determine when the terminus of $\mathbf{L}$ occupies a given point on the line of intersection of the sphere and parabolic cylinder, we must obtain an integral involving t^*

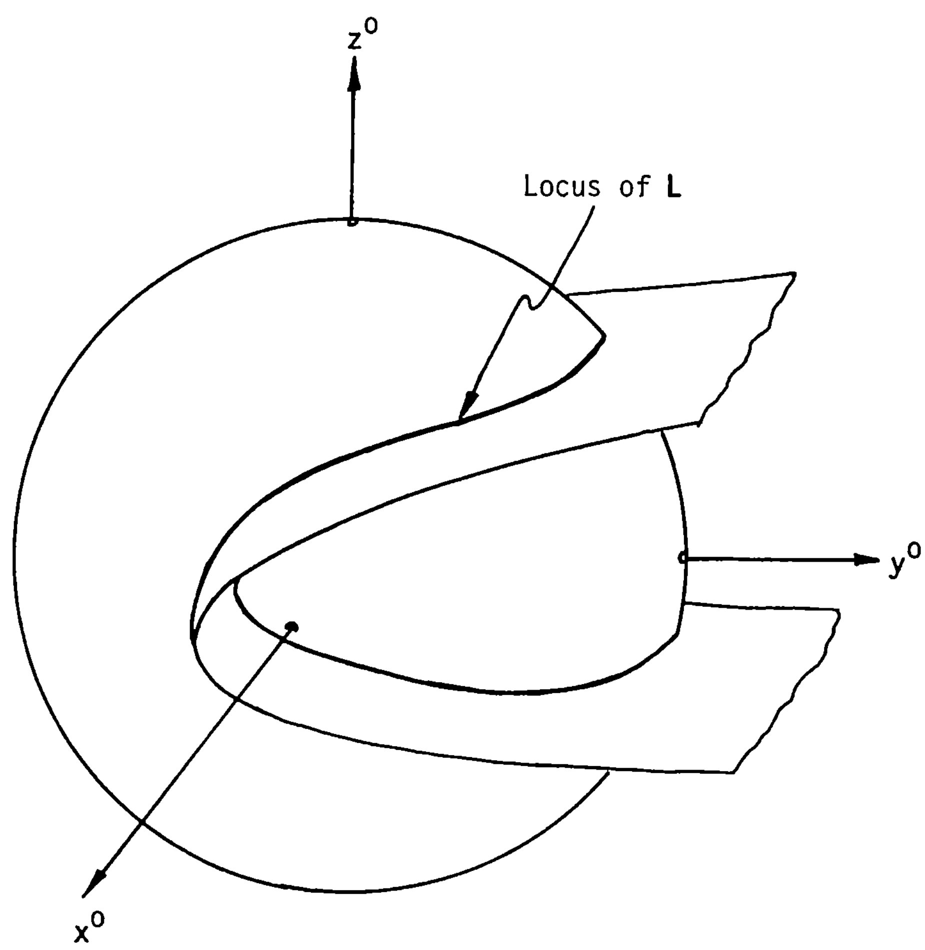

Fig. 4. Geometrical representation of the integrals $L_{y^o} = \hat{a}L_{z^o}{}^2 + \hat{b}L_{z^o} + \hat{c}$ and $L_{x^o}{}^2 + L_{y^o}{}^2 + L_{z^o}{}^2 = L_2{}^2$.

explicitly. From (47) and (50), we have

$$\mathrm{d}L_3/\mathrm{d}t^* = \varepsilon^2 b_o L_{x^o}, \tag{53}$$

where $L_{x^o}^2 = L_2^2 - L_{y^o}^2 - L_3^2$. Using (51), (52), and (53), we get

$$\mathrm{d}L_3/\mathrm{d}t^* = \pm\, \varepsilon^2 b_o \sqrt{L_2^2 - L_3^2 - (\hat{a}L_3^2 + \hat{b}L_3 + \hat{c})^2}, \tag{54}$$

which, as Holland and Sperling (1969) have pointed out, indicates that L_3 is a function of elliptic functions of t^*. It has not, to the author's knowledge, been pointed out, however, that the roots of the quartic equation $g(L_3) = L_{x^o}^2 = L_2^2 - L_3^2 - (\hat{a}L_2 + \hat{b}L_3 + \hat{c})^2 = 0$ are the values of L_3 at which the projections of the parabolic cylinder and the sphere on the yz^o-plane ($x^o = 0$) intersect. This is an important piece of information, since an understanding of the nature of the roots of $g(L_3) = 0$ is necessary for the integration of (54).

