On commutative Grothendieck categories having a Noetherian cogenerator

By

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It is well-known the following theorem, discovered independently in 1939 by C. Hopkins [11] and J. Levitzki [12]: any right Artinian ring with identity element is right Noetherian. Some attempts were made in the last years with a view to generalize this theorem to arbitrary Grothendieck categories. Thus, in 1969 J. E. Roos [16] gave an example of a locally Artinian Grothendieck category which is not locally Noetherian. However, if R is a commutative ring with unit element and $\mathscr{C}$ is a quotient category of the category Mod-R of R-modules by an arbitrary localizing subcategory $\mathscr{T}$ of Mod-R (i.e. $\mathscr{C}$ is a commutative Grothendieck category), it was proved in [3; 4.7] that if $T(R)$ is an Artinian object in $\mathscr{C}$ then $T(R)$ is also a Noetherian object, where T: Mod-$R \to$ Mod-$R/\mathscr{T}$ is the canonical functor. The following problem was raised also in [3; 4.8]: does this result hold for a noncommutative ring R with unit element? This question was solved affirmatively by M. L. Teply and R. W. Miller [20; 1.4]. A very short and elegant proof of the result of Teply and Miller was given by C. Năstăsescu [14; 1.3], who proved the following more general theorem: if $\mathscr{C}$ is an arbitrary Grothendieck category which has an Artinian generator U, then U is Noetherian. This statement seems to be the most natural way to place the Hopkins-Levitzki theorem in the general setting of Grothendieck categories.

The aim of the present paper is to study the dual situation from the Hopkins-Levitzki theorem in Năstăsescu's version, i.e. the case of a Grothendieck category having a Noetherian cogenerator. Thus, for a commutative Grothendieck category $\mathscr{C}$ we prove that $\mathscr{C}$ has a Noetherian cogenerator if and only if $\mathscr{C}$ has an Artinian generator. As a corollary we obtain that for a commutative Grothendieck category $\mathscr{C}$ holds the dual of Hopkins-Levitzki theorem: each Noetherian cogenerator of $\mathscr{C}$ is Artinian. We end the paper with a list of open questions.

I wish to thank Dr. C. Năstăsescu for many stimulating conversations.

1. Noetherian cogenerators and minimal cogenerators. Throughout this paper $\mathscr{C}$ will denote a Grothendieck category, i.e. an abelian category with exact direct limits and with a generator.

Recall that an object C of $\mathscr{C}$ is said to be a *cogenerator* if for each nonzero morphism $f\colon X \to Y$ in $\mathscr{C}$ there exists a morphism $g\colon Y \to C$ such that $g \circ f \neq 0$. A *generator* of $\mathscr{C}$ is defined dually. It is clear that if C is an injective object of $\mathscr{C}$, then C is a cogenerator if and only if for each nonzero object X of $\mathscr{C}$ there exists a nonzero morphism $f\colon X \to C$.

It is well-known that in a Grothendieck category each object X has an injective hull, denoted in the sequel by $E(X)$. It follows then immediately that each Grothendieck category has an injective cogenerator.

Recall that an object S of $\mathscr{C}$ is said to be simple if $S \neq 0$ and S has no other subobjects than 0 and S. We shall denote throughout this paper by $\mathrm{Sim}(\mathscr{C})$ a representative system of all isomorphism classes of simple objects in $\mathscr{C}$. Since $\mathscr{C}$ has a generator, $\mathrm{Sim}(\mathscr{C})$ is a set, possibly empty.

1.1. Lemma. *If $\mathscr{C}$ has a nonzero Not eherian cogenerator C, then each nonzero object X of $\mathscr{C}$ has a maximal subobject.*

Proof. There exists a nonzero morphism $f\colon X \to C$, hence $D = \mathrm{Im}(f) \neq 0$. Let $g\colon X \to D$ be the morphism canonically deduced from f. Since C is a Noetherian object, it follows that D is Noetherian, and hence D has a maximal subobject D'. Then $D/D' = S$ is a simple object of $\mathscr{C}$, and $p \circ g\colon X \to S$ is a nonzero epimorphism, $p\colon D \to D/D'$ being the canonical epimorphism. Therefore $\mathrm{Ker}(p \circ g)$ is a maximal subobject of X.

