On the complexity of d-dimensional Voronoi diagrams

By

VICTOR KLEE

Introduction. For n points $p_1, \ldots, p_n$ of Euclidean d-space E^d, the associated *Voronoi diagram* $V(p_1, \ldots, p_n)$ is a sequence $(P_1, \ldots, P_n)$ of convex polyhedra covering E^d, where P_i consists of all points of E^d that have p_i as a nearest point in the set $\{p_1, \ldots, p_n\}$. Thus

$$P_i = \{x \in E^d : \|x - p_i\| \leqq \|x - p_j\| \text{ for all } j\} = \bigcap_{j \neq i} H_{ij},$$

where

$$H_{ij} = \{x \in E^d : \langle p_j - p_i, x\rangle \leqq \tfrac{1}{2}(\|p_j\|^2 - \|p_i\|^2)\}.$$

Note that H_{ij} is the closed halfspace which contains p_i and whose bounding hyperplane passes through the midpoint of the segment $[p_i, p_j]$ and is perpendicular to that segment.

For $0 \leqq k < d$, let $\Phi_k(p_1, \ldots, p_n)$ denote the number of sets S such that S is a k-dimensional face of at least one of the polyhedra P_i. Then $\Phi_k(p_1, \ldots, p_n)$ is a natural measure of the complexity of the diagram, and the cases $k = 0$ and $k = d - 1$ are of special interest. Let $M_k(d, n)$ denote the maximum of $\Phi_k(p_1, \ldots, p_n)$ as $(p_1, \ldots p_n)$ ranges over all n-tuples of distinct points of E^d. A routine application of Euler's theorem shows

$$M_0(2, n) = 2n - 5 \quad \text{and} \quad M_1(2, n) = 3n - 6 \quad \text{for all} \quad n > 2.$$

Here it is proved that

$$(1) \qquad M_{d-1}(d, n) = \binom{n}{2} \qquad \text{for} \quad d > 3, \quad \text{all } n,$$

$$(2) \qquad 1 \leqq \liminf_{n\to\infty} \frac{M_0(d,n)}{n^r/r!} \leqq \limsup_{n\to\infty} \frac{M_0(d,n)}{n^r/r!} \leqq 2 \quad \text{for even} \quad d = 2r,$$

$$(3) \qquad \frac{1}{re} < \liminf_{n\to\infty} \frac{M_0(d,n)}{n^r/r!} \leqq \limsup_{n\to\infty} \frac{M_0(d,n)}{n^r/r!} \leqq 1 \quad \text{for odd} \quad d = 2r - 1.$$

Our method can also be used to obtain inequalities for the other M_k's.

Theorems. Not suprisingly, all our proofs are based on properties of neighborly polytopes. A d-polytope (that is, a bounded d-dimensional convex polyhedron) is said to be *neighborly* if each set of $\lfloor d/2 \rfloor$ of its vertices is the vertex-set of a face. This implies that for $1 \leqq j \leqq \lfloor d/2 \rfloor$, each set of j vertices is the vertex set of a $(j-1)$-face. For discussions and constructions of neighborly polytopes, see Gale [4] and Grünbaum [5]. (We use $\lfloor x \rfloor$ and $\lceil x \rceil$ respectively for the largest integer $\leqq x$ and the smalest integer $\geqq x$.)

Theorem 1. *If* $1 \leqq j \leqq \lfloor d/2 \rfloor$ *then* $M_{d-j+1}(d, n) = \binom{n}{j}$ *for all* n.

