Algebra Universalis, 33 (1995) 516–532 0002-5240/95/040516-17$01.50 + 0.20/0
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The equational theory of union-free algebras of relations

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Abstract. We solve a problem of Jónsson [12] by showing that the class $\mathscr{R}$ of (isomorphs of) algebras of binary relations, under the operations of relative product, conversion, and intersection, and with the identity element as a distinguished constant, is not axiomatizable by a set of equations. We also show that the set of equations valid in $\mathscr{R}$ is decidable, and in fact the set of equations true in the class of all positive algebras of relations is decidable.

Call an algebra of similarity type $\langle 2, 2, 0, 0, 2, 1, 0\rangle$ a *positive subreduct* of an algebra of (binary) relations, or simply a *positive set relation algebra*, if it has the form

$$\mathfrak{A} = \langle A, \cup, \cap, \emptyset, U \times U, |, ^{-1}, I_U\rangle,$$

where A is a set of binary relations (on some *base set* U) that is closed under the operations of union, $\cup$, intersection, $\cap$, relative product, $|$, and conversion, $^{-1}$, and that contains the empty relation, $\emptyset$, the universal relation, $U \times U$, and identity relation, I_U, on U. Similarly, let's call an algebra of the form

$$\mathfrak{A} = \langle A, \cup, |, ^{-1}, I_U\rangle$$

a *subpositive*, or *union-free*, *set relation algebra*. Let $\mathscr{P}$ and $\mathscr{R}$ be the classes of algebras isomorphic to the positive and subpositive set relation algebras respectively.

The class $\mathscr{R}$ was studied by Jónsson in [12], where it was shown, among other things, that $\mathscr{R}$ is a quasivariety, i.e., it is axiomatizable by an infinite set of conditional equations (or universal Horn sentences). Concerning other possibilities

Presented by B. M. Schein.
Received March 15, 1993; accepted in final form April 13, 1994.
H. Andréka's research was supported by Hungarian National Research Fund No. 1911 and No. 2258.

for axiomatizing $\mathscr{R}$, Jónsson raised two problems: is $\mathscr{R}$ finitely axiomatizable, and is it a variety, i.e., is it axiomatizable by a set of equations. Haiman [11] answered the first question negatively. In this paper we shall answer the second question negatively: $\mathscr{R}$ is not a variety. We shall also show that the equational theory of $\mathscr{R}$ is decidable. The class $\mathscr{P}$ was studied by Andréka and Németi. Using standard methods of algebraic logic (see, e.g., [14]) they showed that it is a quasivariety, and in [4] they showed that it is not a variety. Using the method of proof of Comer [10], it can be shown that $\mathscr{P}$ is not finitely axiomatizable. In the present paper, we shall prove that the equational theory of $\mathscr{P}$ is decidable. At the end of the paper we shall discuss some extensions of these results.

A *labeled graph* is a structure $\mathfrak{G} = \langle V, E \rangle$, where V is an arbitrary set and $E \subseteq V \times N \times V$; here, N is the set of positive integers. V is called the set of *vertices* and E is called the set of *labeled edges*. An element (u, k, v) of E is called a *(directed) edge of* $\mathfrak{G}$ *from u to v with label k*, and will be represented graphically by $\underset{u}{\bullet} \xrightarrow{k} \underset{v}{\bullet}$. Since all graphs considered in this paper will have labels, we shall simply refer to them as *graphs.*

Let $\mathfrak{G}_1 = \langle V_1, E_1 \rangle$ and $\mathfrak{G}_2 = \langle V_2, E_2 \rangle$ be graphs. Recall that a homomorphism from $\mathfrak{G}_1$ to $\mathfrak{G}_2$ is a mapping h from V_1 to V_2 that preserves directed edges, i.e., if (u, k, v) is in E_1, then $(h(u), k, h(v))$ is in E_2. We write $\mathfrak{G}_1 \to \mathfrak{G}_2$ to indicate that there exists a homomorphism from $\mathfrak{G}_1$ to $\mathfrak{G}_2$. Since $\mathfrak{G}_1$ is a structure without operations, a congruence relation on $\mathfrak{G}_1$ is just an equivalence relation on V_1. If Θ is such an equivalence relation, then by the quotient graph $\mathfrak{G}_1/\Theta$ we mean the structure $\langle V_1/\Theta, E_1/\Theta \rangle$, where V_1/Θ is the set of equivalence classes of Θ and $E_1/\Theta = \{(u/\Theta, k, v/\Theta) : (u, k, v) \in E_1\}$.

The *sum* or *disjoint union*, $\mathfrak{G}_1 \oplus \mathfrak{G}_2$, of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ is the graph

$$\mathfrak{G}_1 \cup \bar{\mathfrak{G}}_2 = \langle V_1 \cup \bar{V}_2, E_1 \cup \bar{E}_2 \rangle,$$

where $\bar{\mathfrak{G}}_2$ is the graph with iniverse $\bar{V}_2 = V_2 \times \{V_1\}$ that is canonically isomorphic to $\mathfrak{G}_2$ via the mapping $v \to \bar{v} = (u, V_1)$, i.e., $\bar{\mathfrak{G}}_2$ is a canonically chosen isomorphic copy of $\mathfrak{G}_2$ whose universe is disjoint from that of $\mathfrak{G}_1$. We shall follow the usual custom of abusing notation by writing "v" instead of "$\bar{v}$" for elements of $\mathfrak{G}_1 \oplus \mathfrak{G}_2$ that come from $\mathfrak{G}_2$.

A *2-pointed graph* is a graph with two distinguished vertices, i.e., a structure $\mathfrak{H} = \langle V, E, i, o \rangle$, where $\mathfrak{G} = \langle V, E \rangle$ is a graph, and i and o are two (not necessarily different) distinguished vertices, called the *input* and *output* vertex respectively. We sometimes denote $\mathfrak{H}$ by $\langle \mathfrak{G}, i, o \rangle$. The notions of a homomorphism and a congruence relation extend automatically from graphs to 2-pointed graphs. Of course, homomorphisms preserve input and output vertices. We shall usually speak simply of *graphs* whenever it is clear from the context whether or not the graphs under consideration are 2-pointed.

We now define some further constructions on graphs that will serve, in the context of this paper, as graph-theoretic analogues of operations on binary relations. Given two graphs $\mathfrak{H}_1 = \langle \mathfrak{G}_1, i_1, o_1 \rangle$ and $\mathfrak{H}_2 = \langle \mathfrak{G}_2, i_2, o_2 \rangle$, we define their relative product, $\mathfrak{H}_1 ; \mathfrak{H}_2$, and their Boolean product, $\mathfrak{H}_1 \cdot \mathfrak{H}_2$, as quotients of $\mathfrak{G}_1 \oplus \mathfrak{G}_2$. In the case of relative product, we take U to be the smallest equivalence relation on the universe of $\mathfrak{G}_1 \oplus \mathfrak{G}_2$ that identifies o_1 with i_2, and we set

$$\mathfrak{H}_1 ; \mathfrak{H}_2 = \langle \mathfrak{G}_1 \oplus \mathfrak{G}_2/\Theta, i_1/\Theta, o_2/\Theta \rangle.$$

Notice that almost all equivalence classes in this quotient are singletons. The possible exceptions are i_1/Θ, o_1/Θ, i_1/Θ, and o_2/Θ, and here are four possibilities:

$$i_1/\Theta = \{i_1\}, \quad o_1/\Theta = i_2/\Theta = \{o_1, i_2\}, \quad o_2/\Theta = \{o_2\} \quad \text{in case } i_1 \neq o_1 \text{ and } i_2 \neq o_2,$$

$$i_1/\Theta = o_1/\Theta = i_2/\Theta = \{i_1, o_1, i_2\}, \quad o_2/\Theta = \{o_2\} \quad \text{in case } i_1 = o_1 \text{ and } i_2 \neq o_2,$$

$$i_1/\Theta = \{i_1\}, \quad o_1/\Theta = i_2/\Theta = o_2/\Theta = \{o_1, i_2, o_2\} \quad \text{in case } i_1 \neq o_1 \text{ and } i_2 = o_2,$$

$$i_1/\Theta = o_1/\Theta = i_2/\Theta = o_2/\Theta = \{i_1, o_1, i_2, o_2\} \quad \text{in case } i_1 = o_1 \text{ and } i_2 = o_2.$$

If u is an element of $\mathfrak{G}_1$ or $\mathfrak{G}_2$ and $u/\Theta = \{u\}$, then we shall identfy u/Θ with u.