1.2. Lemma. *Let C be an arbitrary cogenerator of $\mathscr{C}$. Then*

$$\bigoplus_{S \in \mathrm{Sim}(\mathscr{C})} E(S) \subset C.$$

Proof. If $\mathrm{Sim}(\mathscr{C}) = \emptyset$, there is nothing to prove. So, we can suppose $\mathrm{Sim}(\mathscr{C}) \neq \emptyset$. Let $S \in \mathrm{Sim}(\mathscr{C})$ and $i\colon S \hookrightarrow E(S)$ be the canonical monomorphism. There exists a morphism $f\colon E(S) \to C$ such that $f \circ i \neq 0$, hence $f \circ i$ is a monomorphism (since S is simple), and then f is a monomorphism (since i is essential). But

$$\sum_{S \in \mathrm{Sim}(\mathscr{C})} S \cong \bigoplus_{S \in \mathrm{Sim}(\mathscr{C})} S$$

(i.e. $\mathrm{Sim}(\mathscr{C})$ is an independent set of subobjects of C), hence $(E(S))_{S \in \mathrm{Sim}(\mathscr{C})}$ is an independent family of subobjects of C. (See e.g. [4; 6.17].) Thus we have

$$\bigoplus_{S \in \mathrm{Sim}(\mathscr{C})} E(S) \subset C.$$

1.3. Proposition. *If $\mathscr{C}$ has a nonzero Noetherian cogenerator C, then* $\mathrm{Sim}(\mathscr{C})$ *is a finite non-empty set and $\mathscr{C}$ has a minimal cogenerator $\bigoplus_{S \in \mathrm{Sim}(\mathscr{C})} E(S)$, which is injective and Noetherian.*

Proof. By 1.1 and 1.2, $\mathrm{Sim}(\mathscr{C})$ is a non-empty finite set, hence

$$C_0 = \bigoplus_{S \in \mathrm{Sim}(\mathscr{C})} E(S)$$

is an injective and Noetherian object of $\mathscr{C}$. Let X be a nonzero object of $\mathscr{C}$; by 1.1, X has a maximal subobject X', hence $S' = X/X'$ is a simple object. Let $p: X \to X/X'$

be the canonical epimorphism and $i\colon S' \hookrightarrow C_0$ be the canonical monomorphism. Then $i \circ p\colon X \to C_0$ is a nonzero morphism. Hence C_0 is a cogenerator of $\mathscr{C}$, and C_0 is a minimal cogenerator by 1.2.

1.4. Remarks. (1) If $\mathscr{C}$ is a Grothendieck category, we may have $\operatorname{Sim}(\mathscr{C}) = \emptyset$. For instance, let R be an infinite direct product of copies of a commutative field and let $\mathscr{L}$ be the full subcategory of Mod-R consisted of all Loewy R-modules. Then, the quotient category Mod-$R/\mathscr{L}$ has no simple objects [2; 2.5.2 and 3.6].

(2) In general, if $\mathscr{C}$ is an arbitrary Grothendieck category, then $\bigoplus_{S \in \operatorname{Sim}(\mathscr{C})} E(S)$ is not necessarily a cogenerator of $\mathscr{C}$, i.e. the well-known criterion of B. Osofsky [15; Lemma 1] on cogenerators in a module category does not work in a Grothendieck category. For example, let R be the ring considered above. By [8; Théorème 2], the category Mod-R of R-modules is not locally coirreducible, hence the spectral category $\mathscr{C}$ of the category Mod-R is not discrete [19; p. 133, Proposition 7.3], in other words $\mathscr{C}$ has at least a non-semi-simple object X. Then $X/\operatorname{So}(X)$ is a nonzero object of $\mathscr{C}$, where $\operatorname{So}(X)$ is the socle of X, i.e. the sum of all simple subobjects of X. Let

$$C_0 = \bigoplus_{S \in \operatorname{Sim}(\mathscr{C})} E(S) = \bigoplus_{S \in \operatorname{Sim}(\mathscr{C})} S.$$

It is easy to show that each morphism $f\colon X/\operatorname{So}(X) \to C_0$ is zero, i.e. C_0 is not a cogenerator of $\mathscr{C}$.