Excluding the rare cases in which the parabolic cylinder and the sphere are tangent at a point,† from geometry (see Figures 5a, b), it is obvious that, for real motion to occur, $g(L_3) = 0$ must have either two or four real roots. It follows then, excluding the rare cases as stated above, that there must be two possible forms of the solution to (54).

5.2.1. *Case 1. Two Distinct Real Roots*

When $g(L_3) = 0$ has two real roots, z_1 and z_2, $z_1 > z_2$, and a pair of complex roots, $z_3 = a_1 + ib_1$ and $z_3^* = a_1 - ib_1$ (see Figure 5a), we find that (see Byrd and Friedman, 1954, p. 153)

$$L_3 = \frac{A_1 + A_2 \operatorname{cn} \hat{u}}{A_3 + A_4 \operatorname{cn} \hat{u}}, \tag{55}$$

† These cases correspond to stable or unstable orientations of L.

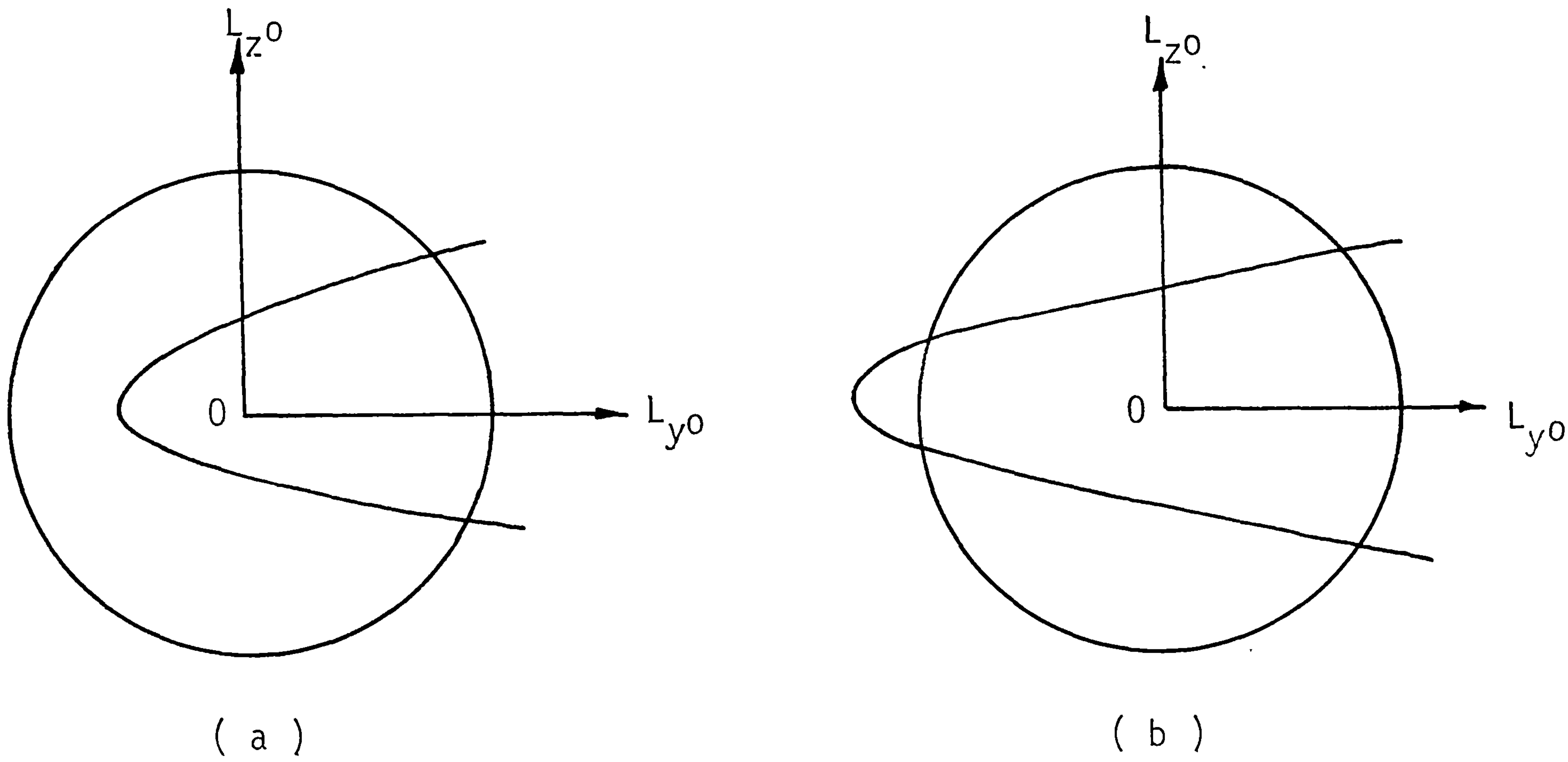

Fig. 5. Projections of the integrals $L_{y^o} = \hat{a}L^2_{z^o} + \hat{b}L_{z^o} + \hat{c}$ and $L^2_{x^o} + L^2_{y^o} + L^2_{z^o} = L_2{}^2$ onto the $L_{y^o}L_{z^o}$-plane.

where

$$A_1 = z_1 G_1 + z_2 G_2, \qquad A_2 = z_2 G_1 - z_1 G_2$$
$$A_3 = G_1 + G_2, \qquad A_4 = G_1 - G_2$$
$$G_1 = \sqrt{(z_1 - a_1)^2 + b_1^2}, \qquad G_2 = \sqrt{(z_2 - a_1)^2 + b_1^2}.$$

In Equation (55), we have also introduced the notation

$$\hat{u} = \hat{\omega}(t^* - C_3), \tag{56}$$

where

