RESONANCE IN REGULAR VARIABLES
I: MORPHOGENETIC ANALYSIS OF THE ORBITS IN THE CASE OF A FIRST-ORDER RESONANCE

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ABSTRACT. Morphogenetic analysis of the orbits of the ideal first-order resonance problem in the neighbourhood of the origin. It is shown that for problems involving central and near-central resonance it is necessary to consider as parameter the cube root of the perturbation instead of the square root used in classical non-central resonance problems.

1. INTRODUCTION

One paradigm of the one-degree-of freedom perturbed Hamiltonians of Celestial Mechanics is the ideal resonance problem, introduced and thoroughly studied by Garfinkel (1966, 1982). There, the disturbing function

$$R = - \varepsilon A(I) \cos \theta, \qquad \varepsilon << 1 \tag{1}$$

is added to a completely integrable Hamiltonian $F_0(I)$.

The use of action-angle variables in the study of the ideal resonance problem limits the validity of the results, since the action I (defined by an integral over a closed path) is singular at $I = 0$. In order to study the resonant motions near the origin, we need to introduce the regular Poincaré variables:

$$k = \sqrt{2I} \cos \theta, \qquad h = \sqrt{2I} \sin \theta. \tag{2}$$

In this paper we consider only a first-order resonance, that is, the case where $A(I)$ is assumed to be linear with respect to $\sqrt{I}$. Then

$$R = - \varepsilon \tau_1 k, \tag{3}$$

where τ_1 is a constant (in fact the angles θ in (1) and (2) do not need to be the same; it is enough to have a linear relation among them).

In forthcoming papers formal solutions of dynamical systems with a central or near-central first-order resonance will be considered. One important feature in the solutions is the use of the cube root of ε as a small parameter in the series expansions. This paper aims at showing the necessity of the use of the cube root of ε instead of the classical square

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root of ε used in action-angle variables analyses, together with a morphogenetic analysis (Thom, 1975) of the orbits of the simple problem stated in the next section.

2. EQUATIONS. CENTRAL AND NON-CENTRAL RESONANCE

We consider the Hamiltonian

$$F = F_0 - \varepsilon\tau_1 k \tag{4}$$

($\tau_1 > 0$) where F_0 is the unperturbed Hamiltonian and $-\varepsilon\tau_1 k$ a linear perturbation. In this paper, for sake of simplicity, except where otherwise stated, we assume $\tau_1 > 0$.

The unperturbed Hamiltonian is assumed to be regular and has the d'Alembert characteristic in canonical form (Henrard, 1974). Therefore its expansion is a power series in the action I. Half integer powers of I may not appear since they would be necessarily coupled to odd multiples of θ and F_0 does not depend on θ. Hence

$$F_0 = \omega^0 I + \frac{1}{2} n^0 I^2 + \ldots, \tag{5}$$

where

$$\omega = \frac{dF_0}{dI}, \qquad n = \frac{d^2F_0}{dI^2} \tag{6}$$

and the supercript 0 means the value of these functions at the point $I = 0$. Figure 1 shows the function F_0 in the neighbourhood of the origin in two main circumstances:

(a) ω^0 and n^0 have the same sign,

(b) ω^0 and n^0 have opposite signs.

In both cases we considered $n^0 > 0$ (otherwise the figures are the same but turned upside down). When $\omega^0 = 0$ the figure is similar to that of (a) but the curvature at the vertex is equal to zero.

The motions that correspond to the unperturbed Hamiltonian are circles drawn with uniform velocity. The frequencies of these motions are given by

$$\omega = \omega^0 + n^0 I + \ldots. \tag{7}$$

Therefore the motions that correspond to (a) have finite frequencies and are direct while the motions that correspond to (b) are retrograde in the neighbourhood of the origin delimited by the minimum of F_0 and direct outside (up to the distance where another extremum of the function, if it exists, is reached). At the minimum (see Figure 2(b)) we have $\omega = 0$ (the resonance is <u>non-central</u>). When $\omega^0 = 0$ (and $n^0 > 0$) the unperturbed motions are direct as in (a); however, the frequency tends toward zero when I goes to zero (<u>central</u> resonance).