In the case of Boolean product, we take Θ to be the smallest equivalence relation on the universe of $\mathfrak{G}_1 \oplus \mathfrak{G}_2$ that identifies i_1 with i_2 and o_1 with o_2, and we set

$$\mathfrak{H}_1 \cdot \mathfrak{H}_2 = \langle \mathfrak{G}_1 \oplus \mathfrak{G}_2/\Theta, i_1/\Theta, o_1/\Theta \rangle.$$

Again, almost all equivalence classes in this quotient are singletons, and the possible exceptions are i_1/Θ, o_1/Θ, i_2/Θ and o_2/Θ. There are two possibilities:

$$i_1/\Theta = i_2/\Theta = \{i_1, i_2\}, \quad o_1/\Theta = o_2/\Theta = \{o_1, o_2\} \quad \text{in case } i_1 \neq o_1 \text{ and } i_2 \neq o_2,$$

$$i_1/\Theta = o_1/\Theta = i_2/\Theta = o_2/\Theta = \{i_1, o_1, i_2, o_2\} \quad \text{in case } i_1 = o_1 \text{ or } i_2 = o_2.$$

The converse, $\mathfrak{H}_1^{\smile}$, of $\mathfrak{H}_1$ is defined to be the graph obtained from $\mathfrak{H}_1$ by switching i_1 and o_1, i.e., by making o_1 the input vertex and i_1 the output vertex, but leaving the set of edges unchanged. See Figure 1.

The equational language of $\mathscr{P}$ consists of an infinite number of variables, $x_1, x_2, \ldots$, the operation symbols $+$, $\cdot$, ;, and $^{\smile}$, denoting the abstract operations of Boolean sum, Boolean product, relative product, and conversion, the constant

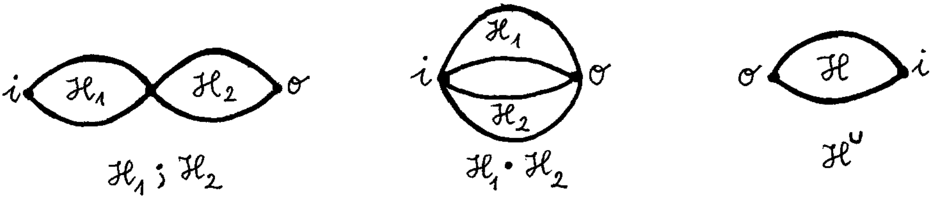

Figure 1. Operations on graphs.

symbols 0, 1, and 1′ denoting the Boolean zero and one, and the identity element, and the equality relation symbol, =. It is sometimes also convenient to admit, in addition to =, a second primitive relation symbol, namely the inequality symbol $\leq$. Of course, every inequality $\sigma \leq \tau$ can be viewed as an equation, namely the equation $\sigma \cdot \tau = \sigma$. The equational language of $\mathscr{R}$ is the same as that of $\mathscr{P}$ except that it does not contain the symbols +, 0, or 1. (We follow the convention that, in expressions lacking parentheses, the symbol ";" has precedence over "·".) Let $T_{\mathscr{P}}$ and $T_{\mathscr{R}}$ be the respective sets of terms (sometimes also called *polynomials*) of these languages. For each term σ with variables among $x_1, \ldots, x_n$, each algebra of relations $\mathfrak{A}$, and each sequence $\bar{R} = (R_1, \ldots, R_u)$ of relations in $\mathfrak{A}$, the value of σ at $\bar{R}$ in $\mathfrak{A}$ is denoted by $\sigma^{\mathfrak{A}}[\bar{R}]$. Of course this is just a binary relation. We say that the inequality $\sigma \leq \tau$, respectively the equality (or equation) $\sigma = \tau$, *holds* in $\mathfrak{A}$, or is *valid* in $\mathfrak{A}$, if $\sigma^{\mathfrak{A}}[\bar{R}] \subseteq \tau^{\mathfrak{A}}[\bar{R}]$, respectively $\sigma^{\mathfrak{A}}[\bar{R}] = \tau^{\mathfrak{A}}[\bar{R}]$, for each sequence $\bar{R}$ of relations from $\mathfrak{A}$ (of the appropriate length), and it holds in $\mathscr{R}$ if it holds in every algebra in $\mathscr{R}$.

For each term σ in $T_{\mathscr{R}}$ we define a graph $\mathfrak{H}_\sigma$. The definition goes by induction on the definition of terms. Fix two arbitrary, distinct elements a and b. For any positive integer k we set $\mathfrak{H}_{x_k} = \langle \{a, b\}, \{(a, k, b)\}, a, b \rangle$. We also set $\mathfrak{H}_{1'} = \langle \{a\}, \varnothing, a, a \rangle$. Finally, for terms σ and τ in $T_{\mathscr{R}}$ we set

$$\mathfrak{H}_{\sigma^\cup} = \mathfrak{H}_\sigma^\cup, \qquad \mathfrak{H}_{\sigma \cdot \tau} = \mathfrak{H}_\sigma \cdot \mathfrak{H}_\sigma \cdot \mathfrak{H}_\tau, \qquad \mathfrak{H}_{\sigma;\tau} = \mathfrak{H}_\sigma ; \mathfrak{H}_\tau .$$

Observe that, for each term σ, the construction of $\mathfrak{H}_\sigma$ is effective, and $\mathfrak{H}_\sigma$ is finite. This will play a role in establishing the decidability of the equational theory of $\mathscr{R}$.