However, if $\mathscr{G}$ is a locally finitely generated Grothendieck category, then

$$\bigoplus_{S \in \operatorname{Sim}(\mathscr{G})} E(S)$$

is a minimal cogenerator of $\mathscr{G}$ [18; 4.5].

2. Noetherian cogenerators relative to a Gabriel topology. Throughout this section R will denote a commutative ring with identity element, and Mod-R the category of unital (right) R-modules. If $M \in$ Mod-R, $x \in M$, and N is a submodule of M, we denote $\operatorname{Ann}_R(x) = \{a \in R \mid xa = 0\}$ and $(N : x) = \{a \in R \mid xa \in N\}$.

Each *Gabriel topology* F on R [19; p. 146] (or topologizing and idempotent filter in Gabriel's sense [9; p. 412]) defines two classes of R-modules:

$$\mathscr{T}_F = \{M \in \text{Mod-}R \mid \operatorname{Ann}_R(x) \in F \quad \text{for all} \quad x \in M\},$$
$$\mathscr{F}_F = \{M \in \text{Mod-}R \mid \operatorname{Ann}_R(x) \notin F \quad \text{for all} \quad x \in M, \quad x \neq 0\}.$$

The pair $(\mathscr{T}_F, \mathscr{F}_F)$ is a *hereditary torsion theory* on Mod-R [19; p. 141], and $\mathscr{T}_F$ is a *localizing subcategory* of Mod-R [9; p. 372]. The hereditary torsion theory $(\mathscr{T}_F, \mathscr{F}_F)$ is uniquely determined by the localizing subcategory $\mathscr{T}_F$. By [19; p. 196] there is a bijective correspondence between the set of all Gabriel topologies on R and the class of all hereditary torsion theories on Mod-R:

$$F \mapsto (\mathscr{T}_F, \mathscr{F}_F),$$

the inverse correspondence being

$$(\mathscr{T}, \mathscr{F}) \mapsto F_{(\mathscr{T},\mathscr{F})} = \{I \mid I \text{ an ideal of } R \text{ such that } R/I \in \mathscr{T}\}.$$

If F is a Gabriel topology on R, we shall simply write $(\mathscr{T}, \mathscr{F})$ for $(\mathscr{T}_F, \mathscr{F}_F)$ if no confusion can occur. If $M \in \mathscr{T}$, M is said to be a *torsion module*, and if $M \in \mathscr{F}$, M is said to be a *torsion-free module*. For each submodule N of an R-module M we denote $\tilde{N} = \{x \in M \mid (N : x) \in F\}$; it is clear that $N = \tilde{N}$ if and only if $M/N \in \mathscr{F}$.

For each $M \in \text{Mod-}R$ we shall use the following notations [2]:

$$C_F(M) = \{N \mid N \text{ submodule of } M \text{ with } M/N \in \mathscr{F}\},$$

$$\text{Spec}_F(R) = \text{Spec}(R) \cap C_F(R),$$

$$\text{Max}_F(R) = \text{the set of maximal elements of } C_F(R)\backslash\{R\} \text{ ordered by inclusion},$$

where $\text{Spec}(R)$ denotes the set of all prime ideals of R. The set $C_F(M)$ ordered by inclusion is a complete modular lattice [19]. M is said to be *F-Noetherian* (resp. *F-Artinian*) if the lattice $C_F(M)$ is Noetherian (resp. Artinian); the ring R is F-Noetherian (resp. F-Artinian) if $C_F(R)$ is a Noetherian (resp. Artinian) lattice.

Let M and V be two R-modules. We say that M is cogenerated by V if M can be embedded in a direct product of copies of V. We shall denote by $\text{Cog}(V)$ the class of all R-modules cogenerated by V. We say that a class $\mathscr{A}$ of R-modules is cogenerated by V if $\mathscr{A} = \text{Cog}(V)$.

For each hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ on Mod-R there exists an injective R-module that cogenerates the torsion-free class $\mathscr{F}$, and conversely, each injective R-module Q defines a Gabriel topology

$$F_Q = \{I \mid I \text{ an ideal of } R \text{ with } \text{Hom}_R(R/I, Q) = 0\};$$

if $(\mathscr{T}_Q, \mathscr{F}_Q)$ denotes the torsion theory associated to F_Q, then $\mathscr{F}_Q$ is cogenerated by Q [19; p. 142, Proposition 3.7]. An injective R-module Q is said to be Σ (resp. Δ)-injective if $C_{F_Q}(R)$ is a Noetherian (resp. Artinian) lattice [6].