$$\hat{\omega} = \varepsilon^2 c_o \sqrt{G_1 G_2}, \tag{57}$$

and C_3 is an arbitrary constant. Furthermore, the modulus $\hat{k}$ of the Jacobian elliptic function cn $\hat{u}$ is given by

$$\hat{k}^2 = [(z_1 - z_2)^2 - (G_1 - G_2)^2]/4G_1 G_2. \tag{58}$$

We may also determine l_3, as a function of t^* for this case. To do this, we use Equations (49), (50), (53), (56), and (57) to get

$$\tan l_3 = -\hat{a}(\mathrm{d}L_3/\mathrm{d}\hat{u})\sqrt{G_1 G_2}/[-(\hat{a}L_3^2 + \hat{b}L_3 + \hat{c})], \tag{59}$$

where

$$\mathrm{d}L_3/\mathrm{d}\hat{u} = -[A_2 A_3 - A_1 A_4]\,\mathrm{dn}\,\hat{u}\,\mathrm{cn}\,\hat{u}/(A_3 + A_4\,\mathrm{cn}\,\hat{u})^2,$$

the quadrant of l_3 being governed by the signs of the numerator and denominator of the right-hand side of (59).

5.2.2. *Case 2. Four Distinct Real Roots*

If the parabola intersects the circle at four distinct points (see Figure 5b), the quartic equation $g(L_3) = 0$, has four distinct real roots, say, $z_1 > z_2 > z_3 > z_4$. The solution for

L_3 in this case takes the form (see Byrd and Friedman, pp. 97 and 133).

$$L_3 = (B_1 + B_2 \operatorname{sn}^2 \hat{u})/(B_3 + B_4 \operatorname{sn}^2 \hat{u}), \tag{60}$$

where

$$\hat{u} = \hat{\omega}(t^* - C_3)$$
$$\hat{\omega} = \varepsilon^2 (c_o/2) \sqrt{(z_1 - z_3)(z_2 - z_4)}.$$