To give some concrete examples of such graphs, let δ, ε, ϱ, and η be the following terms:

(ε) $\varepsilon = x_1 ; x_3 \cdot x_2 ; x_4,$

(δ) $\delta = x_5 ; x_6,$

(ϱ) $\varrho = x_1^\cup ; x_2 \cdot x_3 ; x_4^\cup,$

(η) $\eta = (x_1^\cup ; x_5 \cdot x_3 ; x_6^\cup) ; (x_5^\cup ; x_2 \cdot x_6 ; x_4^\cup).$

Then the graphs of $\mathfrak{H}_{x_k}$, $\mathfrak{H}_{1'}$, $\mathfrak{H}_\delta$, $\mathfrak{H}_\varepsilon$, $\mathfrak{H}_{\varepsilon\cdot\delta}$, $\mathfrak{H}_\varrho$, $\mathfrak{H}_\eta$, and $\mathfrak{H}_{\varrho\cdot\eta}$ can be visualized as follows:

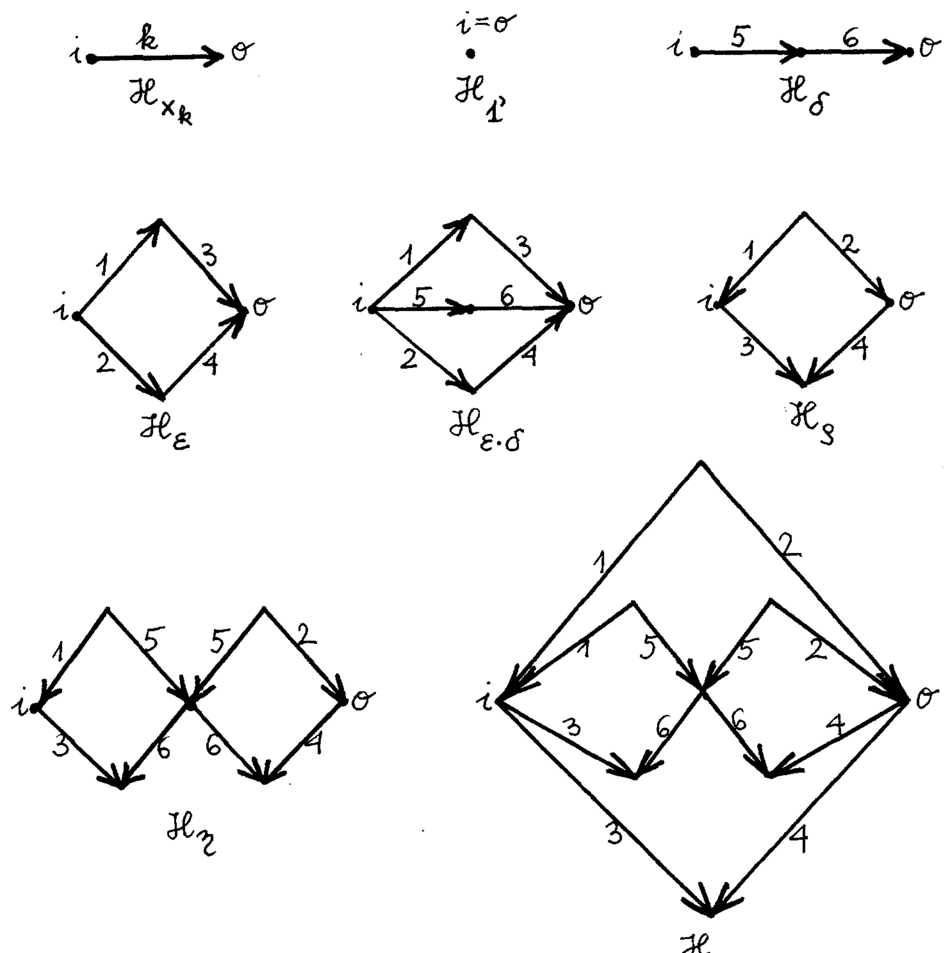

Figure 2(a)–(h). Graphs $\mathfrak{H}_\tau$ associated to terms $\tau \in T_{\mathscr{R}}$.

We now give a theorem that characterizes when an inequality or equality is valid in $\mathscr{R}$.

THEOREM 1. *Let σ and τ be terms in $T_{\mathscr{R}}$. Then the inequality $\sigma \le \tau$ is valid in $\mathscr{R}$ iff there is a homomorphism from $\mathfrak{H}_\tau$ to $\mathfrak{H}_\sigma$. Hence, the equation $\sigma = \tau$ is valid in $\mathscr{R}$ iff there exist homomorphisms from $\mathfrak{H}_\sigma$ to $\mathfrak{H}_\tau$ and from $\mathfrak{H}_\tau$ to $\mathfrak{H}_\sigma$.*

Before proving Theorem 1, we shall establish two lemmas.

LEMMA 2. *Let* $\mathfrak{H}_m = \langle \mathfrak{G}_m, i_m, o_m \rangle$ *be a graph for* $m = 1, 2, 3$.

(i) $\mathfrak{H}_1^{\smile} \to \mathfrak{H}_2^{\smile}$ *iff* $\mathfrak{H}_1 \to \mathfrak{H}_2$.

(ii) $\mathfrak{H}_1 ; \mathfrak{H}_2 \to \mathfrak{H}_3$ *iff there is an element* w *of* $\mathfrak{G}_3$ *such that* $\mathfrak{H}_1 \to \langle \mathfrak{G}_3, i_3, w \rangle$ *and* $\mathfrak{H}_2 \to \langle \mathfrak{G}_3, w, o_3 \rangle$.

(iii) $\mathfrak{H}_1 \cdot \mathfrak{H}_2 \to \mathfrak{H}_3$ *iff* $\mathfrak{H}_1 \to \mathfrak{H}_3$ and $\mathfrak{H}_2 \to \mathfrak{H}_3$.

Proof. We sketch a proof of case (ii). Suppose, first, that h is a homomorphism from $\mathfrak{H}_1 ; \mathfrak{H}_2$ to $\mathfrak{H}_3$. Set $w = h(o_1/\Theta) = h(i_2/\Theta)$, and define h_i, for $i = 1, 2$, by stipulating that $h_i(v) = h(v/\Theta)$ for every v from $\mathfrak{G}_i$. Then h_1 maps $\mathfrak{H}_1$ homomorphically into $\langle \mathfrak{G}_3, i_3, w \rangle$ and h_2 maps $\mathfrak{H}_2$ homomorphically into $\langle \mathfrak{G}_3, w, o_3 \rangle$.

Now suppose that we are given a w from $\mathfrak{H}_3$, and homomorphisms h_1 and h_2 from $\mathfrak{H}_1$ into $\langle \mathfrak{G}_3, i_3, w \rangle$ and from $\mathfrak{H}_2$ into $\langle \mathfrak{G}_3, w, o_3 \rangle$ respectively. Then $h_1(o_1) = h_2(i_2)$, so the mapping h from $\mathfrak{H}_1 ; \mathfrak{H}_2$ to $\mathfrak{H}_3$ defined by $h(v/\Theta) = h_i(v)$ for v from $\mathfrak{G}_i$ is well-defined and a homomorphism. □

The relationship between the graphs $\mathfrak{H}_\sigma$ and the algebras of $\mathscr{R}$ is captured by the following definition and lemma. For an algebra $\mathfrak{A}$ in $\mathscr{R}$ with base set U, a positive integer n, and a sequence $\bar{R} = (R_1, \ldots, R_n)$ of relations from A, we define the graph $\mathfrak{G}(\mathfrak{A}, \bar{R})$ by stipulating

$$\mathfrak{G}(\mathfrak{A}, \bar{R}) = \langle U, \{(u, k, v) : u, v \in U, 1 \le k \le n, (u, v) \in R_k\} \rangle.$$

LEMMA 3. *For each term* σ *in* $T_{\mathscr{R}}$ *and pair of elements* $u, v \in U$ *we have*

$$(u, v) \in \sigma^{\mathfrak{A}}[\bar{R}] \qquad \textit{iff } \mathfrak{H}_\sigma \to \langle \mathfrak{G}(\mathfrak{A}, \bar{R}), u, v \rangle.$$