If $p \in \text{Spec}(R)$, then the Gabriel topology $F_{E(R/p)}$ defined by the injective R-module $E(R/p)$ will be denoted throughout the remainder of this paper by F_p. It is well-known that

$$F_p = \{I \mid I \text{ an ideal of } R \text{ with } I \nsubseteq p\}.$$

A Gabriel topology F is called *semiprime* if $F = \bigcap_{p \in \text{Spec}_F(R)} F_p$, and then, the torsion-free class is cogenerated by $\prod_{p \in \text{Spec}_F(R)} E(R/p)$.

Throughout the remainder of this section F will denote a fixed Gabriel topology on R, and $(\mathscr{T}, \mathscr{F})$ the associated torsion theory on Mod-R.

Let M be an R-module; for each $X \subseteqq R$ and $Y \subseteqq M$ we shall use the following notations:

$$r_R(Y) = \{r \in R \mid Yr = 0\} \quad \text{and} \quad l_M(X) = \{m \in M \mid mX = 0\}.$$

2.1. Proposition. *Let M be an R-module, $M \in \mathscr{F}$, X an ideal of R, and Y a submodule of M. Then*

$$l_M(X) = l_M(\tilde{X}) = \widetilde{l_M(X)} \quad \textit{and} \quad r_R(Y) = r_R(\tilde{Y}) = \widetilde{r_R(Y)}.$$

Proof. Since $X \subseteqq \tilde{X}$, then $l_M(\tilde{X}) \subseteqq l_M(X)$. Let $y \in l_M(X)$ and $r \in \tilde{X}$; then $(X:r) = I \in F$, hence $rI \subseteqq X$, so $(yr)I = y(rI) \subseteqq yX = 0$, and therefore $yr = 0$ since $M \in \mathscr{F}$. Then $y \in l_M(\tilde{X})$, and so $l_M(X) \subseteqq l_M(\tilde{X})$.

Let now $y \in \widetilde{l_M(X)}$; then $(l_M(X):y) = I \in F$, hence $yI \subseteqq l_M(X)$, hence $(yX)I = (yI)X = 0$. Since $M \in \mathscr{F}$, this implies $yX = 0$, that is $y \in l_M(X)$. Therefore

$$l_M(X) = \widetilde{l_M(X)}.$$

Since $Y \subseteqq \tilde{Y}$, we have $r_R(\tilde{Y}) \subseteqq r_R(Y)$. Let $r \in r_R(Y)$ and $y \in \tilde{Y}$; then $(Y:y) = I \in F$, hence $yI \subseteqq Y$, hence $(yr)I \subseteqq Yr = 0$, from which follows $yr = 0$, that is $r \in r_R(\tilde{Y})$. Hence $r_R(Y) \subseteqq r_R(\tilde{Y})$.

Since $r_R(Y) = \bigcap_{y \in Y} \mathrm{Ann}_R(y)$ and $\mathrm{Ann}_R(y) \in C_F(R)$ for all $y \in M$ (M is torsion-free!), one has $r_R(Y) \in C_F(R)$, i.e. $r_R(Y) = \widetilde{r_R(Y)}$.

If $M \in \mathscr{F}$ is a fixed R-module, we can consider by 2.1 the pair of mappings:

$$\alpha\colon C_F(R) \to C_F(M) \quad (\alpha(X) = l_M(X)),$$
$$\beta\colon C_F(M) \to C_F(R) \quad (\beta(Y) = r_R(Y))$$

between the complete lattices $C_F(R)$ and $C_F(M)$.

It is clear that α and β define a *Galois connection* [19; p. 77] between $C_F(R)$ and $C_F(M)$. Let us denote by

$$\overline{C_F(R)} = \{\beta(Y) \mid Y \in C_F(M)\} \quad (\text{resp. by} \quad \overline{C_F(M)} = \{\alpha(X) \mid X \in C_F(R)\})$$

the closed elements of $C_F(R)$ (resp. $C_F(M)$). It is then clear that α and β induce anti-isomorphisms between the lattices $\overline{C_F(R)}$ and $\overline{C_F(M)}$.