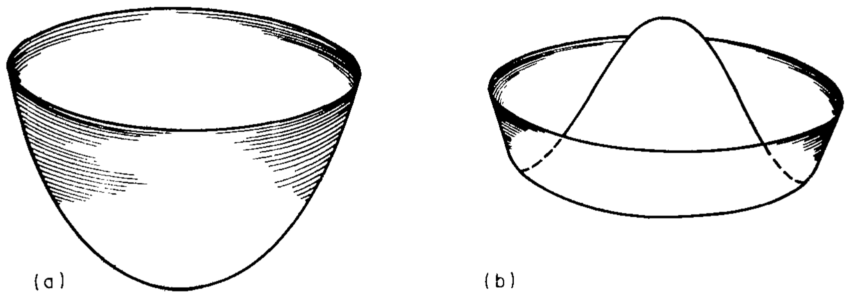

Fig. 1. The unperturbed Hamiltonian $F_0(k, h)$.
(a) $\omega^0 > 0$, (b) $\omega^0 < 0$ (in both cases $n^0 > 0$).

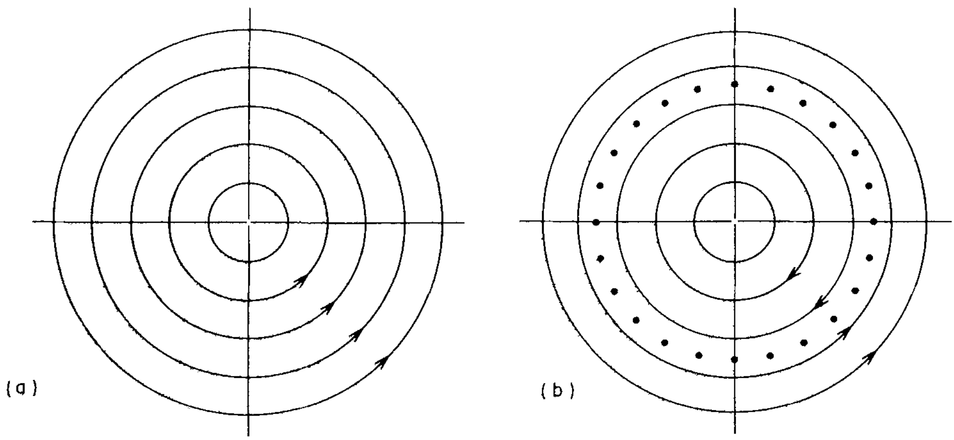

Fig. 2. Orbits defined by the unperturbed Hamiltonian in the plane (k, h). The dots represent a continuous sequence of equilibrium solutions.

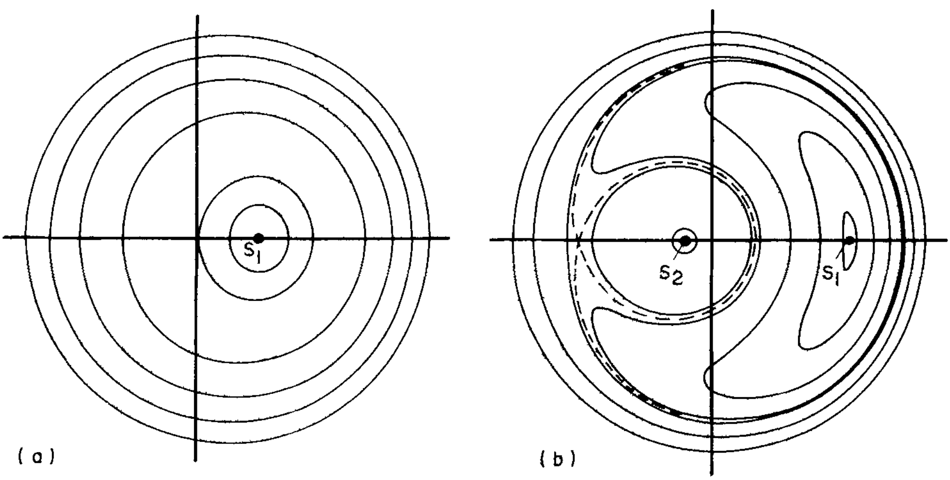

Fig. 3. Orbits defined by the perturbed Hamiltonian in the plane (k, h).

