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# TRANSLATION PLANES OF LARGE DIMENSION ADMITTING NONSOLVABLE GROUPS

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In this article, the question is considered whether there exist finite translation planes with arbitrarily small kernels admitting nonsolvable collineation groups. For any integer $N$, it is shown that there exist translation planes of dimension $> N$ and order $q^3$ admitting $GL(2,q)$ as a collineation group.

## 1. INTRODUCTION

One of the main outstanding problems concerning finite translation planes is the following: Given any finite group $G$, is there a translation plane which admits $G$ as a group of collineations?

Ostrom [14] has shown that if $G$ is an elementary abelian 2-group all of whose involutions are Baer acting on an odd order translation plane of vector dimension $2d$ over the kernel $K$ then $|G|$ must divide $d$. $d$ is called the Ostrom dimension of the plane.

The effect of this result is to exclude the possibility of many simple groups acting on a translation plane with small Ostrom dimension $d$. For example, if $PSL(2,u)$, $u \neq 2$ acts on a translation plane of odd order, the Klein 4-subgroups satisfy the above condition so that 4 divides $d$ (Ostrom [14], Cor. 2).

Fink and Kallaher [3] have shown that if a simple group $G$ acts irreducibly on the associated vector space over the kernel $K$ and $(|G|, \text{characteristic } K) = 1$ then the possibilities for $G$ to be a sporadic simple group are quite limited (see Fink, Kallaher [3], Theorem (4.3)).

Actually, it is easy enough to produce translation planes of Ostrom dimension $d$ with $d$ arbitrarily large. For example, the André planes suffice. However, with the exception of the Desarguesian and Hall planes, the associated collineation groups are solvable.

There are also the sequences of translation planes of Kantor [11] obtained using the procedures of "slicing," "extending" and "spreading" which can produce translation planes of large dimension. In these cases, the collineation group gradually disappears as the dimension increases.

Let $\pi$ be a derivable translation plane of order $q^2$ and kernel $K \cong GF(q)$ (Ostrom dimension 2) which admits a derivable net $D$ whose Baer subplanes which are incident with the zero vector are not all $K$-subspaces. Then Johnson and Ostrom [10] show that the kernel of the derived translation plane is the maximal subfield $L$ of $K$ such that the indicated Baer subplanes are $L$-subspaces (see also Johnson [9]). Hence, if $q = p^r$, the Ostrom dimension conceivably can jump from 2 to $2r$ by derivation (see Johnson [9] for examples where this occurs). However, the collineation groups which can be obtained by this procedure are generally solvable.

Concerning nonsolvable groups, Foulser [5] has shown that derivation can produce translation planes of order $q^4$ and Ostrom dimension 4 admitting $SL(2,q)$.

Kantor [11] has also determined a class of even order translation planes of order $q^3$ and Ostrom dimension 3 admitting $SL(2,q)$.

Bartolone and Ostrom [1] also construct planes of order $q^3$ and dimension 3 admitting $SL(2,q)$.

Finally, there are the unusual planes of Kantor [12] of order $q^6$ and Ostrom dimension 2 which admit $SL(2,q^2)$.

However, this is essentially where the examples stop. That is, there are no examples of translation planes of Ostrom dimension greater than 4 which admit nonsolvable collineation groups.

A question related to the foregoing discussion is the following:

If $N$ is any integer, is there a translation plane of Ostrom dimension $\geq N$ which admits a nonsolvable collineation group?

In this article, we answer the previous question and provide alternative constructions of the planes of Kantor and Bartolone-Ostrom.

We show the following:

COROLLARY. *Let $N$ be any integer. Then there is a translation plane of order $q^3$ that admits $GL(2,q)$ as a collineation group and which has Ostrom dimension $\geq N$.*

This result follows from:

THEOREM (see (6.2)). *For any prime power $q$ and for any integer $n > 1$ and $n \not\equiv 0 mod 3$, there is a translation plane of order $q^{3n}$ admitting $GL(2,q^n)$ as a collineation group with Ostrom dimension $n$.*

Furthermore, when $n = 2$, we may reconstruct Kantor's planes of order $q^6$ admitting $GL(2,q^2)$ and Ostrom dimension 2. When $n = 3$, we show how to obtain the planes of Bartolone-Ostrom (using the Sandler semifields).

But, note the vast number and variety of exotic translation planes. For example, when $n = 5$, we may obtain translation planes of order $q^{15}$ and Ostrom dimension 5 admitting $GL(2,q^5)$ or for $n = 31$, translation planes of order $q^{93}$ admitting $GL(2,q^{31})$ and Ostrom dimension 31.

The group $GL(2,h)$ involved acts as it would on the associated Desarguesian plane of order $h^3$.