Proof. The cases in which $n \leqq d+1$ are left to the reader. With $n \geqq d+2$, let $w_1, \ldots, w_{n-1}$ be the vertices of a neighborly d-polytope Q in E^d such that the origin is interior to Q. Then for $1 \leqq j \leqq \lfloor d/2 \rfloor$, each j facets ($(d-1)$-faces) of the polar polytope

$$Q^0 = \{x \in E^d : \langle w_i, x \rangle \leqq 1 \quad \text{for} \quad 1 \leqq i < n\}$$

intersect in a $(d-j)$-face of Q^0. For $1 \leqq i < n$, let $p_i = 2w_i/\|w_i\|^2$. Then for each $x \in E^d$,

$$\langle w_i, x \rangle = 1 \Leftrightarrow \langle p_i, x \rangle = \tfrac{1}{2}\|p_i\|^2 .$$

Let $p_n = 0$ and $(P_1, \ldots, P_n) = V(p_1, \ldots, p_n)$. Since the affine hulls of Q^0's facets are the sets of the form $\{x : \langle w_i, x \rangle = 1\}$ for $1 \leqq i \leqq n$, it follows that $P_n = Q^0$ and for $1 \leqq i < n$ the intersection $P_i \cap P_n$ is a facet F_i of P_n.

Let $\mathfrak{I}$ ⟨resp. $\mathfrak{I}'$⟩ consist of all j-sets $I \subset \{1, \ldots, n\}$ such that $n \in I$ ⟨resp. $n \notin I$⟩, and for each $\mathfrak{I} \in \mathfrak{I} \cup \mathscr{I}'$ let $G_I = \bigcap_{i \in I} P_i$. If $I \in \mathfrak{I}$ then $G_I = \bigcap_{i \in I \sim \{n\}} F_i$, *a* $(d-j+1)$-face of P_n. If $I \in \mathfrak{I}'$ then $G_I \cap P_n$ is a $(d-j)$-face of P_n and for each $i \in I$ is the intersection with P_n of a $(d-j+1)$-face of P_i. Since different members of $\mathfrak{I}$ ⟨resp. $\mathfrak{I}'$⟩ give rise to distinct sets G_I ⟨resp. $G_I \cap P_n$⟩, the stated conclusion follows. □

A polytope is *simplicial* if all its facets are simplices. It is known [4] that all neighborly d-polytopes are simplicial when d is even, and [5] that the number of facets of a simplicial neighborly d-polytope with n vertices is

$$\gamma(d,n) = \binom{n - \lfloor (d+1)/2 \rfloor}{n-d} + \binom{n - \lfloor (d+2)/2 \rfloor}{n-d}.$$

McMullen [7] proved that $\gamma(d, n)$ is the maximum number of facets of d-polytopes with n vertices and hence, dually, of vertices of d-polytopes with n facets.

A d-polyhedron is *simple* if it has at least one vertex and each of its vertices is incident to precisely d edges or, equivalently, to precisely d facets ($(d-1)$-faces). A d-dimensional Voronoi diagram $V(p_1, \ldots, p_n)$ is *simple* if it has at least one vertex and each vertex is incident to precisely $d+1$ of the P_i's; this implies that all the P_i's are simple.

Theorem 2. *If* $p_1, \ldots, p_n$ *are distinct points of* E^d *such that the Voronoi diagram* $V(p_1, \ldots, p_n) = (P_1, \ldots, P_n)$ *is simple and* u *of the* P_i*'s are unbounded, then*

$$\Phi_0(p_1, \ldots, p_n) \leqq \gamma(d+1, n) + d - u .$$

Proof. The assertion is obvious when $d = 2$, so we assume $d > 2$. A theorem of Davis [2] then guarantees the existence of a real-valued convex function f on E^d such that each P_i is a set X which is maximal with respect to there being an affine function on E_d that agrees with f on X. The epigraph $\{(x, \tau) : \tau \geqq f(x)\}$ is a simple $(d+1)$-polyhedron that has precisely n facets, u of which are unbounded. It then follows from an extension [6] of McMullen's theorem that the number of vertices of the epigraph, and hence of $V(p_1, \ldots, p_n)$, is at most $\gamma(d+1, n) + d - u$. □

Theorem 3. *If $n > d+1$ then $\gamma(d, n-1) \leqq M_0(d, n) < \gamma(d+1, n)$.*

Proof. To establish the lower bound, carry out the construction of Theorem 1 with a neighborly polytope Q that is simplicial. Then Q has $\gamma(d, n-1)$ facets, so $\gamma(d, n-1)$ is also the number of vertices of the polar polytope $Q^0 = P_n$.