Proof. The proof is by a simple induction on terms. We shall treat two cases as examples, that of a variable, and that of the relative product of two terms. Set $\mathfrak{G} = \mathfrak{G}(\mathfrak{A}, \bar{R})$. Then

$$\begin{aligned} (u, v) \in x_k^{\mathfrak{A}}[\bar{R}] \quad & \text{iff } (u, v) \in R_k \\ & \text{iff } (u, k, v) \text{ is an edge of } \mathfrak{G} \quad \text{by definition of } \mathfrak{G}, \\ & \text{iff } \mathfrak{H}_{x_k} \to \langle \mathfrak{G}, u, v \rangle \quad \text{by definition of } \mathfrak{H}_{x_k}. \end{aligned}$$

Now assume that the lemma holds for a given σ and τ. Then

$$\begin{aligned} (u, v) \in \sigma;\ \tau^{\mathfrak{A}}[\bar{R}] \quad & \text{iff } (u, w) \in \sigma^{\mathfrak{A}}[\bar{R}] \text{ and } (w, v) \in \tau^{\mathfrak{A}}[\bar{R}] \quad \text{for some } w, \\ & \text{iff } \mathfrak{H}_\sigma \to \langle G, u, w \rangle \text{ and } \mathfrak{H}_\tau \to \langle G, w, v \rangle \quad \text{for some } w, \\ & \text{iff } \mathfrak{H}_{\sigma;\tau} \to \langle G, u, v \rangle. \end{aligned}$$

The second and third equivalences follow by the induction hypothesis and Lemma 2(ii) respectively. The other cases are quite similar, and are left to the reader. □

We are ready to prove Theorem 1.

Proof of Theorem 1. Assume, first of all, that there is a homomorphism from $\mathfrak{H}_\tau$ to $\mathfrak{H}_\sigma$. Let $\mathfrak{A}$ be any algebra of $\mathscr{R}$, say with base set U, and $\bar{R}$ any sequence of elements from $\mathfrak{A}$. Fix two arbitrary elements u, v in U, and suppose that $(u, v) \in \sigma^{\mathfrak{A}}[\bar{R}]$. Then $\mathfrak{H}_\sigma \to \langle \mathfrak{G}(\mathfrak{A}, \bar{R}), u, v\rangle$, by Lemma 3. Hence, $\mathfrak{H}_\tau \to \langle \mathfrak{G}(\mathfrak{A}, \bar{R}), u, v\rangle$, since $\mathfrak{H}_\tau \to \mathfrak{H}_\sigma$. Therefore, $(u, v) \in \tau^{\mathfrak{A}}[\bar{R}]$, again by Lemma 3. This proves that the inequality $\sigma \leq \tau$ is satisfied by $\bar{R}$ in $\mathfrak{A}$. Since $\mathfrak{A}$ and $\bar{R}$ were arbitrary, $\sigma \leq \tau$ is valid in $\mathscr{R}$.

For the converse implication, assume that the inequality $\sigma \leq \tau$ is valid in $\mathscr{R}$, and suppose that $\mathfrak{H}_\sigma = \langle V, E, x, y\rangle$. For each k with $1 \leq k \leq n$, set

(1) $R_k = \{(u, v) \colon (u, k, v) \in E\}$.

Let $\mathfrak{A} \in \mathscr{R}$ be any algebra with base set V such that $R_1, \ldots, R_n$ are in $\mathfrak{A}$. By (1) and the definition of $\mathfrak{G}(\mathfrak{A}, \bar{R})$ we have

(2) $\mathfrak{H}_\sigma = \langle \mathfrak{G}(\mathfrak{A}, \bar{R}), x, y\rangle$.

Since this trivially implies $\mathfrak{H}_\sigma \to \langle \mathfrak{G}(\mathfrak{A}, \bar{R}), x, y\rangle$, we get $(x, y) \in \sigma^{\mathfrak{A}}[\bar{R}]$ by Lemma 3. Hence, $(x, y) \in \tau^{\mathfrak{A}}[\bar{R}]$, by our assumption. Applying Lemma 3 again, and using also (2), we get $\mathfrak{H}_\tau \to \mathfrak{H}_\sigma$, as desired. □

Regarding Theorem 1 and its proof, we would like to make a few remarks. First, let $\mathfrak{G} = \langle V, E\rangle$ be the disjoint sum of all the graphs $\mathfrak{G}_\sigma$, where σ ranges over the terms in $T_{\mathscr{R}}$ with variables among $x_1, \ldots, x_n$ and $\mathfrak{G}_\sigma$ is the non-pointed reduct of $\mathfrak{H}_\sigma$. For each k with $1 \leq k \leq n$ define R_k as in (1) of the proof of Theorem 1, and let $\mathfrak{A}$ be the union-free set relation algebra generated by $R_1, \ldots, R_n$. Just as in (2), we have $\mathfrak{G} = \mathfrak{G}(\mathfrak{A}, \bar{R})$, i.e.,

(3) $\mathfrak{G}(\mathfrak{A}, \bar{R})$ is the disjoint sum of the graphs $\mathfrak{G}_\sigma$, $\sigma \in T_{\mathscr{R}}$.

Using (3), it is not hard to prove that

(4) $\mathfrak{A}$ is the free algebra over $\mathscr{R}$ with n free generators $R_1, \ldots, R_n$.

Indeed, suppose that $\bar{R}$ satisfies an inequality $\sigma \leq \tau$ in $\mathfrak{A}$. To show that $\sigma \leq \tau$ is valid in $\mathscr{R}$, it suffices to show that $\mathfrak{H}_\tau \to \mathfrak{H}_\sigma$, by Theorem 1. Let $\mathfrak{H}_\sigma = \langle \mathfrak{G}_\sigma, i, o\rangle$.

From (3) we see that the identity function is a monomorphism from $\mathfrak{H}_\sigma$ to $\langle \mathfrak{G}(\mathfrak{A}, \bar{R}), i, o \rangle$. Therefore, by Lemma 3, we have $(i, o) \in \sigma^{\mathfrak{A}}[\bar{R}]$. Since $\bar{R}$ satisfies $\sigma \leq \tau$ in $\mathfrak{A}$, we get $(i, o) \in \tau^{\mathfrak{A}}[\bar{R}]$, and therefore $\mathfrak{H}_\tau \to \langle \mathfrak{G}(\mathfrak{A}, \bar{R}), i, o \rangle$, by Lemma 3. Let h be such a homomorphism.

For a moment, let us treat the edges of graphs as undirected. Under this point of view, it is easy to show, by induction in terms ζ, that the graphs $\mathfrak{G}_\zeta$ are all connected, and that these are precisely the connected components of $\mathfrak{G} = \mathfrak{G}(\mathfrak{A}, \bar{R})$. Since $\mathfrak{G}_\tau$ is connected in this sense, h must map $\mathfrak{G}_\tau$ to a connected component of $\mathfrak{G}(\mathfrak{A}, \bar{R})$. But i and o are in the image of h, and they belong to $\mathfrak{G}_\sigma$. Thus, h must map $\mathfrak{G}_\tau$ to $\mathfrak{G}_\sigma$, and hence also $\mathfrak{H}_\tau$ to $\mathfrak{H}_\sigma$, as desired.

Our second remark is that, by the proof of Theorem 1, $\mathscr{R}$ has the finite model property, i.e., an equation fails in $\mathscr{R}$ iff it fails in a finite algebra in $\mathscr{R}$. In fact, an equation that fails in $\mathscr{R}$ must fail in an algebra with a finite base set.