3. THE PERTURBED MOTIONS

Since the Hamiltonian given by Equation (4) does not depend on the independent variable, the trajectories in the phase-plane are solutions of the equations $F = C$. These curves are the projections of the sections of the surfaces shown in Figure 1 by the planes $C + \varepsilon\tau_1 k$. It must be kept in mind that ε is a small positive parameter and that, as a consequence, the sectioning planes have only a very small inclination. The families of curves that correspond to the surfaces shown in Figure 1 are shown in Figure 3. The denomination of the perturbed motions in case (b), in this paper, is the same as used in Sessin and Ferraz-Mello (1984) and is summarized as follows:

The horseshoes and ovals around S_1 are librations: the polar angle oscillates around a fixed value (0 in Figure 3(b)) and the family of curves has no topological equivalent among the unperturbed curves of Figure 2; they correspond to a bifurcation produced by the resonance in the curves of Figure 2(b), and substitute the sequence of dots. The curves where the polar angle has a monotonic motion are circulations, inner and outer; they are topological equivalents of the curves in Figure 2(b). The limaçon-like curve through the saddle point on the left is the separatrix. Finally, there are the small ovals around S_2; from the kinematical point of view these motions are librations since the polar angle oscillates around a fixed value (π in Figure 3(b)) but they belong to the same family as the inner circulations since there is no topological separation among them. For sake of precise identification, and taking into account these facts, we call them paradoxal librations. This set of curves is an analytic continuation of the inner circulations and form with them only one family of structuraly stable orbits.

Let us now consider the transition cases between (a) and (b). Figure 4 shows what happens along the k-axis. In the transition case the tangent plane touches F_0 at an inflexion point C where points A, B and D of case (b) coalesce. Figure 5 shows the resulting curves, which are the catastrophe set that separates the families of curves (a) and (b). In this figure the whole inner branch of the separatrix in Figure 3(b) coalesces into the saddle point, that becomes the cusp C. Inner circulations disappear and there remains only librations and outer circulations. Now, one may note that Figure 5 shows librations that envelope the origin; therefore, they correspond to motions where the polar angle circulates. In analogy to what has been discussed previously, for sake of precise identification these motions were called paradoxal circulations: they circulate but they are analytic continuation of librations and form with them only one family of structuraly stable orbits. One may also note that in the transition from (b) to (a) paradoxal circulations appear a little before the catastrophe, when the coalescing inner branch of the separatrix crosses the origin.

The families of curves shown in Figure 3 have been discussed by several authors (Jefferys, 1966, Message, 1966; Henrard and Lemaître, 1983; Sessin and Ferraz-Mello, 1984). Similar situations involving simultaneously first- and second-order resonances have been discussed in connection with the motion

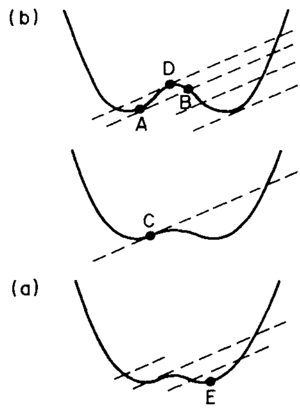

Figure 4. Transition from (a) to (b) shown in the plane $h = 0$.

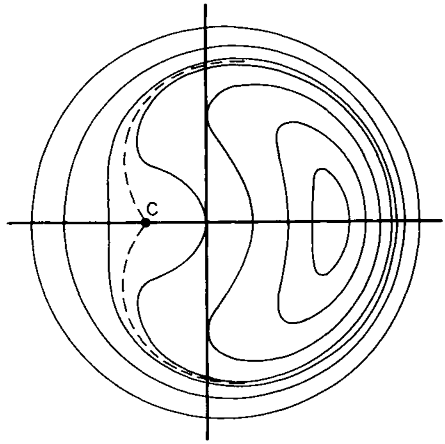

Figure 5. Catastrophe set of the orbits defined by the perturbed Hamiltonian when $\varepsilon = \varepsilon_L$ or $\omega^0 = -\omega_L^0$.

of minor planets near the Kirkwood gap 2:1 (Andoyer, 1903) and with the motion of an Earth satellite in the vicinity of the critical inclination (Aoki, 1963; Jupp, 1980).