In some sense, our results are bi-products of our analysis of the following question:

(*) Which spreads of order $q^3$ admit $GL(2,q)$ acting as it does on the Desarguesian plane of order $q^3$?

We have characterized all the planes satisfying (*) in [7]. In the present article, as mentioned above, we introduce a technique inspired by Bartolone and Ostrom ([1, section 3]), for constructing large numbers of spreads satisfying (*). Our method depends upon the existence of a fixed- point-free nonprojective collineation $\sigma$, of the Desarguesian projective plane $PG(2,q)$; whenever such a $\sigma$ exists, we shall be able to construct a spread $\pi_\sigma$ satisfying (*). It turns out that by varying $\sigma$, while keeping $q$ fixed, we can alter the size of the kern $\pi_\sigma$, and hence also $dim\pi_\sigma$: this is what gives Theorem (6.2) above. In addition, we have enough $\pi_\sigma$ to establish that for every prime power $q = p^m > p$, there exists a translation plane of order $q^3$ admitting $SL(2,q)$; up to now, this fact has only been proved for even $q$, and when $q$ is a square or a cube (cf. Kantor [11], [12] and Bartolone-Ostrom [1]).

THEOREM (see (6.7)). *Let $q = p^m > p$ be any strict prime power. Then there is a non-Desarguesian spread $\pi$ of order $q^3$ that admits $GL(2,q)$ acting as it does on the Desarguesian spread of order $q^3$.*

REMARK. All the known spreads of order $p^3$ admitting $GL(2,p)$ are Desarguesian. Moreover, by Bartolone-Ostrom ([1, section 2]), if $GL(2,p)$ acts in a "Desarguesian manner" on a spread $\pi$ of order $p^3$, then $\pi$ is necessarily Desarguesian. More precisely, if $V = \mathbb{F}_p^6$ admits $G = GL(2,p)$ such that $G$ fixes a Desarguesian spread $\Gamma$ on $V$ then all other $G$-invariant spread $\Gamma$ on $V$ are necessarily Desarguesian.

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## 2. SPREAD-SETS

In this section we describe some notation and results concerning spread-sets that we shall use. Throughout the article, a vector space of type $V = \mathbb{F}_p^N$ is the vector space of row $N$-tuples over the prime field $GF(p)$.

(2.1) DEFINITION. $\mathcal{M}$ is a *spread-set* on $\mathbb{F}_p^N$, if $\mathcal{M} \subseteq GL(n,p) \cup \{0\}$ is a set of $p^N$ matrices such that

(i) $0,\ I \in \mathcal{M}$ ;

(ii) $X, Y \in \mathcal{M} \Rightarrow X - Y \in GL(n,p)$ whenever $X \neq Y$.

$\mathcal{M}$ is an ***additive*** spread-set if, in addition to being a spread-set, $\mathcal{M}$ is a group under matrix addition (or equivalently $\mathcal{M}$ is additively closed).

A *spread* on $\pi = \mathbb{F}_p^N \oplus \mathbb{F}_p^N$ is a collection $\Gamma$ of $p^N + 1$ rank $N$ subspaces of $\pi$ such that any two distinct members (or "components") of $\Gamma$ intersect trivially (e.g., [13, chapter 1]). Any spread-set $\mathcal{M}$ on $\mathbb{F}_p^N$ has associated with it a spread $\Gamma_M$, which we shall describe using the following notation.

(2.2) NOTATION. Let $\pi = \mathbb{F}_p^N \oplus \mathbb{F}_p^N$, and $M$ any $N \times N$ matrix over $GF(p)$. Then we shall use the notation $y = xM$ to denote the subspace $\bar{M}$ of $\pi$ given by:

$$\bar{M} = \{(x, xM) : x \in \mathbb{F}_p^N\} \ .$$

Also, the space $0 \oplus \mathbb{F}_p^N$ is written as $x = 0$. If $\mathcal{M}$ is a spread-set on $\mathbb{F}_p^N$ then the corresponding spread is

$$\Gamma_{\mathcal{M}} = \{y = xM : M \in \mathcal{M}\} \cup \{x = 0\} \ .$$

It is easily verified that $\Gamma_{\mathcal{M}}$ is a genuine spread, and the subring of $Hom(\pi, +)$ that leaves invariant each component of $\Gamma_{\mathcal{M}}$ can be canonically identified with the subring of $Hom(\mathbb{F}_p^N, +)$ that centralizes $M$; in fact, by Schur's lemma, both centralizers are fields since $M$ generates an irreducible group. Thus one obtains

(2.3) RESULT (cf. Foulser [4]). If $\mathcal{M}$ is a spread-set on $\mathbb{F}_p^N$ then the centralizer of $\mathcal{M}$ in $Hom(\mathbb{F}_p^N, +)$ is a field $K$, and $K^*$ is isomorphic to the group of kern homologies of the corresponding spread $\Gamma_M$.

