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CHARACTERIZATION OF SEMINUCLEAR SETS IN A FINITE PROJECTIVE PLANE

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Let C be a set of $q+a$ points in the desarguesian projective plane of order q, such that each point of C is on exactly 1 tangent, and one $a+1$-secant ($a>1$). Then either $q=a+2$ and C consists of the symmetric difference of two lines, with one further point removed from each line, or $q=2a+3$ and C is projectively equivalent to the set of points $\{(0,1,s),(s,0,1),(1,s,0): -s \text{ is not a square in } GF(q)\}$.

INTRODUCTION

There exists an interesting configuration of 9 points in $PG(2,7)$ which has the property that each point lies on a unique tangent (and hence also on a unique 3-secant). The interest in this particular configuration stems from the following theorem: [Blokhuis & Bruen [1]]:

The minimal number of lines blocked by $q+2$ points in $PG(2,q)$, q odd and at least 7 equals $(q+2)(q+1)/2+(q+2)/3$. Moreover for $q>7$ equality occurs if and only if the set of points has the property that each point is on a unique tangent.

A set of $q+2$ points with this property is called a *seminuclear set* in [1].

In this note it is shown that the only q for which a seminuclear set exists in the desarguesian projective plane $PG(2,q)$ are $q=4$ and $q=7$. In fact we shall prove the following result

THEOREM. *Let C be a set of $q+a$ points in the desarguesian projective plane of order q, such that each point of C is on exactly 1 tangent, and one $a+1$-secant ($a>1$). Then either $q=a+2$ and C consists of the symmetric difference of two lines, with one further point removed from each line, or $q=2a+3$ and C is projectively equivalent to the set of points $\{(0,1,s),(s,0,1),(1,s,0): -s$ is not a square in $GF(q)\}$.*

In both of the examples mentioned in the theorem we see that the tangents at the different points on an $(a+1)$-secant of the configuration are concurrent, in fact their point of intersection lies on the remaining (two) $(a+1)$-secant(s). It is this property which is shown to hold for any set of the specified type (in a desarguesian plane); as a consequence no other examples exist.

A THEOREM OF CEVA

Let $P =< (x,y,z) >$ be a point and $q =< [a,b,c] >$ a line in $PG(2,q)$. The point P is on the line q if $ax+by+cz=0$ or in other terms: $< q,P >= 0$. If no confusion is possible we write qP in stead of $< q,P >$. Note that if P is not on the line q then qP is not well-defined, since its value depends on the choice of homogeneous coordinates. Still we shall use the expression, but only in formulas that are homogeneous of degree 0 in all variables, or in situations where the coordinates have been fixed.

THEOREM 1 [(CEVA) see e.g. [2, p. 89]]. *Let p, q and r be three lines through the (non-collinear) points P, Q and R respectively, (but such that p does not contain Q or R etc.). Then the lines p, q and r are concurrent if and only if*

$$\frac{pQ.qR.rP}{pR.qP.rQ} = -1.$$

(Note that Ceva's theorem is no longer valid if P, Q and R are collinear).

THE STRUCTURE OF SEMINUCLEAR SETS

In the following, C will be a set of $q+a$ points in $PG(2,q)$ such that each point of C is on precisely one tangent and one $(a+1)$-secant of C.

PROPOSITION 2. *Let l be an $(a+1)$-secant of C. The tangents of C at the different points of $l \cap C$ are concurrent.*

Proof. Let P, Q and R be three points of C on l. We shall fix coordinates in such a way that
(1) $R = P + Q$;
(2) $pQ = 1$ for each line p through P other than l;
(3) $qP = -1$ for each line q through Q other than l.
As a consequence of this the line $p+q$ passes through R. As X runs through $C \setminus l$, PX runs through the lines through P different from l, except the tangent: $p := t(P)$. Similarly QX (resp. RX) runs through the lines through Q (resp. R) except the tangent $q = t(Q)$ (resp. $r = t(R)$). We fix coordinates for r such that $rQ = -rP = 1$. Let S be an arbitrary point of the plane outside the line l. If p_1 and p_2 are two different lines through P, then $p_1S \neq p_2S$ since equality would imply that P, Q and S are on the line $p_1 - p_2$. Now we show that

$$m := p + \sum_{X \in C \setminus l} PX = \mathbf{0}, \quad \text{if } q > 2.$$

To see this, note that $mP = 0$, $mQ = 1 + ... + 1$ (q times)$= 0$ and $mS = 0$, since this equals the sum of all elements of $GF(q)$ and $q > 2$. Since P, Q and S are independent, $m = \mathbf{0}$. In the same way we find

$$q + \sum_{X \in C \setminus l} QX = \mathbf{0} \quad \text{and} \quad r + \sum_{X \in C \setminus l} RX = \mathbf{0}.$$

However, since $RX = PX + QX$ for all $X \in C \setminus l$, it follows that $r = p + q$; the tangents at P, Q and R are concurrent.

The point of intersection of the tangents of C at the different points of $l \cap C$ is called the *nucleus* of the line l.