4. A QUANTITATIVE APPROACH

A quantitative approach of the morphogenesis described in Section 3 may be obtained through an analysis of the curves, starting from the unperturbed case, that is, from $\varepsilon = 0$. For sake of simplicity F_0 is restricted to its leading terms; also, as the centers and saddles are on the k-axis we may take $h = 0$ and consider only the restriction of the function to this axis. We then have

$$F(k, 0) = \frac{1}{2}\,\omega^0 k^2 + \frac{1}{8}\,n^0 k^4 - \varepsilon\tau_1 k = C \tag{8}$$

and the centers and saddles are given by the cubic equation

$$k^3 + \frac{2\omega^0}{n^0}\,k - \frac{2\varepsilon\tau_1}{n^0} = 0. \tag{9}$$

The condition for having three real solutions is

$$\varepsilon \le \varepsilon_L = \left| \frac{-\omega^0\ k^*}{3\tau_1} \right| , \tag{10}$$

where

$$k^* = \sqrt{\frac{-8\omega^0}{3n^0}} \tag{11}$$

The roots of Equation (9) for $\varepsilon = 0$ and $\varepsilon = \varepsilon_L$, and the corresponding values of C are easy to calculate. These values are shown in Table I where, for sake of simplicity, we used

$$C^* = \frac{(\omega^0)^2}{n^0}. \tag{12}$$

The results for a generic ε are shown in Figure 6.

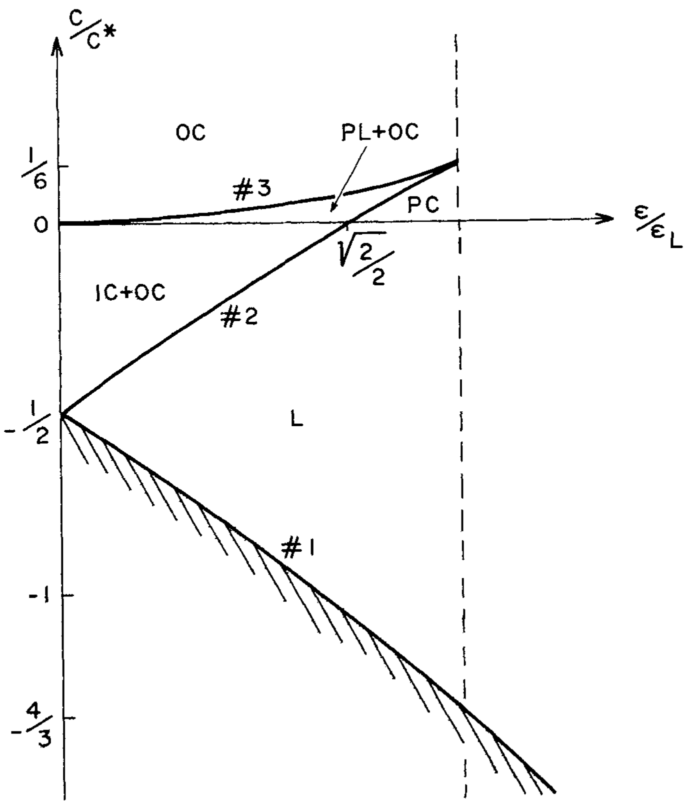

Fig. 6. Locus of the singular points in the plane (C, ε). Points #1 and #3 are centers and #2 is a saddle. (L = libration, C = circulation, I = inner, O = outer, P = paradoxal).

It is noteworthy to say that the quantities k^*, ε_L and C^* introduced in the calculations are natural units for the problem under consideration. Their utilisation leads to the equations

$$\frac{C}{C^*} = \frac{8}{9}\left(\frac{k}{k^*}\right)^4 - \frac{4}{3}\left(\frac{k}{k^*}\right)^2 - \frac{8}{9}\frac{\varepsilon}{\varepsilon_L}\left(\frac{k}{k^*}\right) \tag{13}$$

instead of Equation (8), and

$$4\left(\frac{k}{k^*}\right)^3 - 3\left(\frac{k}{k^*}\right) - \frac{\varepsilon}{\varepsilon_L} = 0 \tag{14}$$

instead of Equation (9). In these equations the sign before $\varepsilon/\varepsilon_L$ must be changed if $\omega^0\tau_1 > 0$.