The modulus, $\hat{k}$, of $\operatorname{sn}^2 \hat{u}$ is given by

$$\hat{k}^2 = [(z_1 - z_2)(z_3 - z_4)(z_1 - z_3)(z_2 - z_4)],$$

and, depending on between which two roots L_3 librates, the B_j have two possible forms. If $z_1 \leq L_3 \leq z_2$, then $B_1 = z_1(z_2 - z_4)$, $B_2 = z_4(z_1 - z_2)$, $B_3 = z_2 - z_4$, and $B_4 = z_1 - z_2$.If $z_4 \leq L_3 \leq z_3$, $B_1 = z_4(z_1 - z_3)$, $B_2 = z_1 - z_3$, and $B_4 = z_3 - z_4$. Values of l_3 may be obtained from

$$\tan l_3 = -(\hat{a}/2)(\mathrm{d}L_3/\mathrm{d}\hat{u})(z_1 - z_3)(z_2 - z_4)/[-(\hat{a}L_3^2 + \hat{b}L_3 + \hat{c})], \tag{61}$$

where

$$\mathrm{d}L_3/\mathrm{d}\hat{u} = 2[B_2B_3 - B_1B_4] \operatorname{cn} \hat{u} \operatorname{sn} \hat{u} \operatorname{dn} \hat{u}/(B_3 + B_4 \operatorname{sn}^2 \hat{u})^2,$$

and L_3 is given by (60).

Now, $L_3 = L_2 \cos\hat{\theta}_H$ and $l_3 = \hat{\psi}_H$, and the angles $\hat{\theta}_H$ and $\hat{\psi}_H$ define the orientation of L in space. We will use these results in the next section to predict the orientation of the rotational angular momentum of an artificial satellite as a function of time.

The coordinates l_1 and l_2 exhibit secular and long-period perturbations. By using the integral (55) and the Hamiltonian $\hat{F}$, we may write

$$\begin{aligned} \mathrm{d}l_1/\mathrm{d}t^* &= \hat{n}_1 + D_1L_3^2 \\ \mathrm{d}l_2/\mathrm{d}t^* &= \hat{n}_2 + D_2L_3^2 + D_3/(L_2 + L_3) + D_4/(L_2 - L_3), \end{aligned} \tag{62}$$

where

$$\begin{aligned}
\hat{n}_1 &= [(\bar{B} - \bar{C})/\bar{B}\bar{C}]\, L_1 + \varepsilon^2 \tfrac{3}{8} (1 - e^2)^{-3/2} (\partial \Delta_0/\partial L_1) \\
\hat{n}_2 &= L_2/\bar{A} - \varepsilon^2 \tfrac{3}{8} (1 - e^2)^{-3/2} [\partial \Delta_0/\partial L_2 + c_0 L_2] \\
D_1 &= -\varepsilon^2 \tfrac{9}{8} (1 - e^2)^{-3/2} (\partial \Delta_0/\partial L_1)/L_2^2 \\
D_2 &= -\varepsilon^2 \tfrac{9}{8} (1 - e^2)^{-3/2} [\partial \Delta_0/\partial L_2)/L_2^2 - 2c_0/L_2] \\
D_3 &= -\varepsilon^2 b_0 (\hat{a}L_2^2 - \hat{b}L_2 + \hat{c})/2 \\
D_4 &= -\varepsilon^2 b_0 (\hat{a}L_2^2 + \hat{b}L_2 + \hat{c})/2 \\
\partial \Delta_0/\partial L_1 &= (2\Delta_0 - \Delta^*)/L_1 \\
\partial \Delta_0/\partial L_2 &= -(2\Delta_0 - \Delta^*)/L_2 \\
\Delta^* &= [(\bar{A} - \bar{B})(\bar{A} - \bar{C})/\bar{A}] \frac{(E - K)K - E(E - k'^2K)/k'^2}{k^2K^2}.
\end{aligned}$$

If either of the solutions, (55) or (60), for L_3 is substituted into Equations (62), l_1 and l_2 may be obtained by quadrature. In fact, analytical expressions for l_1 and l_2 may be obtained (Cochran, 1970). These expressions involve elliptic integrals of the second and third kinds and elliptic functions, and are not very well suited for computations. However, by merely knowing the forms of the solutions for l_1 and l_2, we can conclude that the angles of spin and precession of the satellite will experience both secular and long-period perturbations.