We now turn to another version of Foulser's results, in the context of additive spread-sets.

(2.4) DEFINITION. Let $\mathcal{M}$ be an additive spread-set on $(\mathbb{F}_p^N, +)$. For any $x, y \in \mathbb{F}_p^N$, define

$$x \circ y = xM_y$$

where $M_y$ is the unique element of $\mathcal{M}$, the slope of $y$, whose first column is $y$ (transpose). We call $D_{\mathcal{M}} = (\mathbb{F}_p^N, +, 0)$ the semifield *associated* with the additive spread-set $\mathcal{M}$.

REMARK. It is well known, and readily verified, that $D_{\mathcal{M}}$ actually is a semifield.

According to the conventions of [6], the right nucleus $N_r(D)$, of any semifield $D$, is canonically isomorphic to the kern of the translation plane coordinatized by $D$. Thus by Result (2.3), we have

(2.5) RESULT. Let $D_{\mathcal{M}}$ be the semifield associated with the additive spread-set $\mathcal{M}$, defined on $\mathbb{F}_p^N$. Then

$$C_{GL(N,p)}(M) \cup \{0\} \cong N_r(D_{\mathcal{M}}) .$$

## 3. IRREDUCIBLE PAIRS

Let $\sigma$ be a nonprojective fix-point-free collineation of a Desarguesian projective plane $PG(2,q)$; thus $\sigma \in P\Gamma L(3,q)$, and $\sigma$ fixes no point of $PG(2,q)$. We shall eventually show that any such $\sigma$ gives rise to a spread $\pi_\sigma$ whose order is $q^3$ and admits $GL(2,q)$ as an automorphism group. However, we shall not describe $\pi_\sigma$ directly in terms of $\sigma$, but rather in terms of a coordinatizing "irreducible pair," which we define now.

(3.1) DEFINITION. Let $V = \mathbb{F}_p^{3m}$ be regarded as a vector space over a field of matrices

$$F \subseteq GL(3m,p) \cup \{0_{3m}\}$$

such that $F \cong GF(q)$, where $q = p^m$. Now if $T \in GL(3m,p)$, we call $(T,F)$ an *irreducible pair* on $V$ if

(i) $T$ normalizes but does not centralize $F$; and

(ii) $T$ does not fix any rank-one $F$-subspace of $V$.

Thus $T \in P\Gamma L(V,F)$ and fixes no projective point of $PG(2,F)$. So $T$ induces a nonprojective fixed-point-free collineation $\sigma$ of $PG(2,q)$, and we regard this $\sigma$ as being *coordinatized* by $(T,F)$. Conversely, it is obvious that any $fpf$ nonprojective collineation $\sigma$, of $PG(2,q)$, can be coordinatized by at least one irreducible pair $(T,F)$ on $V = \mathbb{F}_p^{3m}$, where $q = p^m$. Hence we normally deal only with irreducible pairs $(T, F)$, rather than with the collineations $\sigma$ that coordinatize them. But we begin by proving a lemma that explains the connection between $\sigma$ and $(T, F)$. The lemma justifies our terminology by showing that $T$ acts irreducibly; it also brings definition 1 into line with our earlier definition of irreducible pairs [8].

(3.2) LEMMA. *Let $(T, F)$ be an irreducible pair on $V = \mathbb{F}_p^{3m}$, with $F \cong GF(q)$. Then $T$ does not fix any proper nonzero $F$-subspace of $V$.*

PROOF. Let $g$ be the collineation of $PG(2, q)$ defined by the action of $T$ on $V$, regarded as an $F$-space. Then $g$ is an $fpf$ collineation of $PG(2, q)$. But an $fpf$ collineation of a projective plane cannot fix any line ([6 Theorem 13.1]); so $T$ cannot fix a rank-two subspace of $V$, and the lemma follows.

We now obtain an additive spread-set from any irreducible pair (and hence, indirectly, from any $fpf$ collineation $\sigma$ of $PG(2, q)$ such that $\sigma \in P\Gamma L(3, q) - PGL(3, q)$). The short proof of the result has been included for the convenience of the reader, even though a more general result is obtained in [8].