TABLE I: Roots of Equation (11).

<table>
<tr><th rowspan="2">Root #</th><th colspan="2">$\varepsilon = 0$</th><th colspan="2">$\varepsilon = \varepsilon_L$</th></tr>
<tr><th>k/k^*</th><th>C/C^*</th><th>k/k^*</th><th>C/C^*</th></tr>
<tr><td>1</td><td>$\sqrt{3}/2$</td><td rowspan="2">$-\frac{1}{2}$</td><td>1</td><td>$-\frac{4}{3}$</td></tr>
<tr><td>2</td><td>$-\sqrt{3}/2$</td><td rowspan="2">$-\frac{1}{2}$</td><td rowspan="2">$\frac{1}{6}$</td></tr>
<tr><td>3</td><td>0</td><td>0</td></tr>
</table>

The point where the branch #2 cuts the ε-axis is such that $C = 0$; that is, from Equation (13), since $k \neq 0$,

$$2\left(\frac{k}{k^*}\right)^3 - 3\left(\frac{k}{k^*}\right) - 2\frac{\varepsilon}{\varepsilon_L} = 0 .$$

This equation combined with Equation (14) leads to

$$\frac{\varepsilon}{\varepsilon_L} = -\frac{k}{k^*}$$

and then

$$\frac{\varepsilon}{\varepsilon_L} = \frac{\sqrt{2}}{2} .$$

It is also elementary to see that the branches #2 and #3 are tangent at $\varepsilon = \varepsilon_L$, that the branches #1 and #2 have opposite inclinations at $\varepsilon = 0$, and that the branch #3 has a horizontal tangent at $\varepsilon = 0$.

The results of this section may be presented in a different way (Figure 7), taking as parameter the unperturbed frequency at the origin (ω^0) instead of ε, that is, through the change in shape of the unperturbed Hamiltonian, while the linear perturbation is kept fixed. The condition for having three real roots now is written as

$$\omega^0 \leq -\omega_L^0 = -\frac{3}{2}\left(n^0\varepsilon^2\tau_1^2\right)^{1/3} . \tag{15}$$

The roots in the point $\omega^0 = -\omega_L^0$ are the same as already given in Table I; we have just to translate them in terms of the appropriate variables. The results are shown in Table II where for sake of simplicity we used

$$k^{**} = \left(\frac{\varepsilon\tau_1}{n^0}\right)^{1/3}, \qquad C^{**} = \varepsilon\tau_1 \; k^{**}. \tag{16}$$

When these quantities are used, Equations (8) and (9) become

$$\frac{C}{C^{**}} = \frac{1}{8}\left(\frac{k}{k^{**}}\right)^4 + \frac{3}{4}\;\frac{\omega^0}{\omega_L^0}\left(\frac{k}{k^{**}}\right)^2 - \left(\frac{k}{k^{**}}\right) \tag{17}$$

and

$$\left(\frac{k}{k^{**}}\right)^3 + 3\;\frac{\omega^0}{\omega_L^0}\left(\frac{k}{k^{**}}\right) - 2 = 0. \tag{18}$$

There results for $\omega^0 = 0$ the values $k/k^{**} = \sqrt[3]{2}$ and $C/C^{**} = -(3/4)\ \sqrt[3]{2}\,(= -0.945)$.