Since the potential V is composed of periodic functions, it is not surprising that we do not find secular perturbations in momenta L_j. Furthermore, referring to the transformations $(\mathbf{P}, \mathbf{Q}) \to (\mathbf{x}, \mathbf{y})$, $(\mathbf{x}, \mathbf{y}) \to (\boldsymbol{\xi}, \boldsymbol{\eta})$ and $(\boldsymbol{\xi}, \boldsymbol{\eta}) \to (\mathbf{L}, \mathbf{l})$ it is apparent that the P_j, to the order of the present theory, do not experience secular perturbations. Hence, the angles $\theta' = \cos^{-1}(P_1/P_2)$ and $\theta_H = \cos^{-1}(P_3/P_2)$, are not secularly perturbed by the gravity-gradient torque.

6. Comparison of Theoretical Results and Observations

The theoretical results derived in the preceding section have been used to predict the long-term changes in the rotational motion of the Pegasus A satellite. This satellite 'tumbled' rapidly, so the assumption $\omega \gg n$ is valid. However, it was not spin-stabilized against gravity-gradient and other environmental torques. These torques produced decidedly non-Eulerian type motion, which was inferred from data obtained from on-board sun and horizon sensors.

During the period of time we will consider, Pegasus A was spinning about its axis of maximum moment of inertia, so that its angular momentum and body-fixed z'-axis were colinear. The data obtained by the on-board sensors therefore may be used to describe changes in the rotational angular momentum. Since the data presented here (Holland, 1969) was obtained by *statistically* averaging acquired data over six-hour time periods and since the theoretical results were obtained by an *analytical* averaging process, what is compared here are two descriptions of the motion of **L**, the averaged rotational angular momentum of the satellite.

The analytical descriptions of $\hat{\theta}_H = \cos^{-1}(L_3/K_2)$ and $\hat{\psi}_H = l_3$ were obtained using Equations (55) and (59). The initial conditions used were $h = 5.842 \times 10^5 \text{kg-m}^2/\text{min}$, $\psi_H = 310°$, $\theta_H = 88°$, $\phi_H = \phi' = \theta' = 0$. Other pertinent data used were $A = 1.03068 \times 10^5 \text{kg-m}^2$, $B = 3.33455 \times 10^5 \text{kg-m}^2$, $C = 3.94992 \times 10^5 \text{kg-m}^2$, $n = 3.71°/\text{min}$, $e = 0.1617$, $I = 31.7°$ and $\dot{\Omega} = -6.152°/\text{day}$. The initial conditions for θ_H and ψ_H were corrected for quasi-long period discrepancies using Equations (40) and (42).

Theoretical and observed time histories of $\hat{\theta}_H$ and $\hat{\psi}_H$ are presented in Figures 6 and 7. The small black triangles denote observed values while theoretical results are shown as solid lines. Excellent agreement can be observed for the time period of 17 days covered by the data. Deviations of no more than 10° in $\hat{\theta}_H$ and $\hat{\psi}_H$ are noted.

The predicted period of the motion of the terminus of **L** on the sphere (Equation (52)) for the case examined is 22 days. Agreement of theory with observations during this time period is very good; however, since forces of other than a gravitational origin affect the satellite's motion, good agreement cannot usually be maintained for time periods longer than one or two predicted periods of the motion of **L**. In particular, it has been observed that the **L**-vector for the satellite Pegasus A passed from the state