(3.3) PROPOSITION. *Let $(T, F)$ be an irreducible pair on $V = \mathbb{F}_p^{3m}$. Then*

$$\Delta_{T,F} = F + FT + FT^2$$

*is an additive spread-set on $V$.*

PROOF. If not, then: $\exists \alpha_1, \alpha_2, \alpha_3 \in \mathcal{F}$, not all zero, such that $\alpha_0 + \alpha_1 T + \alpha_2 T^2$ is singular. Thus,

$$\exists x \in V - \{0\} \text{ such that } \sum_{i=0}^{2} \alpha_i x T^i = 0 .$$

Now if $\alpha_2 = 0$ then $T$ fixes a one-space over $F$, and if $\alpha_2 \neq 0$ then $T$ leaves invariant the $F$-space generated by $\{x, Tx\}$, contradicting Lemma 2. Thus, $\Delta_{T,F}$ has $q^3 - 1$ nonzero elements, and these are all nonsingular. The result follows, as $\Delta_{T,F}$ is obviously an additive group.

The proposition permits us to use the following

(3.4) NOTATION. If $(T, F)$ is an irreducible pair on $(\mathbb{F}_p^{3m}, +)$ then

$$D_{T,F} = (\mathbb{F}_p^{3m}, +, \cdot)$$

is the semifield associated with the spread $\Delta_{(T,F)}$.

We now characterize these $D_{T,F}$ as being the semifields satisfying the following (slightly redundant) system of axioms.

(3.5) DEFINITION. A semifield $D$ of order $q^3$ is a cyclic semifield (relative to $GF(q)$) if
(I) $N_m(D) = N_\ell(D) = F \cong GF(q)$ where $N_m(D)$, $N_\ell(D)$ denotes the middle and left nucleus of $D$ respectively; and
(II) $\exists t \in D - F$ and $\sigma \in AutGF(q)$–{identity} such that
(i) $ft = tf^\sigma \ \forall f \in F$;
(ii) $t(tx) = t^2x \ \forall x \in D$; and
(iii) $D = F + Ft + Ft^2$.

(3.6) PROPOSITION. *If $(T, F)$ is an irreducible pair on $\mathbb{F}_p^{3m}$, $D_{(T,F)}$ is a cyclic semifield of order $q^3 = p^{3m}$. Conversely, any cyclic semifield of order $q^3$ is isomorphic to a $D_{(T,F)}$, for some irreducible pair; $(T, F)$, on $\mathbb{F}_p^{3m}$.*

PROOF. Let $D$ be a cyclic semifield of order $q^3$, and choose a $GF(p)$-basis $F$ of $D$, such that $1_D$ is the first element of $F$. We can now identify $(D, +)$ with $V = (\mathbb{F}_p^{3m}, +)$, via the canonical isomorphism determined by $F$. Moreover, each map of type

$$\bar{d} : D \longrightarrow D$$
$$x \longmapsto dx$$

is in $GL(3m, p) \cup \{0_m\}$, and so $\bar{F} \cong F \cong GF(q)$ is a field of matrices over which $V$ is a rank three vector space. Let $T$ be the matrix $\bar{t}$, and observe the condition (i), of Definition

3.5(II), implies that $T$ normalizes, but does not centralize $F$: We are making crucial use of the fact that Definition (3.5) (I) implies

$$\bar{u}\bar{x} = \overline{ux} \text{ and } \bar{x}\bar{u} = \overline{xu} \qquad \forall \bar{u} \in \mathcal{F},\ \bar{x} \in D\ . \tag{a}$$

Next observe that $T$ cannot fix any rank-one $F$-subspace of $(V,+)$, for otherwise, $T - \alpha$ is singular for some $\alpha \in \mathcal{F}$, contradicting the fact that $\bar{D}$ is an additive spread-set. Thus $(T,F)$ is an irreducible pair on $\mathbb{F}_p^{3m}$. It remains to verify that

$$\bar{D} = F + FT + FT^2\ . \tag{b}$$

By Definition (3.5)(II)(ii), $T^2 \in \bar{D}$, and now by equation (a) above, and the fact that $\bar{D}$ is an additive group, we get

$$\bar{D} \supseteq F + FT + FT^2$$

and since, by Proposition (3.3), the RHS is also a spread-set of order $q^3 = |\bar{D}|$, equation (b) holds; and so $D \cong \Delta_{(T,F)}$, as required. The converse, that any $D_{(T,F)}$ is a cyclic semifield, is proved using standard arguments similar to the above (cf. [8, Lemma 5]).