Third, in our opinion Theorem 1 gives a useful tool for investigating the equational theory of $\mathscr{R}$. For example, it follows at once from Theorem 1 that, for every equation $\sigma = \tau$ valid in $\mathscr{R}$, exactly the same variables must occur in σ and τ. Indeed, a variable x_k occurs in σ, respectively τ iff $\mathfrak{H}_\sigma$, respectively $\mathfrak{H}_\tau$, contains an edge labeled k. Furthermore, if $\mathfrak{H}_\sigma \to \mathfrak{H}_\tau$ and $\mathfrak{H}_\tau \to \mathfrak{H}_\sigma$, then a label k occurs in $\mathfrak{H}_\sigma$ iff it occurs in $\mathfrak{H}_\tau$.

Finally, since the graphs $\mathfrak{H}_\sigma$ are finite and effectively constructed, we can effectively decide whether $\mathfrak{H}_\sigma \to \mathfrak{H}_\tau$ and $\mathfrak{H}_\tau \to \mathfrak{H}_\sigma$. This gives us a decision procedure for the equational theory of $\mathscr{R}$.

COROLLARY 4. *The equational theory of $\mathscr{R}$ is decidable.*

It is possible to give another proof of Corollary 4 using the known theorem in [1], pp. 70–71, that the set of logically valid universal-existential sentences (with relation symbols only) is decidable. In fact, this actually proves more.

THEOREM 5. *The equational theory of $\mathscr{P}$ is decidable.*

Proof. Let $\mathscr{L}$ be the first-order language with a denumerable number of binary predicates $\mathbf{R}_1, \mathbf{R}_2, \ldots,$ and individual variables $v_1, v_2, \ldots.$ Following Tarski-Givant [17], p. 28, we define, for each term σ in $T_{\mathscr{P}}$ and each pair of indices i, j in $\{1, 2, 3\}$, a formula $G(\sigma, i, j)$ of $\mathscr{L}$ with the two free variables v_i and v_j as follows:

$$G(x_k, i, j) = v_i \mathbf{R}_k v_j,$$

$$G(0, i, j) = (v_i \neq v_i),$$

$$G(1, i, j) = (v_i = v_i),$$

$$G(1', i, j) = (v_i = v_j),$$

$$G(\sigma^{\cup}, i, j) = G(\sigma, j, i),$$

$$G(\sigma + \tau, i, j) = G(\sigma, i, j) \vee G(\tau, i, j),$$

$$G(\sigma \cdot \tau, i, j) = G(\sigma, i, j) \wedge G(\tau, i, j),$$

$$G(\sigma; \tau, i, j) = \exists v_m[G(\sigma, i, m) \wedge G(\tau, m, j)]$$
where m is the first positive integer $\neq i, j$.

For each term σ, let φ_σ be the formula $G(\sigma, 1, 2)$. Now let $\mathfrak{A}$ be any algebra of $\mathscr{P}$, say with base set U, let $\bar{R} = \langle R_1, \ldots, R_n \rangle$ be any sequence of relations from $\mathfrak{A}$, and let σ, τ be any terms of $T_{\mathscr{P}}$ with variables among $x_1, \ldots, x_n$. Using the definition of G and induction on terms, it is easy to prove that

$$(u, v) \in \sigma^{\mathfrak{A}}[R_1, \ldots, R_n] \qquad \text{iff } \langle U, R_1, \ldots, R_n \rangle \models \varphi_\sigma[u, v].$$

Hence,

$$\mathfrak{A} \models (\sigma = \tau)[R_1, \ldots, R_n] \qquad \text{iff } \langle U, R_1, \ldots, R_n \rangle \models \forall v_1 \forall v_2 (\varphi_\sigma \leftrightarrow \varphi_\tau).$$

In particular, the equation $\sigma = \tau$ is valid in $\mathscr{P}$ iff the sentence $\forall v_1 \forall v_2(\varphi_\sigma \leftrightarrow \varphi_\tau)$ is universally valid. Now φ_σ and φ_τ are built up from atomic formulas and formulas of the form $v_i \neq v_i$ using only $\vee$, $\wedge$ and $\exists$. Thus, each of them is equivalent to an effectively constructible existential formula in prenex normal form. Hence, $\forall v_1 \forall v_2(\varphi_\sigma \leftrightarrow \varphi_\tau)$ is equivalent to an effectively constructible universal-existential sentence ψ. Since the logical validity of ψ is decidable, it is decidable whether or not $\sigma = \tau$ is valid in $\mathscr{P}$. □

The above theorem can be improved to cover other positive classes of algebras of logic. For example, by modifying the proof slightly, we can show that the equational theory of the class of positive cylindric set algebras of any given finite dimension n is decidable.

We now turn our attention to the principal result of this paper.

THEOREM 6. *$\mathscr{R}$ is not a variety.*

Again, we begin with a lemma. Let us say that a graph $\mathfrak{H} = \langle V, E, i, o \rangle$ has an *M_3-subgraph* if there are vertices u, v, and w in V such that u, v, w, i, and o are pairwise distinct, and there are (not necessarily distinct) labels $k_0, \ldots, k_5$ such that (u, k_0, i), (u, k_1, w), (u, k_2, o), and (i, k_3, v), (w, k_4, v), (o, k_5, v) are all edges of $\mathfrak{H}$

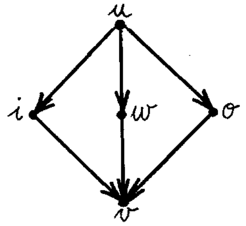

Figure 3. The subgraph M_3.

(see Figure 3). (The notation "M_3" is used because of the similarity of this subgraph to the lattice M_3.)

LEMMA 7. *For no term σ of $T_{\mathfrak{R}}$ does $\mathfrak{H}_\sigma$ contain an M_3-subgraph.*

Proof. The proof is by induction on terms. For σ either x_k or $1'$, this follows from the definition of $\mathfrak{H}_\sigma$. The inductive clauses are immediate consequences of (1)–(3) below.

Let $\mathfrak{H}_m = \langle \mathfrak{G}_m, i_m, o_m \rangle$ be a graph for $m = 1, 2$.

(1) If $\mathfrak{H}_1^\smile$ contains an M_3-subgraph, then so does $\mathfrak{H}_1$.

(2) If $\mathfrak{H}_1 \cdot \mathfrak{H}_2$ contains an M_3-subgraph, then so does $\mathfrak{H}_1$ or $\mathfrak{H}_2$.

(3) If $\mathfrak{H}_1 ; \mathfrak{H}_2$ contains an M_3-subgraph, then so does $\mathfrak{H}_1$ or $\mathfrak{H}_2$.

Indeed, (1) is clear, since the definition of an M_3-subgraph is symmetric with respect to the input and output vertices, and $\mathfrak{H}_1^\smile$ differs from $\mathfrak{H}_1$ only in that the input and output vertices have been interchanged.

For (2), suppose that $\mathfrak{H}_3 = \mathfrak{H}_1 \cdot \mathfrak{H}_2$ has an M_3-subgraph, say (u, k_0, i_3), (u, k_1, w), (u, k_2, o_3), and (i_3, k_3, v), (w, k_4, v), (o_3, k_5, v) are the edges of this subgraph. Then u is either in V_1 or V_2, by definition of V_3. Without loss of generality, we may suppose it is in V_1. Now the edge (u, k_1, w) is also either in E_1 or in E_2, by definition of E_3. Because V_1 and V_2 are disjoint, and u is in $V_1 \backslash \{i_1, o_1\}$, the edge must be in E_1. In a similar fashion, using the definition of $\mathfrak{H}_1 \cdot \mathfrak{H}_2$, we see that each of the edges (u, k_0, i_1), (u, k_2, o_1), (i_1, k_3, v), (w, k_4, v), and (o_1, k_5, v) are in E_1. Thus, $\mathfrak{H}_1$ contains an M_3 subgraph.

