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A Variational Expression for the Relative Entropy

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Abstract. We prove that for the relative entropy of faithful normal states φ and ω on the von Neumann algebra M the formula

$$S(\varphi,\omega)=\sup\{\omega(h)-\log\varphi^h(I)\colon h=h^*\in M\}$$

holds.

In general von Neumann algebras the relative entropy was defined and investigated by Araki [1, 3]. After Lieb had proved the joint convexity of the relative entropy in the type I case [10] several proofs appeared in the literature and they all benefited from the operator convexity of the function $t\to -\log t$ [8, 11]. Improving a result of Pusz and Woronowicz [14] Kosaki [9] obtained a variational formula for the relative entropy, which allows to extend the notion also to C^*-algebras. The expression we are going to deal with is of a different kind. It shows that the relative entropy $S(\varphi,\omega)$ as a function of φ is the conjugate convex function (i.e., Legendre transform) of the convex function $h\to\log\varphi^h(I)$, where φ^h denotes the inner perturbation of the state φ by the selfadjoint operator h. The perturbed state φ^h was used by Araki to extend the Golden-Thompson inequality ([7, 18], see also [15]) to traceless von Neumann algebras. Approaching our variational expression for the relative entropy we generalize the Golden-Thompson-Araki inequality [2] essentially and we state also the exact condition for the equality.

If φ and ω are faithful normal states of the von Neumann algebra M then the relative entropy is defined by means of the relative modular operator $\Delta(\varphi,\omega)$. If Ω is the vector representative of ω in the natural positive cone P then

$$S(\varphi,\omega)=-\langle\log\Delta(\varphi,\omega)\Omega,\Omega\rangle\,.$$

The variational expression of Kosaki says that

$$S(\varphi,\omega)=\sup\sup\left\{\log n-\int_{1/n}^{\infty}t^{-1}\omega(y(t)^*y(t))+t^{-2}\varphi(x(t)x(t)^*)dt\right\},$$

where $y(t)=I-x(t)$, the first sup is taken over the positive integers and the second one is over all step functions $x:[1/n,\infty)\to M$ such that the range of x is finite and $x(t)=I$ for t large enough.

For a cyclic and separating vector $\Phi\in P$ and a selfadjoint element $h\in M$ the perturbed vector Φ^h is defined by

$$\Phi^h=\sum_{n=0}^{\infty}\int_0^{1/2}dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{n-1}}dt_n\Delta^{t_n}\Delta^{t_{n-1}-t_n}\ldots\Delta^{t_1-t_2}h\Phi,$$

where Δ is the modular operator of φ. The perturbed functional is the nonnormalized vector functional corresponding to Φ^h. The inequality

$$\|\Phi^h\|^2\geqq\exp\varphi(h)$$

reduces to the Golden-Thompson inequality if the algebra admits a faithful normal trace.

If φ and ω are faithful normal states on the von Neumann algebra M then ω is of the form φ^h for some $h=h^*\in M$ provided that there are some constants $\lambda,\mu>0$ such that $\varphi\leqq\lambda\omega\leqq\mu\varphi$ [1]. This h is called the relative Hamiltonian.

Proposition 1. *Let φ and ω be faithful normal states on the von Neumann algebra M and $h=h^*\in M$. Then*

$$\log\varphi^h(I)\geqq\omega(h)-S(\varphi,\omega),$$

and the equality holds if and only if $\omega=\varphi^h/\varphi^h(I)$.

Proof. By Theorem 3.10 of [3] we have $S(\varphi^h,\omega)=S(\varphi,\omega)-\omega(h)$. The monotonicity of the relative entropy gives that $S(\varphi^h,\omega)\geqq\omega(I)[\log\omega(I)-\log\varphi^h(I)]$. Theorem 4 of [12] tells us that here the equality holds if and only if

$$[D\varphi^h,D\omega]_t=(\varphi^h(I)/\omega(I))^{it}\qquad(t\in\mathbb{R}),$$

that is, $\varphi^h=\lambda\omega$ with a $\lambda\in\mathbb{R}^+$ such that $\varphi^h(I)=\lambda\omega(I)$.

Corollary 2. $\log\varphi^h(I)=\sup\{\omega(h)-S(\varphi,\omega)$: ω is a faithful normal state$\}$.