## 4. $GL(2,q)$ SPREADS OF ORDER $q^3$

In this section, $V = \mathbb{F}_p^{3m}$ is a vector space of row tuples admitting an irreducible pair $(T,F)$, with $F \cong GF(q)$ and $q = p^m$. In addition, $\pi = V \oplus V$ is regarded as a $GL(2,q)$-module, under the action of $G = GL(2,F)$ defined as follows:

$$g : \pi \longrightarrow \pi$$

$$(x,y) \longmapsto (xa + yc, xb + yd)$$

whenever

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$$

Hence the action of $g$ on $y = xM$ (cf. notation) is given by:

(4.1) REMARK. If $M$ is a $3m \times 3m$ matrix over $GF(p)$, then

$$g : y = xM \longmapsto y = x(a + Mc)^{-1}(b + Md)$$

whenever

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G \quad \text{and} \quad a + Mc \ \text{ is nonsingular} .$$

Thus, if we choose a field of matrices $\mathcal{M} \supset \mathcal{F}$ such that $\mathcal{M} \cong GF(q^3)$, then Remark 4.1 shows that $G$ fixes the Desarguesian spread $\Gamma_{\mathcal{M}}$ (cf. notation), while leaving invariant the partial spread

$$\delta_F = \{y = xF : F \in \mathcal{F}\} \cup \{x = 0\} .$$

Some properties of the action of $G$ on $\delta_F$ are summarized below.

(4.2) NOTATION. In (4.3), (4.4), (4.5), we shall use the notation $G = GL(2, F)$ and $\delta_F = \{y = xF : F \in \mathcal{F}\} \cup \{x = 0\}$.

(4.3) PROPOSITION. *Every $\ell \in \Delta_F$ is the fixed space of a Sylow p-subgroup of G; hence G induces a 3-transitive group on the components of $\delta_F$.*

PROOF. The upper and lower triangular Sylow $p$-subgroups of $G$ are elation groups of $\Gamma_M$ that have axes (resp.) $y = 0$ and $x = 0$. Thus $y = 0$ and $x = 0$ are co-orbital and now by Remark (4.1), $G$ is transitive on $\delta_F$. Hence each component of $\delta_F$ is an elation axis. The 3-transitivity of $G$ is now an immediate consequence of the fact that $GL(2, q)$ has only one transitive representation on $q + 1$ objects.

We now extend $\delta_F$ to a $G$-invariant non-Desarguesian spread on $\pi$.

(4.4) THEOREM. *$\pi_{T,F} = Orb_G(y = xT) \cup \delta_F$ is a G-invariant non-Desarguesian spread. Further,*

$$kern \pi_{(T,F)} \cong C_{GL(3m,p)}(\{T\} \cup \mathcal{F}) \cup \{0_{3m}\} .$$

PROOF. We first verify that the $G$-orbit of $y = xT$ consists of subspaces that do not meet any member of $\delta_F$ at a nonzero point. Since $\delta_F$ itself is a $G$-orbit, we only need to check

$$y = xT \cap (\cup\delta_F) = \{0\} . \tag{i}$$

Now (i) can only fail if

$$xT = xF \exists F \in \mathcal{F} \quad \text{and} \quad x \in \pi - \{0\} \Rightarrow T - F \text{ is singular} .$$

But this contradicts the fact $\{T\} \cup \mathcal{F}$ is contained in an additive spread, viz., $\Delta_{T,F}$ (Proposition 3.3). Thus $\theta = Orb_G(y = xT)$ is a collection of subspaces of $\pi$, each of order $q^3$, such that $\cup\theta \cap \cup\delta_F = \{0\}$.

We now check that $\theta$ is a partial spread by showing that

$$(y = xT)G \cap (y = xT) \neq \{0\} \Rightarrow G \quad \text{fixes} \quad y = xT . \tag{ii}$$

Thus consider any $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $G = GL(2, F)$ such that

$$(y = xT)g \cap (y = xT) \neq \{0\} .$$

So by Remark (4.1),

$$y = x(a + Tc)^{-1}(b + Td) \cap y = xT \neq \{0\}$$

since $a + Tc$, being part of a spread-set (Proposition (3.3)), is nonsingular. Thus

$$x[(a + Tc)^{-1}(b + Td) - T] = 0 \quad \exists x \neq 0 .$$

Hence,

$$(b + Td) - (a + Tc)T \text{ is singular} . \tag{iii}$$

Now since $T$ normalizes but does not centralize $F$ (Definition 3.1)), $F$ admits a field automorphism $\sigma(\neq \text{identity})$ such that

$$FT = TF^\sigma \quad \exists F \in \mathcal{F} .$$

Thus (iii) gives:

$$b + T(d - a^\sigma) + T^2c^\sigma \quad \text{is singular} .$$

But this element lies in the additive spread-set $\mathcal{F} \oplus \mathcal{F}T \oplus \mathcal{F}T^2$ (Proposition (3.3)), and hence must be zero, because it is singular. Thus $b = c = 0$ and $d = a^\sigma$. Hence $g = \begin{bmatrix} a & 0 \\ 0 & a^\sigma \end{bmatrix}$ and by Remark (4.2), any such $g$ must fix $y = xT$. Thus (ii) is proved, and we have also computed the global stabilizer of $y = xT$ in $GL(2, F)$ to be