We are interested into finding R-modules M, $M \in \mathscr{F}$, for which $\overline{C_F(R)} = C_F(R)$. For this, let us recall the following

2.2. Definition [20]. An R-module M is said to be an F-cogenerator if $M \in \mathscr{F}$ and $\mathscr{F} \subseteqq \mathrm{Cog}(M)$.

2.3. Lemma. *If M is an F-cogenerator, then $\overline{C_F(R)} = C_F(R)$.*

Proof. Let $I \in C_F(R)$; then $R/I \in \mathscr{F}$, hence there exists a set Z and a monomorphism $f\colon R/I \hookrightarrow M^Z$. Then $I = r_R(Y)$, where Y is the set of all coordinates of $f(1+I)$. If Y_1 is the submodule generated by Y in M, and $Y_2 = \tilde{Y}_1$, then by 2.1 we have $I = r_R(Y) = r_R(Y_1) = r_R(Y_2) = \beta(Y_2) \in \overline{C_F(R)}$, since $Y_2 \in C_F(M)$.

2.4. Theorem. *The ring R is F-Artinian if and only if there exists an F-Noetherian F-cogenerator R-module.*

Proof. Let M be an F-Noetherian F-cogenerator. Then $C_F(M)$ is a Noetherian lattice, hence $\overline{C_F(M)}$ is a Noetherian lattice, and then $\overline{C_F(R)}$ is an Artinian lattice by Galois connection. By 2.3, $\overline{C_F(R)} = C_F(R)$, and so $C_F(R)$ is an Artinian lattice, i.e. R is an F-Artinian ring.

Conversely, if R is an F-Artinian ring, then $\mathrm{Spec}_F(R) = \mathrm{Max}_F(R)$ is a finite set by [3; 4.22 and 4.23], and F is a semiprime Gabriel topology by [3; 4.2], hence $\mathscr{F}$ is cogenerated by the injective module $\bigoplus_{p \in \mathrm{Max}_F(R)} E(R/p)$, which is F-Noetherian by [3; 4.31].

2.5. Corollary. *Each F-Noetherian F-cogenerator R-module is F-Artinian.*

2.6. Corollary. *Let Q be an injective R-module, $Q \in \mathcal{F}$. If Q is F-Noetherian, then Q is Δ-injective.*

Proof. Since $Q \in \mathcal{F}$, it follows $F \subsetneqq F_Q$, hence Q is F_Q-Noetherian by [2; 1.8]. Clearly Q is an F_Q-cogenerator, hence R is an F_Q-Artinian ring by 2.4, i.e. Q is Δ-injective.

3. The main results. Let $\mathscr{C}$ be a Grothendieck category and G an arbitrary generator of $\mathscr{C}$. Let $R = \mathrm{Hom}_{\mathscr{C}}(G, G)$ be the ring of endomorphisms of G. By the Gabriel-Popescu theorem [10] there exists a localizing subcategory $\mathscr{L}_G$ of the category Mod-R of right R-modules such that $\mathscr{C}$ is equivalent to the quotient category Mod-$R/\mathscr{L}_G$ [9].

A Grothendieck category $\mathscr{C}$ is called *commutative* [1], [2; 3.4] if there exists a generator of $\mathscr{C}$ having the ring of endomorphisms commutative. It follows that the commutative Grothendieck categories are exactly the categories equivalent to quotient categories of module categories over commutative rings with unit by localizing subcategories.

Let F be a Gabriel topology on R and $(\mathcal{T}, \mathcal{F})$ the associated hereditary torsion theory on Mod-R. We shall denote by T_F: Mod-$R \to$ Mod-$R/\mathcal{T}$ the canonical functor and by S_F: Mod-$R/\mathcal{T} \to$ Mod-R the right adjoint of T_F [9; p. 369]. Almost all properties of an R-module M relative to the Gabriel topology F can be translated in "absolute" properties of the object $T_F(M)$ in the Grothendieck category Mod-$R/\mathcal{T}$, and conversely. For instance, an R-module M is F-Noetherian (resp. F-Artinian) if and only if $T_F(M)$ is a Noetherian (resp. Artinian) object in the category Mod-$R/\mathcal{T}$ [2; 1.3].