Corollary 3 (cf. [2]). *The function $h\to\log\varphi^h(I)$ is convex on M^{sa}.*

Theorem 4. *Let $\alpha:M_0\to M$ be a unital 2-positive mapping between the von Neumann algebras M_0 and M, and let φ be a faithful normal state of M. Assume that $\varphi\circ\alpha$ is a faithful normal state of M_0. Then for every $h=h^*\in M_0$, the inequality*

$$\varphi^{\alpha(h)}(I)\leqq(\varphi\circ\alpha)^h(I)$$

holds. Furthermore, the equality implies $\varphi^{\alpha(h)}\circ\alpha=(\varphi\circ\alpha)^h$.

Proof. Let $\omega=\varphi^{\alpha(h)}/\varphi^{\alpha(h)}(I)$. Then

$$\log\varphi^{\alpha(h)}(I)=\omega(\alpha(h))-S(\varphi,\omega)$$

by Theorem 3.10 of [2] again. According to the monotonicity of the relative entropy [9, 11, 16] we have

$$S(\varphi,\omega)\geqq S(\varphi\circ\alpha,\omega\circ\alpha),$$

and application of Proposition 1 gives that

$$\log \varphi^{\alpha(h)}(I) \leqq (\omega \circ \alpha)(h) - S(\varphi \circ \alpha, \omega \circ \alpha) \leqq \log(\varphi \circ \alpha)^h(I).$$

If the latest inequality is actually an equality, then $\omega \circ \alpha = \lambda(\varphi \circ \alpha)^h$, that is $\varphi^{\alpha(h)} \circ \alpha = \lambda(\varphi \circ \alpha)^h$.

Corollary 5. *If $N \subset M$ and $h = h^* \in N$, then for a faithful normal state φ on M we have*

$$\varphi^h(I) \leqq (\varphi|N)^h(I),$$

and the equality holds if and only if $\sigma^\varphi{}_t(h) \in N$ for every $t \in \mathbb{R}$. In particular, if N is commutative, then $\varphi^h(I) \leqq \varphi(\exp h)$ and $\sigma^\varphi{}_t(h) = h$ for every $t \in \mathbb{R}$ is a necessary and sufficient condition for the equality.

Proof. We learn from the proof of the previous theorem that $\varphi^h(I) = (\varphi|N)^h(I)$ implies $S(\varphi^h, \varphi) = S(\varphi^h|N, \varphi|N)$, and due to Theorems 4 and 6 of [12] this is equivalent to the condition $\sigma^\varphi{}_t(h) \in N$ for every $t \in \mathbb{R}$.

For a commutative N we have $\psi^h(I) = \psi(\exp h)$ for every state ψ on N and

$$\{a \in N \colon \sigma^\varphi{}_t(a) \in N \text{ for every } t \in \mathbb{R}\} = \{a \in N \colon \sigma^\varphi{}_t(a) = a \text{ for every } t \in \mathbb{R}\}.$$

Corollary 5 is an extension of the Golden-Thompson-Araki inequality, which was proved in [2] by different methods. Our proof is based on the monotonicity of the relative entropy. Roughly speaking, the equality in Corollary 5 may occur only in a trivial way. It is so also in Theorem 4. The condition $\varphi^{\alpha(h)} \circ \alpha = (\varphi \circ \alpha)^h$ is very restrictive and its equivalent (formulated in terms of the modular groups) may be extracted from Theorems 2 and 8 of [13].

Theorem 6. *Let (p_n) be a sequence of projections in M such that $p_n \to I$ strongly. If $M_n = p_n M p_n + \mathbb{C}(I - p_n)$, then*

$$S(\varphi|M_n, \omega|M_n) \to S(\varphi, \omega)$$

as $n \to \infty$ for every faithful normal states φ and ω on M.

Proof. Due to the monotonicity we have $S(\varphi|M_n, \omega|M_n) \leqq S(\varphi, \omega)$. Using Kosaki's formula we assume that

$$\log n - \int_{1/n}^{\infty} t^{-1}\omega(y(t)^* y(t)) + t^{-2}\varphi(x(t)x(t)^*)dt$$

approximates $S(\varphi, \omega)$ for an appropriate step function $x \colon [1/n, \infty) \to M$ with $x(t) = I$ for t large enough. Set $x_n(t) = p_n x(t) p_n + (I - p_n)$ and $y_n(t) = I - x_n(t)$. Then

$$S(\varphi, \omega) \geqq S(\varphi|M_n, \omega|M_n) \geqq \log n - \int_{1/n}^{\infty} t^{-1}\omega(y_n(t)^* y_n(t)) + t^{-2}\varphi(x_n(t)x_n(t)^*)dt,$$

and since

$$\int_{1/n}^{\infty} t^{-1}\omega(y_n(t)^* y_n(t)) t^{-2}\varphi(x_n(t)x_n(t)^*)dt \to \int_{1/n}^{\infty} t^{-1}\omega(y(t)^* y(t)) + t^{-2}\varphi(x(t)x(t)^*)dt,$$

we can conclude the theorem.