$$\{\mathrm{Diag}(a, a^\sigma) : a \in \mathcal{F}^*\} .$$

Thus, $Orb_G(y = xT)$ consists of $|G|/q - 1 = q^3 - q$ components. Hence

$$\delta_F \cup Orb_G(y = xT)$$

is a partial spread on $\pi$ with $(q^3 - q) + (q + 1) = q^3 + 1$ components; and so $\delta_F \cup Orb_G(y = xT)$ is a $G$-invariant spread, which we call $\pi_{T,F}$. The kern of this spread is obtained (up to isomorphism) by Foulser's result (2.3). In particular, since $\{T\} \cup \mathcal{F}$ does not generate an abelian group (multiplicatively), $\pi_{T,F}$ cannot be Desarguesian.

(4.5) COROLLARY. *Let $\pi_{T,F}$ be the spread of order $q^3$ associated with $(T, F)$, an irreducible pair on $V = \mathbb{F}_p^{3m}$, $p^m = q$. Then $\pi_{T,F}$ admits $GL(2, q)$ such that*

(1) *The Sylow $p$-subgroups of $GL(2, q)$ are elation groups of order $q$, and the corresponding $q + 1$ elation axes are precisely the members of $\delta_F$.*

(2) *$GL(2, q)$ is 3-transitive on $\delta_F$, and transitive on the other $q^3 - q$ components of $\pi_{T,F}$.*

(3) *$\delta_F$ is invariant under $Aut\pi_{T,F}$, and the elation groups in $\pi_{T,F}$ are precisely those in $GL(2, q)$.*

PROOF. Parts (1) and (2) are included in Theorem (4.4) and Proposition (4.2). If (3) were false, then every component of $\pi_{T,F}$ would be a nontrivial elation group forcing it to be Desarguesian, by the Hering-Ostrom theorem [13, p. 178], contrary to Theorem 3.

Before giving examples of $\pi_{T,F}$'s, it is desirable to summarize what we have achieved so far; in particular, to restate the connections between the various entities that we have introduced: *fpf* nonprojective collineations, irreducible pairs, cyclic semifields, and $\pi_{T,F}$'s.

## 5. CONNECTIONS ESTABLISHED

Let $V = \mathbb{F}_p^{3m}$, $p^m = q$. Then $V$ admits an irreducible pair $(T, F)$, $F \cong GF(q)$, iff $PG(2,q)$ admits an $fpf$ collineation $\theta \in P\Gamma L(3,q) - PGL(3,q)$; the existence of $(T, F)$ is also equivalent to the existence of a cyclic semifield $D_{T,F}$, of order $q^3$, whose spread set is given by $\Delta_{T,F} = \mathcal{F} \oplus \mathcal{F}T \oplus \mathcal{F}T^2$. Hence,

$$N_r(D_{T,F}) \cong C_{Hom(V,+)}(\{T\} \cup \mathcal{F}) \ . \tag{i}$$

Next consider the action of $GL(2,q)$ on $\pi = V \oplus V$, defined by the standard action of $G = GL(2,F)$ on $\pi$, by matrix multiplication; thus, $G$ has the "Desarguesian" action on $\pi$, permuting the components of a Desarguesian spread $\Gamma_M$ on $\pi$, where $M \cong GF(q^3)$ is a field of matrices containing $F$. Further, $G$ leaves invariant the Desarguesian partial spread

$$\delta_F = \{y = xf : f \in \mathcal{F}\} \cup \{x = 0\}$$

and acts as an automorphism group of the following spread:

$$\pi_{T,F} = \delta_F \cup Orb_G(y = xT) \ .$$

$G$ has exactly two orbits on the components of $\pi_{T,F}$; the $q+1$ components of $\delta_F$ is one of them, and the other orbit consists of the remaining $q^3 - q$ components of $\pi_{T,F}$. Further:

$$Kern \pi_{T,F} \cong N_r(D_{T,F}) \ . \tag{ii}$$

To provide examples of $\pi_{T,F}$, we therefore only need to give examples of irreducible pairs of cyclic semifields with appropriate parameters. Such examples will now be given.

## 6. EXAMPLES

The matrix version of the following theorem corresponds to a large class of irreducible pairs. (A more general version of the theorem is proved in [8].)