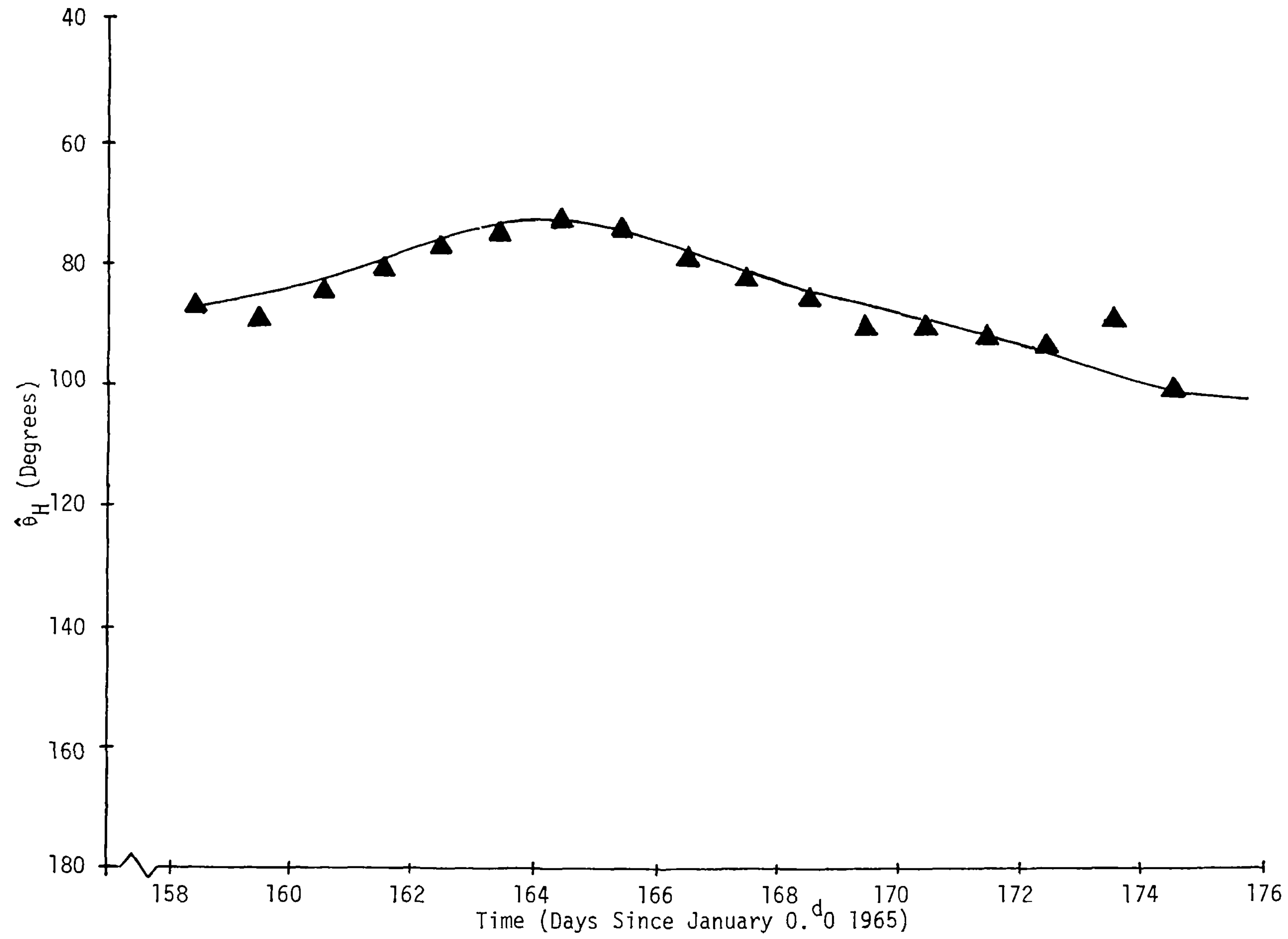

Fig. 6. Predicted and observed motion in $\hat{\theta}_H$.