For (3), assume first that $i_1 \neq o_1$ and $i_2 \neq o_2$. Suppose that (u, k_0, i_3) and (u, k_2, o_3) are edges in $\mathfrak{H}_3 = \mathfrak{H}_1 ; \mathfrak{H}_2$. By definition of $\mathfrak{H}_1 ; \mathfrak{H}_2$, the only vertex that can have an edge to both i_3 and o_3 is the vertex $o_1/\Theta = i_2/\Theta$. Thus, $u = o_1/\Theta = i_2/\Theta$. Similarly, if (i_3, k_3, v) and (o_3, k_5, v) are also edges in $\mathfrak{H}_3$, then $v = o_1/\Theta = i_2/\Theta = u$. Because the points u and v must be distinct in an M_3-subgraph, this shows that $\mathfrak{H}_3$ can have no such subgraph.

Assume now that $\mathfrak{H}_1 ; \mathfrak{H}_2$ contains an M_3-subgraph. Then $i_3 \neq o_3$. Assume that $i_1 = o_1$. Then $i_2 \neq o_2$ by $i_3 \neq o_3$. Thus if a vertex u has "straight edges" to both i_3 and

o_3, then u has to be in V_2 and the edges have to be in E_2. By applying this argument again, we obtain that u, v, w all have to be in V_2, and $\mathfrak{H}_2$ contains an M_3-subgraph. The case when $i_2 = o_2$ is completely analogous. □

We are ready now to prove Theorem 6.

Proof of Theorem 6. Let φ be the equation

$$x_1; x_3 \cdot x_2; x_4 \le x_5; x_6$$

and ψ the equation

$$x_1^{\smile}; x_2 \cdot x_3; x_4^{\smile} \le (x_1^{\smile}; x_5 \cdot x_3; x_6^{\smile}); (x_5^{\smile}; x_2 \cdot x_6; x_4^{\smile}).$$

Notice that φ and ψ are just the equations $\varepsilon \le \delta$ and $\varrho \le \eta$, where ε, δ, ϱ, and η are the terms defined in (ε), (δ), (ϱ), and (η).

(1) The conditional equation $\varphi \to \psi$ holds in $\mathscr{R}$.

Indeed, let $\mathfrak{A}$ be in $\mathscr{R}$, and suppose that $\bar{R} = \langle R_1, \ldots, R_6 \rangle$ is a sequence of elements from $\mathfrak{A}$ satisfying the equation $\varepsilon \le \delta$. Let (i, o) be in $R_1^{-1} \mid R_2 \cap R_3 \mid R_4^{-1}$, say u and v are such that (u, i), (u, o), (i, v), and (o, v) are in R_1, R_2, R_3, and R_4 respectively (see Figure 2(f)). Then (u, v) is in $R_1 \mid R_3$ and in $R_2 \mid R_4$. Hence, by our assumption, (u, v) is in $R_5 \mid R_6$, say (u, w) is in R_5 and (w, v) is in R_6 (see Figure 2(d) and (c)). Now we easily check that (i, w) is in $R_1^{-1} \mid R_5$ and in $R_3 \mid R_6^{-1}$, and that (w, o) is in $R_5^{-1} \mid R_2$ and in $R_6 \mid R_4^{-1}$ (see Figure 2(g)). Hence, (i, o) is in

$$(R_1^{-1} \mid R_5 \cap R_3 \mid R_6^{-1}) \mid (R_5^{-1} \mid R_2 \cap R_6 \mid R_4^{-1}),$$

as was to be shown. This proves (1).

Let $\mathfrak{T}$ be the term algebra on the variables $x_1, x_2, \ldots, x_6$ for the language of $\mathscr{R}$, or, equivalently, the absolutely free algebra on six generators for the similarity type $\langle 2, 2, 1, 0 \rangle$. We shall define a congruence relation Ψ on $\mathfrak{T}$ with the following properties:

(2) (σ, τ) is in Ψ whenever $\sigma = \tau$ is valid in $\mathscr{R}$,
(3) $(\varepsilon, \varepsilon \cdot \delta)$ is in Ψ,
(4) $(\varrho, \varrho \cdot \eta)$ is not in Ψ.

Assuming, for the moment, that (2)–(4) hold, we complete the proof as follows.

(5) $\mathfrak{T}/\Psi$ is a model of the equational theory of $\mathscr{R}$.

Indeed, suppose that an equation $\sigma = \tau$ with variables among $x_1, \ldots, x_n$ is true in $\mathscr{R}$, and let $(\gamma_1, \ldots, \gamma_n)$ be any sequence of n terms from $\mathfrak{T}$. Then the equation

$$\sigma(\gamma_1, \ldots, \gamma_n) = \tau(\gamma_1, \ldots, \gamma_n)$$

is also true of $\mathscr{R}$, by substitution. Hence, $(\sigma(\gamma_1, \ldots, \gamma_n), \tau(\gamma_1, \ldots, \gamma_n))$ is in Ψ, by (2. Thus, $(\gamma_1/\Psi, \ldots, \gamma_n/\Psi)$ satisfies $\sigma = \tau$ in $\mathfrak{T}/\Psi$. Since $(\gamma_1, \ldots, \gamma_n)$ was arbitrary, this proves (5).

A similar argument, using (3), (4), and the assignment $(x_1/\Psi, \ldots, x_6/\Psi)$, shows that the implication $\varphi \to \psi$ fails in $\mathfrak{T}/\Psi$. But this implication is true of every algebra in $\mathscr{R}$, by (1). Hence, $\mathfrak{T}/\Psi$ is not in $\mathscr{R}$. Together with (5), this shows that $\mathscr{R}$ cannot be a variety.

We shall define Ψ as an extension of the congruence relation Ψ_1 on $\mathfrak{T}$, where Ψ_1 is:

$$\Psi_1 = \{(\sigma, \tau) \colon \sigma = \tau \text{ is true in } \mathscr{R}\}.$$

To define this extension, we need to introduce some auxiliary notions. We begin with the notion of a φ-extension. By a *φ-extension* of $\mathfrak{H} = \langle V, E, i, o \rangle$ we mean a graph $\mathfrak{H}' = \langle V', E', i, o \rangle$, such that, for some vertex w that is not in V, and some vertices u, v, p, and q that are in V, we have: $V' = V \cup \{w\}$; the edges $(u, 1, p)$, $(u, 2, q)$, $(p, 3, v)$, and $(q, 4, v)$ are all in E; and $E' = E \cup \langle (u, 5, w), (w, 6, v) \rangle$ (see Figure 4). The name "φ-extension" comes from the relationship of the graph $\mathfrak{H}_\varepsilon$ to the graph $\mathfrak{H}_{\varepsilon \cdot \delta}$, and the connection of φ to ε and δ (see Figure 2(d) and (c)).

(6) Let $\mathfrak{H}_m = \langle \mathfrak{G}_m, i_m, o_m \rangle$ be a graph, for $m = 1, 2, 3$. If $\mathfrak{H}_1$ is a φ-extension of $\mathfrak{H}_2$, then $\mathfrak{H}_1^{\smile}$, $\mathfrak{H}_1 \cdot \mathfrak{H}_3$, $\mathfrak{H}_1 ; \mathfrak{H}_3$, and $\mathfrak{H}_3 ; \mathfrak{H}_1$ are φ-extensions of $\mathfrak{H}_2^{\smile}$, $\mathfrak{H}_2 \cdot \mathfrak{H}_3$, $\mathfrak{H}_2 ; \mathfrak{H}_3$, and $\mathfrak{H}_3 ; \mathfrak{H}_2$ respectively.