In this section we shall translate the relative properties of R-modules with respect to a Gabriel topology, established in the previous section, in absolute properties of objects in an arbitrary commutative Grothendieck category.

3.1. Lemma. *Let F be a Gabriel topology on a ring R (not necessarily commutative) and $(\mathcal{T}, \mathcal{F})$ the associated torsion theory on* Mod-R. *Then*

(1) $M \in$ Mod-R *is F-cogenerator* $\Rightarrow T_F(M) \in$ Mod-$R/\mathcal{T}$ *is cogenerator.*

(2) $C \in$ Mod-$R/\mathcal{T}$ *is cogenerator* $\Rightarrow S_F(C) \in$ Mod-R *is F-cogenerator.*

Proof. (1) Let $Y \in$ Mod-$R/\mathcal{T}$; there exists $X \in$ Mod-R, $X \in \mathcal{F}$, such that $Y = T_F(X)$. Since M is an F-cogenerator, there exists a set J and a monomorphism $X \hookrightarrow M^J$; thus we have a monomorphism $T_F(X) \hookrightarrow T_F(M^J)$, since T_F is an exact functor. But $M \in \mathcal{F}$, hence the canonical morphism $M \to S_F T_F(M)$ is a monomorphism. Thus we have a monomorphism $M^J \hookrightarrow (S_F T_F(M))^J$, and hence a monomorphism

$$T_F(M^J) \hookrightarrow T_F((S_F T_F(M))^J).$$

Since the functor S_F preserves limits, one finds

$$T_F((S_F T_F(M))^J) \cong T_F(S_F(T_F(M)^J)) \cong T_F S_F(T_F(M)^J) \cong T_F(M)^J,$$

and therefore we obtained a monomorphism $Y \hookrightarrow T_F(M)^J$; thus $T_F(M)$ is a cogenerator in Mod-$R/\mathcal{T}$.

(2) Let $X \in \mathscr{F}$; since C is a cogenerator, there exists a set J and a monomorphism $T_F(X) \hookrightarrow C^J$, and hence a monomorphism

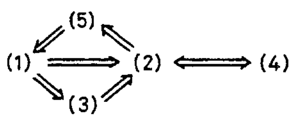
$$X \hookrightarrow S_F T_F(X) \hookrightarrow S_F(C^J) \xrightarrow{\sim} S_F(C)^J.$$

Since $S_F(C) \in \mathscr{F}$, it follows that $S_F(C)$ is an F-cogenerator.

3.2. Theorem. *Let $\mathscr{C}$ be a nonzero commutative Grothendieck category. Then the following statements are equivalent:*

(1) *$\mathscr{C}$ has a Noetherian cogenerator.*
(2) *$\mathscr{C}$ has an Artinian generator.*
(3) *$\mathscr{C}$ has an object of finite length which is simultaneous a generator and a cogenerator of $\mathscr{C}$.*
(4) *If G is a generator of $\mathscr{C}$ having the ring of endomorphisms commutative, then G is Artinian.*
(5) *$\mathscr{C}$ is equivalent to* Mod-A, *where A is a (nonzero) commutative Artinian ring with unit.*

Proof. The sketch of the proof is the following:



(1) ⇒ (2). We can suppose that $\mathscr{C} = \text{Mod-}R/\mathscr{T}$, where $(\mathscr{T}, \mathscr{F})$ is the hereditary torsion theory associated to a Gabriel topology F on the commutative ring R. If C is a Noetherian cogenerator of $\mathscr{C}$, then $S_F(C)$ is an F-Noetherian F-cogenerator R-module by 3.1, hence R is an F-Artinian ring by 2.4. Then clearly $T_F(R)$ is an Artinian generator in Mod-$R/\mathscr{T}$.