(6.1) THEOREM. *Let $\mathbb{F} = GF(q^{3m})$, where $q$ is any prime power, and $m > 1$ satisfies $(3, m) = 1$. Let $\mathbb{P}$ and $\mathbb{N}$ be the subfields of $\mathbb{F}$ given by $\mathbb{P} = (GF(q^m)$, and $\mathbb{N} = GF(q^3)$. Choose $\omega$ to be any primitive element of $\mathbb{F}^*$, and define*

$$\alpha : \mathbb{F} \longrightarrow \mathbb{F}$$
$$x \longmapsto \omega x^{q^3} .$$

Then

(1) $\alpha$ is strictly semilinear map of $\mathbb{F}$ viewed as an $\mathbb{P}$-space, and $\alpha$ does not fix any $\mathbb{P}$-subspace of $\mathbb{F}$ (other than $\{0\}$ and $\mathbb{F}$).

(2) $C_{Hom(\mathbb{F},+)}(\{\alpha\} \cup \bar{\mathbb{P}}) = \bar{\mathbb{N}}$. (N.B.: The notation $\bar{f} : \underset{x \longrightarrow fx}{\mathbb{F} \longrightarrow \mathbb{F}}$ is being used.)

PROOF. For any integer $k > 1$, a routine induction shows that

$$\alpha^k : x \longrightarrow \omega^{(q^{3k}-1)/(q^3-1)} x^{q^{3k}} ,$$

and hence choosing $k = m$ yields

$$\alpha^m = \bar{\nu} \;\text{ where }\; \langle \nu \rangle = \mathbb{N}^* .$$

Hence $W$, any $\alpha$-invariant $\mathbb{P}$-subspace of $(\mathbb{F}, +)$, must also be fixed by $\mathbb{N}$. So $W$ is invariant as an $R$-module, where the ring $R$ is defined by

$$R = \{f(\delta, \beta) : \delta \in \mathbb{P},\ \beta \in \mathbb{N},\ f(x, y) \in \ K[x, y]\}$$

where $K[x, y]$ is the ringe of $K = GF(q)$ polynomials in the indeterminates $x$ and $y$. But as $\mathbb{F}$ is finite, the ring $R$ must be a subfield of $\mathbb{F}$ containing $\mathbb{P}$ and $\mathbb{N}$. Hence $R = \mathbb{F}$, since $\mathbb{P}$ and $\mathbb{N}$ are maximal subfields of $\mathbb{F}$. Thus $W = \mathbb{F}$ or $\{0\}$ and (1) is proved. To prove (2), let $\lambda \in Hom(\mathbb{F}, +)$ be any element centralizing $\{\alpha\} \cup \bar{\mathbb{P}}$. Thus $\lambda$ centralizes $\alpha^m$, which we recall is a generator of $\bar{\mathbb{N}}^*$. Thus $\lambda$ centralizes $\bar{\mathbb{P}} \cup \bar{\mathbb{N}}$, and hence also $\bar{\mathbb{F}}$. But by Schur's lemma $\mathbb{F}$ is its own centralizer, and so $\lambda \in \bar{\mathbb{F}}$. Hence we may write

$$\lambda : \mathbb{F} \longrightarrow \mathbb{F} \quad \exists \ell \in \mathbb{F} .$$
$$x \longrightarrow \ell x$$

But now $\lambda$ centralizes $\alpha$ iff $\ell \in \bar{\mathbb{N}}$. Hence, the theorem is proved.

(6.2) COROLLARY. *Let $V = \mathbb{F}_p^{3N}$, where $p$ is any prime and $N$ is any positive integer such that $N = n_0 n_1$ and $n_1 > 1$ satisfies $3 \nmid n_1$. Then $V$ admits an irreducible pair $(T, F)$ such that the corresponding spread $\pi_{(T,F)}$, of order $p^{3N}$ and admitting $GL(2, p^N)$, satisfies the conditions:*

$$\text{Kern } \pi_{T,F} \cong GF(p^{3n_0}) \text{ and } dim\ \pi_{(T,F)} = n_1 .$$

PROOF. Take the matrix version of Theorem (6.1), relative to any $GF(p)$ basis of $(\mathbb{F}, +)$, and choose $q = p^{n_0}$.

We now turn to a second construction of irreducible pairs: these lead to the Kantor planes of order $q^6$ admitting $GL(2, q^2)$ [12], if $q \neq 2$.

(6.3) THEOREM. *Let $GF(q) \cong K \subseteq \mathbb{F} \cong GF(q^i)$. Let $V = \mathbb{F}^3$. Assume $u$ is a prime $p$-primitive divisor of $q^3 - 1$ where $(i, u) = (i, 3) = 1$. Choose $B \in GL(3, \mathbb{F})$ so that the entries are in $K$ (i.e., in $GL(3, K)$). Assume $|B| = u$. Define $\alpha : x \to x^q B = (x_1^q, x_2^q, x_3^q)B$ for some fixed basis of $V$. Then*

*(1) $(F, \mathbb{F})$ written as matrices relative to some $GF(p)$-basis of $(V, +)$ forms an irreducible pair.*

*(2) The translation plane $\pi_{\alpha,\mathbb{F}}$ corresponding to $(F, \mathbb{F})$ has order $q^{3i}$, admits $GF(2, q^i)$ and has kernel containing $GF(q^3)$ so is of Ostrom dimension $\leq i$.*

*(3) If $i$ is prime, the plane is of Ostrom dimension $i$.*

PROOF. For $\alpha : x \to x^q B$ then $\alpha^2 : x \to x^{q^2} B^2$ since $B \in GL(3, K)$ so that $\alpha^i : x \to x^{q^i} B^i$.