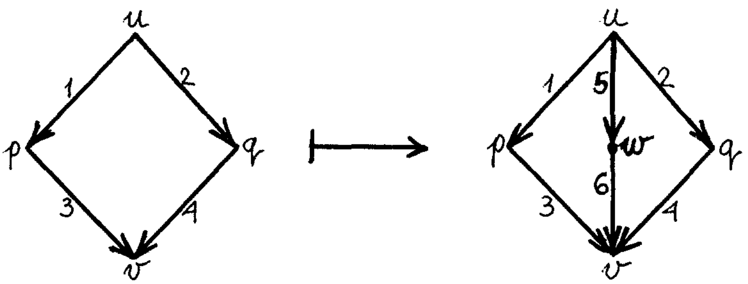

Figure 4. φ-extension.

Indeed, suppose that $\mathfrak{H}_1$ is a φ-extension of $\mathfrak{H}_2$. Since the graphs $\mathfrak{H}_m$ and $\mathfrak{H}_m^{\smile}$, for $m = 1, 2$, are identical except for their input and output vertices, and since the definition of a φ-extension doesn't mention the input and output vertices explicitly, we see that $\mathfrak{H}_1^{\smile}$ must be a φ-extension of $\mathfrak{H}_2^{\smile}$. Furthermore, the notion of a φ-extension involves the addition of a new vertex, different from the input and output vertices, that satisfies special conditions. Now, the Boolean product and the relative product of two graphs is essentially their disjoint union, except for the input and output vertices. Therefore, if the special conditions are satisfied in $\mathfrak{H}_1$ with respect to $\mathfrak{H}_2$, then they are satisfied in $\mathfrak{H}_1 \cdot \mathfrak{H}_3$ with respect to $\mathfrak{H}_2 \cdot \mathfrak{H}_3$, and similarly for the relative products. We leave the details to the reader.

Let Δ be the set of pairs of terms (σ, τ) of $\mathfrak{T}$ such that, either $\mathfrak{H}_\tau$ is a φ-extension of $\mathfrak{H}_\sigma$, or $\mathfrak{H}_\sigma$ is a φ-extension of $\mathfrak{H}_\tau$.

(7) If (σ, τ) is in Δ, then, for every term γ, so are

$$(\sigma^{\smile}, \tau^{\smile}), (\sigma \cdot \gamma, \tau \cdot \gamma), (\gamma \cdot \sigma, \gamma \cdot \tau), (\sigma ; \gamma, \tau ; \gamma), (\gamma ; \sigma, \gamma ; \tau).$$

Indeed, if σ and τ are identical, this is trivial. If $\mathfrak{H}_\sigma$ is a φ-extension of $\mathfrak{H}_\tau$, or vice-versa, then this is an immediate consequence of (6) and the definitions of $\mathfrak{H}_{\sigma \cdot \gamma}$, etc.

The relation Δ is obviously symmetric. Using (7) and the fact that Ψ_1 is a congruence relation, an easy general algebraic argument shows that the transitive closure of $\Psi_1 \cup \Delta$ is a congruence relation on $\mathfrak{T}$. We take Ψ to be just this congruence relation. We now show that (2)–(4) hold for Ψ. We immediately get (2) from the definition of Ψ_1 and the fact that $\Psi_1 \subseteq \Psi$. Furthermore, $\mathfrak{H}_{\varepsilon \cdot \delta}$ is a φ-extension of $\mathfrak{H}_\varepsilon$. Thus, $(\varepsilon, \varepsilon \cdot \delta)$ is in Δ, and hence also in Ψ, so (3) holds. The proof of (4) will require more work.

We shall say that a graph $\mathfrak{H} = \langle V, E, i, o \rangle$ has a *loop* if one of the following conditions holds: $i = o$; an edge of the form (u, k, u) is in E; there is an edge (u, k, v) in E such that, for some $l \neq k$, the edge (u, l, v) is also in E; there is an edge (u, k, v) in E such that, for some l, the edge (v, l, u) is also in E. It is clear that loops are preserved under homomorphisms, i.e., if $\mathfrak{H}_1 \to \mathfrak{H}_2$, and if $\mathfrak{H}_1$ has a loop, then $\mathfrak{H}_2$ also has a loop of the same type.

(8) Let $(\sigma, \tau) \in \Psi_1 \cup \Delta$. If $\mathfrak{H}_\sigma$ has no loop and $\mathfrak{H}_\eta \to \mathfrak{H}_\sigma$, then τ has no loop and $\mathfrak{H}_\eta \to \mathfrak{H}_\tau$.

Suppose, first, that (σ, τ) is in Ψ_1. Then, by Theorem 1 and the definition of Ψ_1, there are homomorphisms from $\mathfrak{H}_\sigma$ to $\mathfrak{H}_\tau$ and conversely. Hence, $\mathfrak{H}_\eta \to \mathfrak{H}_\sigma$ iff $\mathfrak{H}_\eta \to \mathfrak{H}_\tau$, and $\mathfrak{H}_\sigma$ has a loop iff $\mathfrak{H}_\tau$ has a loop. Thus, (8) holds in this case.

Assume, now, that (σ, τ) is in Δ, that $\mathfrak{H}_\sigma$ has no loop, and that $\mathfrak{H}_\eta \to \mathfrak{H}_\sigma$. Suppose that $\mathfrak{H}_\tau$ is a φ-extension of $\mathfrak{H}_\sigma$. Then we obviously have $\mathfrak{H}_\sigma \to \mathfrak{H}_\tau$, and hence $\mathfrak{H}_\eta \to \mathfrak{H}_\tau$. Assume, for contradiction that $\mathfrak{H}_\tau$ has a loop. Let w, u, v, p, and q be as in the definition of a φ-extension. $\mathfrak{H}_\sigma$ has no loop, by assumption. Therefore, since $\mathfrak{H}_\tau$ is a φ-extension of $\mathfrak{H}_\sigma$, a loop in $\mathfrak{H}_\tau$ that involve w. Hence, it must have the form $(u, 5, w)$, $(w, 6, v)$, since these are the only edges in $\mathfrak{H}_\tau$ that involve w. But this is a loop only if $u = v$. Since the latter implies that $(u, 1, p)$, $(p, 3, v)$ is a loop in $\mathfrak{H}_\sigma$, we have reached a contradiction.

Suppose, finally, that $\mathfrak{H}_\sigma$ is a φ-extension of $\mathfrak{H}_\tau$. Then $\mathfrak{H}_\tau \to \mathfrak{H}_\sigma$. Since σ has no loop, by assumption, we see that τ has no loop. It remains to prove that $\mathfrak{H}_\eta \to \mathfrak{H}_\tau$. From the definition of $\mathfrak{H}_\eta = \langle V_\eta, E_\eta, i, o \rangle$, we see that, for some elements c, d, e, f, and g, we have

$$V_\eta = \{i, c, d, e, f, g, o\},$$

$$E_\eta = \{(c, 1, i), (c, 5, e), (i, 3, d), (e, 6, d), (f, 5, e), (f, 2, o), (e, 6, g), (o, 4, g)\}.$$

(See Figure 5.)