(2) ⇒ (5). We can also suppose that $\mathscr{C} = \text{Mod-}R/\mathscr{T}$, where R is a commutative ring, F a Gabriel topology on R, and $(\mathscr{T}, \mathscr{F})$ is the associated torsion theory. Let G be an Artinian generator of $\mathscr{C}$. By [14; 1.3], G is an object of finite length. Let

$$0 = G_0 \subset G_1 \subset \cdots \subset G_n = G$$

be a composition chain of G. Let S be an arbitrary simple object of $\mathscr{C}$. Since G is a generator of $\mathscr{C}$, there exists an epimorphism $f\colon G \to S$. Let $G' = \text{Ker}(f)$; then the chain $0 \subset G' \subset G$ of subobjects of G can be refined to a composition chain. It follows that there exists $1 \leqq k \leqq n$ such that $S \cong G/G' \cong G_k/G_{k-1}$. Therefore $\text{Sim}(\mathscr{C})$ is a finite non-empty set, and then, by [2; 3.6], $\text{Max}_F(R)$ is a finite non-empty set.

By [2; 2.4], $\text{Max}_F(R) \subseteqq \text{Spec}_F(R)$; conversely, let $p \in \text{Spec}_F(R)$. Since G is an Artinian generator of $\mathscr{C}$, $\mathscr{C}$ is a semi-Artinian category, hence $T_F(R/p)$ is a Loewy object of $\mathscr{C}$. Therefore

$$\{p\} = \text{Ass}(S_F T_F(R/p)) \subseteqq \text{Max}_F(R)$$

by [2; 3.7]. Thus $\text{Spec}_F(R) = \text{Max}_F(R)$.

By [3; 4.2], F is a semi-prime Gabriel topology, hence

$$F = \bigcap_{p \in \text{Max}_F(R)} F_p.$$

For each $p \in \mathrm{Max}_F(R)$ let us denote by $(\mathscr{T}_p, \mathscr{F}_p)$ the torsion theory defined by F_p. Then, for each $p \in \mathrm{Max}_F(R)$, $\mathscr{T} \subseteqq \mathscr{T}_p$, hence

$$\text{Mod-}R_p \cong \text{Mod-}R/\mathscr{T}_p \cong (\text{Mod-}R/\mathscr{T})/(\mathscr{T}_p/\mathscr{T}).$$

It follows that Mod-R_p has an Artinian generator, and then R_p is an Artinian ring, i.e. R is F_p-Artinian for each $p \in \mathrm{Max}_F(R)$. Then clearly R is F-Artinian, and moreover,

$$\mathscr{C} = \text{Mod-}R/\mathscr{T} \cong \text{Mod-}U^{-1}R,$$

where U is the multiplicatively closed subset

$$R \setminus \Big(\bigcup_{p \in \mathrm{Max}_F(R)} p \Big)$$

of R [3; 4.5]. The ring $A = U^{-1}R$ is surely Artinian.

(5) $\Rightarrow$ (1) is well-known. (See e.g. [17; Theorem 5].)

(1) $\Rightarrow$ (3). Let C be a Noetherian cogenerator of $\mathscr{C}$. By (1) $\Rightarrow$ (2), $\mathscr{C}$ has also an Artinian generator G. Then $C \oplus G$ is the desired object.

(3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (4). Let G be a generator of $\mathscr{C}$ such that $R = \mathrm{Hom}_{\mathscr{C}}(G, G)$ is a commutative ring. Via Gabriel-Popescu theorem, we may assume that $\mathscr{C} = \text{Mod-}R/\mathscr{T}$, where $(\mathscr{T}, \mathscr{F})$ is the hereditary torsion theory associated to some Gabriel topology F on R, and $G = T_F(R)$. By the proof of (2) $\Rightarrow$ (5), R is F-Artinian, i.e. G is an Artinian object of $\mathscr{C}$.

(4) $\Rightarrow$ (2) since $\mathscr{C}$ has at least a generator having the ring of endomorphisms commutative.

The proof is now complete.

3.3. Corollary. *Let R be a commutative ring, F a Gabriel topology on R, and $(\mathscr{T}, \mathscr{F})$ the associated torsion theory. Then, R is F-Artinian if and only if* Mod-$R/\mathscr{T}$ *has an Artinian generator.*

3.4. Corollary. *Let $\mathscr{C}$ be a commutative Grothendieck category. If G_1 and G_2 are two generators of $\mathscr{C}$ having both the rings of endomorphisms commutative, then G_1 is Artinian if and only if G_2 is Artinian.*

3.5. Corollary (Hopkins-Levitzki dual). *If $\mathscr{C}$ is a commutative Grothendieck category, then each Noetherian cogenerator of $\mathscr{C}$ is Artinian.*