PROPOSITION 3. *Let l and m be two $(a+1)$-secants of C. Then either the nucleus of l is on m and the other way round, or both lines have the same nucleus.*

Proof. Let $P, R_0, \ldots, R_{a-1}$ be the points of C on the line l. Assume we have fixed the coordinates in such a way that $R_i = R_0 + c_i P$, for appropriate c_i. We can then coordinatise the corresponding tangents similarly; that is $r_i = r_0 + c_i p$, since they are concurrent and $pR_0 + r_0 P = 0$. Let S be an arbitrary point of C on the line m, let $R = R_i$ be one of the points of $l \cap C \setminus \{P\}$, and X be a point from $C \setminus l$, different from S. Then by Ceva's theorem:

$$\frac{< XP, R >}{< XP, S >} \cdot \frac{< XR, S >}{< XR, P >} \cdot \frac{< XS, P >}{< XS, R >} = -1. \tag{1}$$

On the other hand we have the product

$$\frac{pR}{pS} \cdot \prod_{X \in C \setminus l \cup \{S\}} \frac{< XP, R >}{< XP, S >} = -1. \tag{2}$$

To see this, note that the left hand side is the product of all nonzero elements of $GF(q)$, since it is a product over all lines through P not containing R or S. In the same way we obtain:

$$\frac{rS}{rP} \cdot \prod_{X \in C \setminus l \cup \{S\}} \frac{< XR, S >}{< XR, P >} = -1. \tag{3}$$

In a slightly different but analogous way we obtain (4) by looking at all lines through S not containing P or R:

$$\frac{sP}{sR} \cdot \prod_{X \in C \setminus l \cup \{S\}} \frac{< XS, P >}{< XS, R >} \cdot \prod_{j \neq i} \frac{r_j P}{r_j R_i} / (\frac{mP}{mR_i})^{a-1} = -1. \tag{4}$$

This is explained as follows: The ratio mP/mR occurs a times in the first half of the product, since there are a points on the line m, other than S; therefore we divide by this ratio $a-1$ times. Similarly the lines joining S to the points $R_j \neq R = R_i$ must be added.

But the ratio $< SR_j, P > / < SR_j, R >$ is the same as r_jP/r_jR. (Compare with the remark after (2). Write

$$M_i := \prod_{j \neq i} \frac{r_j P}{r_j R_i} / (\frac{mP}{mR_i})^{a-1}.$$

We can then deduce from equations (1) to (4) by eliminating $\prod_{X \in C \setminus l \cup \{S\}}$ the fundamental equation:

$$\frac{pR_i.r_iS.sP}{pS.r_iP.sR_i}.M_i = 1.$$

In view of our convenient choice of coordinates of the r_i and R_i it follows that $pR_i = -r_iP$; so we can rewrite the above as follows:

$$\text{(5)} \qquad \frac{sR_i}{sP} = -M_i.\frac{r_iS}{pS}.$$

We now substract the above equations corresponding to i and j and use $R_i - R_j = (c_i - c_j)P$ and $r_iS/pS = r_0S/pS + c_i$.

We get $c_i - c_j = -(r_0S/pS).(M_i - M_j) - c_iM_i + c_jM_j$. What does this mean? It means that we can compute r_0S/pS, unless $M_i - M_j = 0$. Since c_i and c_j are different, this can only occur if both are equal to -1.

(I). Suppose M_i and M_j are different for some i and j. Then for any pair of points $S, T \in m \cap C$ we have $r_0S/pS = r_0T/pT = c$ (say); in other words $S, T \in r_0 - cp$. This is equivalent to m (the line ST) being dependent on p and r_0, i.e. m contains the nucleus of l.

(II). All M_i are equal to -1. In this case it follows immediately from (5) that $sR_i/r_iS = sP/pS$, which implies that the point of intersection of the tangents of points on l is the same as that of tangents of points on m. For, let $sR_i/sP = r_iS/pS = c$. Then $s(R_i - cP) = 0$ and $(r_i - cp)S = 0$. Now since by our choice of coordinates $(r_i - cp)(R_i - cP) = 0$ we get that s is the same line as $r_i - cp$. But in that case s passes through their point of intersection, that is the nucleus of l. Since S was arbitrary, l and m have the same nucleus.

We now proceed to show that (II) does not in fact occur. Starting with our line l and using Proposition 2, we see that the remaining lines are of two types, those with the same nucleus as l, and those whose nucleus is on the line l. The lines with their nucleus on the line l all pass through the nucleus of l. Conversely, if a line m has the same nucleus as l, then the point of intersection of m and l is the other nucleus (there can not be just one nucleus, since that point would be on $q + a$ tangents, but it is on only $q + 1$ lines). Hence we find in this case precisely 2 nuclei, and each $(a + 1)$-secant passes through one of them. Let Q be any point outside the set C. The number of lines through Q intersecting C in an odd number of points has the same parity as C. Take two points of C, such that the nuclei of the $(a + 1)$-secants through them are different. Then their tangents intersect in a point Q which can not be on a $(a + 1)$-secant, and hence Q lies on precisely 2 lines intersecting C in an odd number of points. Now assume that $q > a + 2$. In that case through at least one of the nuclei there is more then one $(a+1)$-secant, and hence also a line missing C and the other nucleus. Intersect this line with an arbitrary tangent through the other nucleus. The point of intersection is on precisely one tangent. Since C cannot be odd and even at the same time we have a contradiction, whence (II) does not occur.

We are now able to prove the result.

THEOREM 4. *Let C be a set of $q+a$ points in the desarguesian projective plane such that each point of C is on exactly 1 tangent, and one $(a+1)$- secant $(a>1)$. Then either $q=a+2$ and C consists of the symmetric difference of two lines, with one further point removed from each line, or $q=2a+3$ and C is projectively equivalent to the set of points $\{(0,1,s),(s,0,1),(1,s,0) : -s$ is not a square in $GF(q)\}$.*

Proof. The $(a+1)$-secants partition the points of C, hence $q+a$ is a multiple of $a+1$. If $q=a+2$ there is nothing to prove, so assume the number of $(a+1)$-secants is at least 3. From proposition 3 and the fact that the second alternative does not occur, we see that the nucleus of any $(a+1)$-secant is on the intersection of all others. Since the nucleus obviously cannnot be on the secant itself, at most two $(a+1)$-secants go through one point. Hence the number of $(a+1)$-secants is precisely three. The last part of the theorem is left as an exercise.

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