Lemma 7. *If φ and ω are positive normal functionals on the von Neumann algebra M, then for every $n \in \mathbb{N}$ there is a projection $p \in M$ such that*

$$\varphi(pap) \leqq 2^n \omega(pap) \qquad (a \in M_+)$$

and

$$\omega(I-p) \leqq 2^{-n} \varphi(I).$$

Proof. Let $\psi_+ - \psi_-$ be the Jordan decomposition of $\varphi - 2^n\omega$ and let p be $\operatorname{supp}\psi_-$ [17]. Then $\varphi(pap) - 2^n\omega(pap) = -\psi_-(pap) \leqq 0$ if $a \in M_+$. On the other hand, $\varphi(I-p) - 2^n\omega(I-p) = \psi_+(I-p) \geqq 0$. So $\omega(I-p) \leqq 2^{-n}\varphi(I-p) \leqq 2^{-n}\varphi(I)$.

Proposition 8. *If φ and ω are faithful normal states on the von Neumann algebra M, then in any strong neighbourhood of the identity there is a projection q such that for some constants $\lambda, \mu \in \mathbb{R}^+$ the estimate*

$$\varphi(qaq) \leqq \lambda\omega(qaq) \leqq \mu\varphi(qaq)$$

holds for every $a \in M_+$.

Proof. We use the previous lemma twice. First, we choose a projection p_n according to the lemma. Then we take the restrictions of φ and ω to the subalgebra $p_n M p_n$ and change the roles. So we get a projection $q_n \leqq p_n$ such that

$$\varphi(q_n a q_n) \leqq 2^n \omega(q_n a q_n), \qquad \omega(q_n a q_n) \leqq 2^n \varphi(q_n a q_n) \qquad (a \in M),$$

and

$$\varphi(p_n - q_n) \leqq 2^{-n} \omega(p_n - q_n) \leqq 2^{-n}, \qquad \omega(I - p_n) \leqq 2^{-n}.$$

To show that $q_n \to I$ strongly it is sufficient to prove that $\varphi(I - q_n) \to 0$ (cf. [6, I. Chap. 4, Proposition 4]). Indeed, $\omega(I - p_n) \to 0$ means that $p_n \to I$ strongly. Hence $\varphi(I - q_n) = \varphi(p_n - q_n) + \varphi(I - p_n) \to 0$.

Now we are in a position to prove the main result of the paper.

Theorem 9. *If φ and ω are faithful normal states on the von Neumann algebra M, then*

$$S(\varphi, \omega) = \sup\{\omega(h) - \log\varphi^h(I) \colon h = h^* \in M\}.$$

If the supremum is attained at $h = h^* \in M$, then $\omega = \varphi^h / \varphi^h(I)$.

Proof. We know both the inequality

$$S(\varphi, \omega) \geqq \omega(h) - \log\varphi^h(I)$$

and the condition for the equality from Proposition 1. A sequence (p_n) of projections is guaranteed by Proposition 8 such that $p_n \to I$ strongly, and on the subalgebra $M_n = p_n M p_n + \mathbb{C}(I - p_n)$ the mutual majorization

$$\varphi(a) \leqq \lambda_n \omega(a) \leqq \mu_n \varphi(a) \qquad (0 \leqq a \in M_n)$$

holds. Due to Theorem 6.3 of [1] the relative Hamiltonian for $\varphi_n = \varphi | M_n$ and $\omega_n = \omega | M_n$ exists. In other words, there is $h_n \in M_n$, $\omega_n = (\varphi_n)^{h_n}$. Hence

$$S(\varphi_n, \omega_n) = \omega(h_n) - \log(\varphi_n)^{h_n}(I),$$

and by Proposition 1 we have

$$S(\varphi_n, \omega_n) \leqq \omega(h_n) - \log \varphi^{h_n}(I).$$

Since $S(\varphi_n, \omega_n) \to S(\varphi, \omega)$ in consequence of Theorem 6 we complete the proof by establishing $\omega(h_n) - \log \varphi^{h_n}(I) \to S(\varphi, \omega)$.

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