Let h map $\mathfrak{H}_\eta$ homomorphically into $\mathfrak{H}_\sigma$, and let w, u, v, p, and q be the vertices of $\mathfrak{H}_\sigma$ as in the definition of a φ-extension. Suppose, for contradiction, that w is in the range of h. As is clear from the labels in Figures 4 and 5, and from the fact that $(u, 5, w)$ and $(w, 6, v)$ are the only edges of $\mathfrak{H}_\sigma$ that involve w, we must have $h(e) = w$, $h(c) = h(f) = u$, and $h(d) = h(g) = v$. For similar reasons, w is not in $\{h(i), h(o), h(c), h(d)\}$. Since τ has no loops, the vertices $h(i)$, $h(o)$, $h(c)$, $h(e)$, and $h(d)$ are pairwise distinct. Using Figure 5 and the fact that h preserves labels, it now follows that $h(i)$, $h(o)$, $h(c)$, $h(e)$, and $h(b)$ are five distinct points that form a M_3-subgraph of $\mathfrak{H}_\sigma$. But this contradicts Lemma 7. We conclude that w cannot be

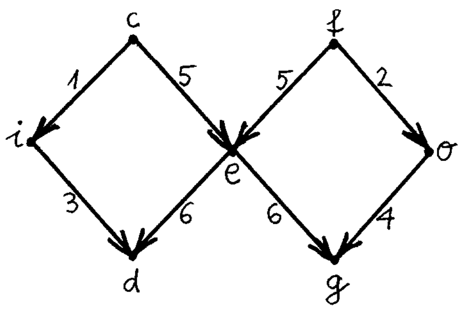

Figure 5. $\mathfrak{H}_\eta$.

in the range of h. Hence, h is actually a homomorphism of $\mathfrak{H}_\eta$ into $\mathfrak{H}_\tau$, is desired. This proves (8).

(8) Let $(\sigma, \tau) \in \Psi$. If $\mathfrak{H}_\sigma$ has no loop and $\mathfrak{H}_\eta \to \mathfrak{H}_\sigma$, then τ has no loop and $\mathfrak{H}_\eta \to \mathfrak{H}_\tau$.

Indeed, suppose that (σ, τ) is in Ψ. Since Ψ is the transitive closure of $\Psi_1 \cup \Delta$, there is a sequence $(\gamma_1, \ldots, \gamma_m)$ of terms of $\mathfrak{T}$ such that σ is γ_1, τ is γ_n, and (γ_i, γ_{i+1}) is in $\Psi_1 \cup \Delta$ for $1 \le i < n$. Applying (8) successively, we get the conclusion of (8) holds for each γ_i. Hence, it holds for τ.

We are ready to prove (4). Clearly, $\varrho \cdot \eta$ does not contain a loop, and $\mathfrak{H}_\eta \to \mathfrak{H}_{\varrho \cdot \eta}$. If $(\varrho, \varrho \cdot \eta)$ were in Ψ, then we could apply (9) to conclude that $\mathfrak{H}_\eta \to \mathfrak{H}_\varrho$. But it is easy to check that there cannot be a homomorphism of $\mathfrak{H}_\eta$ into $\mathfrak{H}_\varrho$ (see Figures 2(g) and (f)). This completes the proof of the theorem. □

Remark. Using the notion of 2-pointed graphs, and the methods of the proof of Theorem 1, we can extend Theorem 1 to the equational theory of $\mathscr{P}$. To handle "union" we use sets of graphs instead of single graphs; in particular, we associate a set $\mathscr{S}_\tau$ of graphs with each term $\tau \in T_{\mathscr{P}}$ in the following way: $\mathscr{S}_x = \{\mathfrak{H}_x\}$, $\mathscr{S}_0 = \emptyset$, $\mathscr{S}_1 = \{\langle\{a, b\}, \emptyset, a, b\rangle\}$, $\mathscr{S}_{\tau+\sigma} = \mathscr{S}_\tau \cup \mathscr{S}_\sigma$, $\mathscr{S}_{\tau \cdot \sigma} = \mathscr{S}_\tau \cdot \mathscr{S}_\sigma$, etc., where $\mathscr{S}_\tau \cdot \mathscr{S}_\sigma = \{\mathfrak{H}_1 \cdot \mathfrak{H}_2 : \mathfrak{H}_1 \in \mathscr{S}_\tau, \mathfrak{H}_2 \in \mathscr{S}_\sigma\}$. If $\mathscr{S}_1, \mathscr{S}_2$ are sets of graphs, then $\mathscr{S}_1 \to \mathscr{S}_2$ iff $(\forall \mathfrak{H}_2 \in \mathscr{S}_2)(\exists \mathfrak{H}_1 \in \mathscr{S}_1)(\mathfrak{H}_1 \to \mathfrak{H}_2)$. With these definitions, the extension of Theorem 1 goes through with almost no change.

Using the extended Theorem 1, one can obtain some known axiomatizations of the classes of subreducts of $\mathscr{P}$ (see e.g. [2, 3, 5–9, 13, 15, 16]). For example, to obtain a description of the equational theory of the $\{\,|\,\}$-reducts (or the $\{o\}$-reducts), we apply Theorem 1. Since we have only $|$, the term-graphs are linear, and each edge contains only one label. (In other words, between any two vertices we have only one edge.) There is a homomorphism between two such graphs iff they are equal. So, $\sigma = \tau$ holds iff $\mathfrak{H}_\sigma = \mathfrak{H}_\tau$. One can show that $\mathfrak{H}_\sigma = \mathfrak{H}_\tau$ iff σ can be obtained from τ by applying the associativity of ;. Thus we obtain a characterization of the equations that are valid in all $\{\,|\,\}$-reducts. A similar argument can be given for $\{\,|, \cap\}$.

If we have only composition $|$ and inverse $^{-1}$, then the graphs consist of a single undirected path, like $\bullet \xrightarrow{x} \bullet \xleftarrow{y} \bullet \xrightarrow{x} \bullet$. There is a homomorphism from one such graph into another only if the length of the first path is at least as big as that of the second. So $\mathfrak{H} \to \mathfrak{G}$ and $\mathfrak{G} \to \mathfrak{H}$ only if the lengths of $\mathfrak{H}$ and $\mathfrak{G}$ are the same, and in this case $\mathfrak{H} \to \mathfrak{G}$ iff $\mathfrak{H} = \mathfrak{G}$. Thus, again $\tau = \sigma$ iff $\mathfrak{H}_\tau = \mathfrak{H}_\sigma$. It can be seen that $\mathfrak{H}_\tau = \mathfrak{H}_\sigma$ iff $\tau = \sigma$ can be derived by using the associativity of ; and the involution properties of $^\smile$ ($x^{\smile\smile} = x$, $(x; y)^\smile = y^\smile; x^\smile$). If we are interested in the "inequality" theory of $\{ | , {}^{-1}\}$ (i.e. in formulas of the form $\tau \leq \sigma$), then we have to add $x \leq x; x^\smile; x$ as an axiom. A similar argument can be given for $\{ | , {}^{-1}, \cup\}$ (see [13], proof of Theorem 4.1).

Related results are given in [8]. Among others, it is shown in [8] that the equational theory of all subreducts (to a given subsimilarity type) of $\mathscr{P}$ is finitely axiomatizable iff not all of the operations $|$, ${}^{-1}$, $\cap$ are contained in these subreducts. In particular, [8] contains another proof that the equational theory of $\mathscr{R}$ is not finitely axiomatizable.

Acknowledgements

We are extremely grateful to the referee for kind and extensive help in bringing the paper to the present form.

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