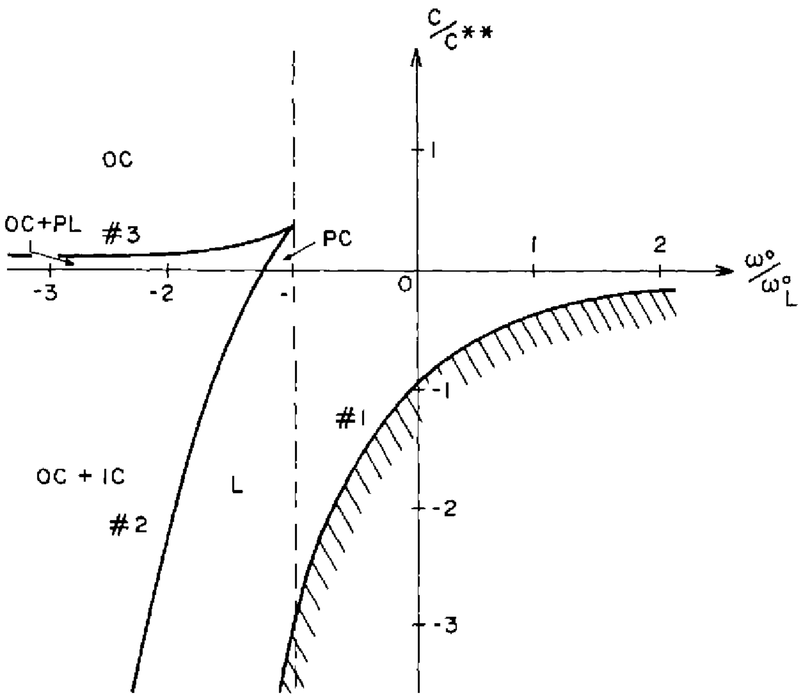

Fig. 7. Locus of the singular points in the plane (C, ω^0).
The symbols are the same of Figure 6.

TABLE II: Roots when $\omega^0 = -\omega_L^0$.

Root #	k/k^{**}	C/C^{**}
1	2	-3
2, 3	-1	$\frac{3}{8}$

5. THE CUBE ROOT OF THE SMALL PARAMETER

It is a classical point in the Mechanics of the Solar System that in case of resonance the formal solutions valid in the neighbourhood of resonant initial conditions are series of powers in the square root of the small parameter. This fact may be considered as a consequence of the Weierstrass' implicit functions theorem (see, e.g. Poincaré, 1893). The quantitative elements of Figure 3b give an a posteriori justification of the use of the powers of the square root of the small parameter in the study of the orbits in the bifurcation created by the perturbation of a non-central resonance with a finite ω^0. Compute the intersections of the separatrix with the k-axis. The separatrix is defined by the planar section that is tangent to the unperturbed Hamiltonian in A (see Figure 4). The corresponding values of k are given by Equation (8). Because of the tangency in A, one of the roots (let it be called $-k_0$) is double. If the other roots are b and c, the fourth degree equation is

$$\frac{1}{8} n^0 (k + k_0)^2 \quad (k - b)(k - c) = 0. \tag{19}$$

The coefficient of the cubic term is $\frac{1}{8} n^0 \ (2k_0 - b - c)$; but there is no cubic term in Equation (8), then

$$b + c = 2k_0. \tag{20}$$

Analogously, comparing the quadratic term of Equations (8) and (19)

$$k_0^2 - 2(b + c)k_0 + bc = \frac{4\omega^0}{n^0} . \tag{21}$$

The comparison of Equations (20) and (21) gives

$$(c - b)^2 = - \frac{16\omega^0}{n^0} - 8 k_0^2 . \tag{22}$$

k_0 is one root of Equation (9). When $\omega^0 < 0$ the roots of Equation (9) are given by

$$k_0 = k^* \cos\frac{x}{3}; \qquad x = \arc\cos\frac{-3\varepsilon\tau_1}{\omega^0 k^*}\ . \tag{23}$$

Since we assumed that ω^0 is finite, the argument of the arc cos is $0(\varepsilon)$ and we may write

$$x \simeq \frac{\pi}{2} + \frac{3\varepsilon\tau_1}{\omega^0 k^*} \qquad (\text{mod } 2\pi) \tag{24}$$

and

$$\text{cox}\,\frac{x}{3} \simeq -\frac{\sqrt{3}}{2} - \frac{\varepsilon\tau_1}{2\omega^0 k^*}\ ; \tag{25}$$

the choice of the determination ($x/3$ close to $5\pi/6$) is fixed by the fact that the saddle is the leftmost root of Equation (9). Using this approximation for k_0 in Equations (20) and (21) we get

$$c - b \simeq 4\sqrt{\frac{\tau_1\varepsilon}{k_0 n^0}}\ , \tag{26}$$

that is, $|c - b| = 0\ (\sqrt{\varepsilon})$. The corresponding variation in the action I is

$$\Delta I = \sqrt{2I}\ |c - b| = 0(\sqrt{\varepsilon})$$

since I is finite. Therefore, the excursion of the action around the equilibrium value is $0(\sqrt{\varepsilon})$, and could not be given by a series expansion in the powers of ε.