4. The noncommutative case. The previous results, established for a commutative Grothendieck category are far to be true for an arbitrary Grothendieck category. Thus, even for a noncommutative ring R, the dual of the Hopkins-Levitzki theorem (3.5) does not hold in general in Mod-R. To see this, let k be a universal differential field of characteristic zero with derivation D; the ring $R = k[y, D]$ of differential polynomials over k in the derivation D is among others a principal right ideal domain, is not a field, and the category Mod-R of unital right R-modules has a simple injective cogenerator S [5]. Then $C = R \oplus S$ is a Noetherian generator and cogenerator in Mod-R, which is clearly not Artinian. However, the minimal cogenerator S of Mod-R, which is Noetherian, is also Artinian.

4.1. Proposition. *Let R be a noncommutative ring with unit element. If the category* Mod-R *has a Noetherian cogenerator, then the Jacobson radical J of R is nilpotent.*

Proof. Let C be a Noetherian cogenerator of the category Mod-R.

It is well known that the mappings

$$X \mapsto l_C(X),$$
$$Y \mapsto r_R(Y)$$

define a Galois connection between the lattice of two-sided ideals of R and the lattice of R-submodules of C. Since C is a cogenerator of Mod-R, each two-sided ideal of R is a closed element, hence the lattice of two-sided ideals of R is an Artinian lattice (because C is a Noetherian R-module). It follows that if we consider the descending chain of two-sided ideals of R

$$J \supseteqq J^2 \supseteqq J^3 \supseteqq \cdots \supseteqq J^m \supseteqq \cdots,$$

we must have $J^k = J^{k+1}$ for some k.

Hence

$$C J^k = C J^{k+1} = (C J^k) J.$$

By Nakayama's lemma, $CJ^k = 0$, in other word $J^k \subseteqq r_R(C)$. But $r_R(l_C(X)) = X$ for each two-sided ideal X of R, by Galois connection; hence $r_R(l_C(0)) = r_R(C) = 0$, and so $J^k = 0$.

5. Some problems. 1. It is known that each injective Noetherian module over a commutative ring with unit element is Artinian [7; 3.3] or [14; 4.2]. Does this result extend to a commutative Grothendieck category $\mathscr{C}$, i.e. are injective Noetherian objects of $\mathscr{C}$ Artinian? Equivalently, this problem can be formulated in relative terms: let F be a Gabriel topology on the commutative ring R, $(\mathscr{T}, \mathscr{F})$ the associated torsion theory on Mod-R, and $Q \in \mathscr{F}$ an injective F-noetherian R-module; is Q an F-Artinian module? An intermediate result was established in 2.6: such a Q is necessarily Δ-injective. Let us mention that R. Miller and D. Turnidge [13] have produced an example of an injective Noetherian R-module which is not Artinian (R is of course noncommutative!).

2. Let $\mathscr{C}$ be a Grothendieck category having a Noetherian cogenerator; then the minimal injective cogenerator $\bigoplus_{S \in \mathrm{Sim}(\mathscr{C})} E(S) = C_0$ is also Noetherian. Is C_0 Artinian? This problem, which is an weak form of 3.5, has an affirmative answer in the following particular case: $\mathscr{C} = $ Mod-R, where R is a right Noetherian, fully right bounded, and right classical ring. Indeed, each localizing subcategory of Mod-R is stable under taking injective hulls by [19; chap. VII, 3.4 and 4.4], hence C_0 is a Noetherian Loewy R-module. It follows that C_0 is an Artinian R-module.

3. It would be interesting to determine the Grothendieck categories $\mathscr{C}$ for which $\bigoplus_{S \in \mathrm{Sim}(\mathscr{C})} E(S)$ is a cogenerator.

4. Does the results from the section 3 of this paper extend to fully bounded Noetherian Grothendieck categories [21]?

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Eingegangen am 10. 12. 1979

Added in proof (*June* 1980): (1) I have recently learned that my Proposition 4.1 was obtained earlier by R. W. Miller and D. R. Turnidge in their paper "Co-Artinian rings and Morita duality", Israel J. Math. **15**, 12–26 (1973).

(2) C. Năstăsescu has recently obtained an affirmative answer to my Problem 1: any injective Noetherian object of a commutative Grothendieck category is Artinian.

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