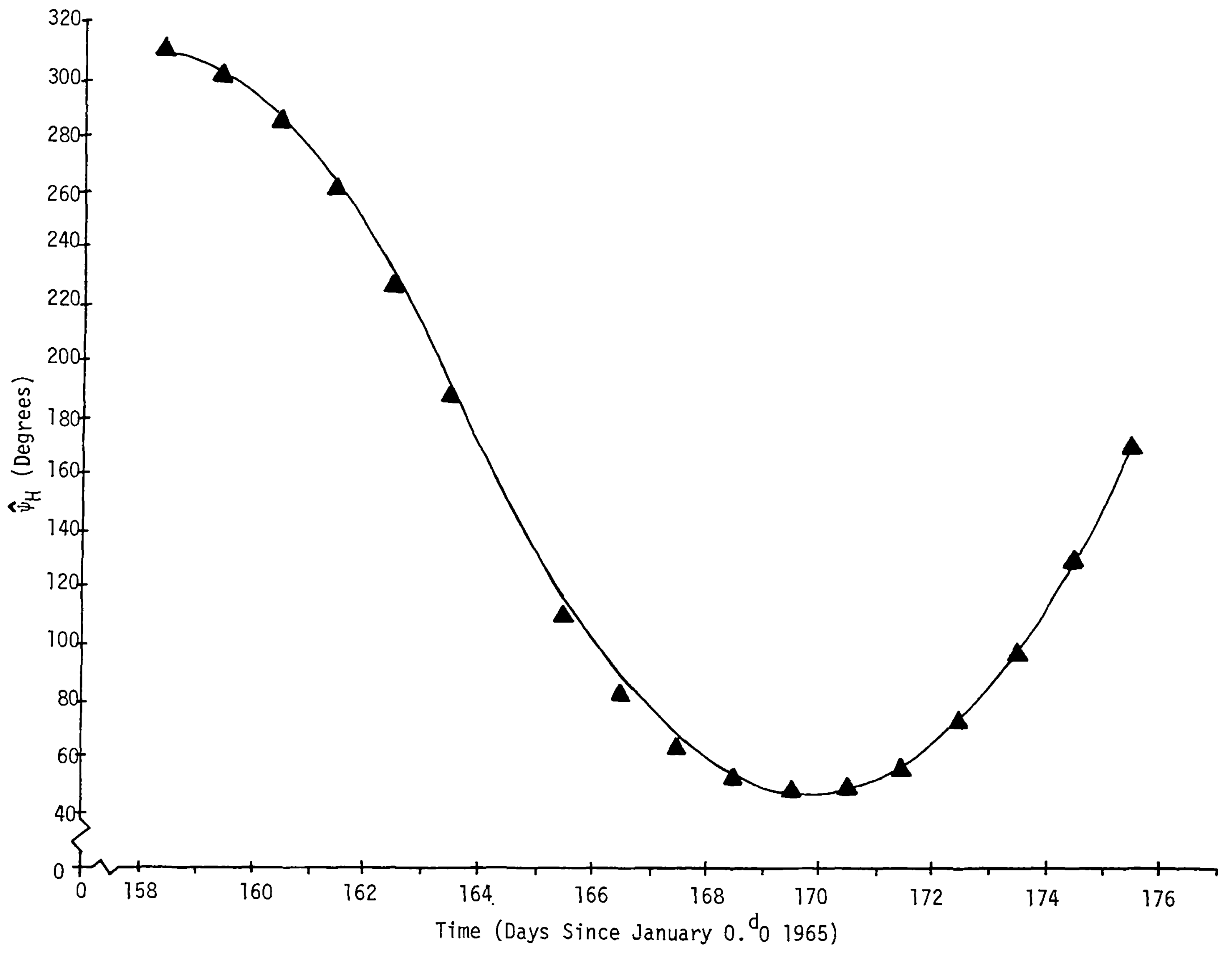

Fig. 7. Predicted and observed motion in $\hat{\psi}_H$.

given in Figures 6 and 7 into a state in which circulation in $\hat{\psi}_H$ occurred. Meanwhile, h remained essentially constant. From the theory developed here, we conclude that the parabolic cylinder (Equation (49)) must have been altered or at least shifted so that it completely penetrated the sphere. These changes are, of course, not predicted by the present theory.

Another change observed in the rotational motion of Pegasus A was the transition from rotation, essentially about the axis of minimum moment of inertia, to rotation about the axis of maximum moment of inertia. As previously stated, the corresponding secular change in the Eulerian angle θ' is not predicted by a gravitational theory.

7. Conclusion

A problem of rotational motion has been studied by applying a new method of general perturbations based on the use of Lie series and complete integrals of Hamilton-Jacobi equations to generate canonical transformations. The model for the problem is that of a triaxial, rigid satellite the center of mass of which moves in a precessing Keplerian ellipse about an attracting point mass. The theoretical long-term rotational motion of the satellite Pegasus A has been determined and excellent agreement between theory and observations has been obtained over a period of time of approximatley three weeks.

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