A similar situation happens when the resonance is near central. In such a case libration may happen only for $|\omega^0| \geq \omega^0_L = 0(\varepsilon^{2/3})$ and the characteristic size in the phase plane is given by $k^{**} = 0(\varepsilon^{1/3})$. If Equation (22) is used as before, we observe that the width of the libration zone in the k-axis is $0(\varepsilon^{1/3})$ when $\omega^0 = 0\ (\varepsilon^{2/3})$. Indeed, in this case the argument of the arc cos in Equation (23) is finite, and there follows $k_0 = 0\ (\sqrt{-\omega^0}) = 0\ (\varepsilon^{1/3})$. Then, from Equation (22), $c - b = 0(\varepsilon^{1/3})$. Therefore the excursion of the regular variables around the origin is $0\ (\varepsilon^{1/3})$, and certainly cannot be given by a series expansion in the powers of ε or of $\sqrt{\varepsilon}$.

$\omega^0 = 0$. In this case Equation (8) becomes

$$F(k,\ 0) = \frac{1}{8}\, n^0 k^4 - \varepsilon\tau_1 k = C.$$

For the oval that passes through the origin in Figure 3(a), we have $C = 0$. The second root of $F(k,\ 0) = 0$ in this case is given by

$$k^3 = \frac{8\varepsilon\tau_1}{n^0} \tag{27}$$

i.e. the perturbed solution makes a great excursion from the origin up to distances of the order of $\varepsilon^{1/3}$, and the solution must be searched using this power of ε as a small parameter.

6. APPLICATION: THE PLANETARY RESONANCE 2 : 1

In a recent paper, Sessin and Ferraz-Mello (1984) have obtained the equation for the motion of two planets with periods commensurable in the ration 2 : 1. In that paper, the classical square root of the parameter was used; however, in order to have correct results it was necessary to treat the order of the terms in a non-rigid way, for instance, putting together in F_1 terms of first order and terms of order $O(m'/M)^{5/4}$. The use of the cube root of the disturbing masses as a parameter allows us to obtain the equations of Section 3 of that paper in a rigourous way. We assume that the modified values of the eccentricities (e_{j0}) and of the sines of the half inclinations (s_{j0}) are of order $O(m'/M)^{1/3}$ and that the variables that give the departure to the exact resonance of the adopted canonical variables: x/x_{20}, y_j/x_{20}, and z_j/x_{20} are of the order $O(m'/M)^{2/3}$. We then obtain for the leading term of the Hamiltonian:

$$F_{4/3} = F_{02}\left(\frac{x}{x_{20}}\right)^2 - \frac{m'}{M}\left[P_{30}e_1' \cos(\theta + \tilde{\omega}_1) - P_{40}e_2' \cos(\theta + \tilde{\omega}_2)\right]$$

which is the same function as F_1 of the paper, except for an additive constant. The next part of the Hamiltonian, $F_{5/3}$, is almost the same as $F_{3/2}$ of that paper. Because of the stricter rules for the definition of the orders the terms whose coefficients are F_{03}, P_{30}, P_{31}, P_{40}, and P_{41}, of that paper, are of order $O(m'/M)^2$ and must be dropped from Equation (18) of that paper in order to obtain $F_{5/3}$.

7. CONCLUSION

The fact that the orbits of the first-order ideal resonance problem are defined as projections of plane sections of a simple revolution surface allows us to obtain a description of the totality of motions through elementary methods. The quantitative analysis of the orbits shows that the width of the libration in near-central resonance is of the order of the cube root of the perturbation instead of the square root as in non-central resonance. This result will be used in forthcoming papers of this series as a basis for adopting power series in the cube root of the small parameter to represent the formal solution of some resonant problems.

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