Thus $\langle B^i \rangle = \langle B \rangle$ as $(i, u) = 1$. If $\alpha$ is reducible so is $\alpha^i = \theta$. Asume $W$ is a nonzero $\mathbb{F}$-subspace which is invariant under $\alpha$ and hence $\theta$. Thus,

$$|W| = q^i \text{ or } q^{2i} .$$

Since $u | q^3 - 1$, if also $u | q^i - 1$ or $q^{2i} - 1$ then $u | (q^3 - 1, q^z - 1) = q^{(3,z)} - 1$ where $z = i$ or $2i$. But, $(3, i) = 1 = (3, 2i)$. Hence, $\theta$ fixes $u$ nonzero vector of $W$. Thus, Fix $\theta \neq 0$. By

Maschke's Theorem, let $C$ be an $\mathbb{F}$-space complement of Fix $\theta$ invariant under $\theta$ (note that Fix $\theta$ is an $\mathbb{F}$-space since $\theta$ is linear over $\mathbb{F}$). But, since $|C| = q^i$ or $q^{2i}$, it follows that Fix $\theta \cap C \neq 0$—a contradiction.

Hence, $\alpha$ is irreducible. This proves (1).

Note that $x \to xB$ centralizes $\alpha$ so that the kernel of $\alpha$ and $\mathbb{F}$ contains the smallest field $L$ containing $B$ and $K$, $L \cong GF(q^3)$.

Hence the kernel has order $q^{3j}$ where $3j|3i$. If $i$ is prime, $j = 1$ since the plane is not Desarguesian. This proves (2) and (3).

Note if $i = 2$, we obtain:

(6.4) THEOREM (cf. Kantor [12]). *If $q > 2$, then there is an irreducible pair $(T, F)$ such that the corresponding translation plane $\pi_{T,F}$ has order $q^6$, admits $GL(2, q^2)$, and is two-dimensional over its Kern $\cong GF(q^3)$* (Ostrom dimension 2).

(6.5) REMARK. That $\pi_{T,F}$ is a Kantor plane follows from Kantor [12, Remark 4] and the properties of $\pi_{T,F}$ that we have established in Section 4.

Since irreducible pairs are equivalent to cyclic semifields, we can use known cyclic semifields of order $q^3$ to construct non-Desarguesian spreads of order $q^3$ admitting $GL(2, q)$ (cf., Proposition 3.6). In particular, the Sandler semifields of order $q^3$ [2, p. 243] imply that

(6.6) THEOREM. (cf., Bartolone-Ostrom [1, section 3]). *If $q$ is a cubic prime-power then there is an irreducible pair $(T, F)$ such that the corresponding spread $\pi_{T,F}$ of order $q^3$ admitting $GL(2, q)$ is three-dimensional over its kern* (Ostrom dimension 3).

We leave it to the interested reader to check that the irreducible pairs arising from Sandler's cyclic semifields lead to the Bartolone-Ostrom plane indicated above. Combining Theorem (6.5) with Corollary 2, we see that $\pi_{T,\mathcal{F}}$'s exist for all orders $q^3$, if $q \neq$ prime. Thus we have (cf., Corollary (4.5)):

(6.7) THEOREM. *Let $q = p^m > p$, be any prime-power. Then there is a translation plane $\pi$ of order $q^3$, such that $Aut\pi \subset GL(2,q)$ and $\pi$ is non-Desarguesian. $\pi$ can be chosen so that $Aut\pi$ has exactly two orbits on the translation axis: one of length $q+1$, and one of length $q^3 - q$. The smaller orbit consists of all the points of $\pi$ through which there pass an affine line which is the axis of a nontrivial elation.*

(6.8) REMARKS. In (6.2) and (6.3), we have given some constructions of translation planes of order $q^{3j}$ with kernel $GF(q^{3j_0})$ where $j_0|j$ that admit $GL(3,q^j)$ as a collineation group. By varying the irreducible pair chosen, it is possible to construct nonisomorphic planes with isomorphic kernels. The exact ennumeration of the isomorphism classes is an open problem.

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