For the upper bound, note that whenever $p_1, \ldots, p_n$ are points of E^d (with $n > d$), they can be perturbed slightly so that the diagram $V(p_1, \ldots, p_n)$ becomes simple and its number of vertices does not decrease. (A formal proof can be based on a semicontinuity theorem of [3].) Then use Theorem 2, noting that the number of unbounded P_i's must exceed d. □

Note that

$$\gamma(d, n) = \frac{n}{n-r}\binom{n-r}{r} \text{ for even } \quad d = 2r$$

and

$$\gamma(d, n) = 2\binom{n-r}{r-1} \qquad \text{for odd} \quad d = 2r-1.$$

Thus Theorem 3 yields the following corollary, which in turn implies (2).

Corollary 1. *For even $d = 2r$ and for $n > d+1$,*

$$\frac{n-1}{n-1-r}\binom{n-1-r}{r} \leqq M_0(d, n) < 2\binom{n-1-r}{r}.$$

To establish (3) we use an idea of Preparata [8] in conjunction with some special neighborly polytopes.

Theorem 4. *If d is odd, $s > d$ and $t \geqq 1$, then $M_0(d, s+t) \geqq t\gamma(d-1, s-1)$.*

Proof. Let $d = 2r+1$, and for each angle θ let

$$x(\theta) = (\sin\theta, \cos\theta, \sin 2\theta, \cos 2\theta, \ldots, \sin r\theta, \cos r\theta) \in E^{2r}.$$

Let

$$C_r = \{x(\theta) : 0 \leqq \theta \leqq 2\pi\},$$

a simple closed curve on the sphere in E^{2r} that is centered at 0 and has radius $\sqrt{r}$. This curve was studied by Carathéodory [1] and also by Gale [4], who observed that the convex hull con X is a neighborly $(2r)$-polytope for each finite set X of more than $2r$ points of C_r. Grünbaum [5] noted this is easily proved with the aid of Scott's identity [11] asserting that if $\delta(\theta_1, \ldots, \theta_d)$ is the determinant of the matrix whose ith row consists of a 1 followed by $x(\theta_i)$, then

$$\delta(\theta_1, \ldots, \theta_d) = 2^{2r^2} \prod_{1 \leqq i < j \leqq d} \sin \tfrac{1}{2}(\theta_j - \theta_i).$$

For $1 \leqq i \leqq d$, let $\alpha_i = 2\pi(i-1)/d$ and $w_i = x(\alpha_i)$. From neighborlines and a remark of Gale [4], and also from Scott's identity, it follows that the convex hull of the w_i's is a $(2r)$-simplex. Since $\sum_1^d w_1 = 0$, the origin is interior to the simplex. For the given $s > d$, let $w_{d+1}, \ldots, w_s$ be distinct points of $C_r \sim \{w_1, \ldots, w_d\}$. For $1 \leqq i < s$, let $p_i = 2w_i/\|w_i\|^2$, so that $\|p_i\| = 2/\sqrt{r}$, and let $p_s = 0$. With

$$(P_1, \ldots, P_s) = V(p_1, \ldots, p_s),$$

P_s is the polar of the neighborly $(2r)$-polytope con $\{w_1, \ldots, w_{s-1}\}$ and hence P_s has $\gamma(d-1, s-1)$ vertices. Let $q_i = (\sqrt{r}/2)\, p_i$ for $1 \leqq i \leqq s$, so that $q_s = 0$, $\|q_i\| = 1$ for $1 \leqq i < s$, and the polytope

$$K = \{x \in E^{2r} : \|x\| \leqq \|x - q_i\| \text{ for } 1 \leqq i < s\}$$

is equal to $(\sqrt{r}/2)\, P_s$.

Now let E^{2r} be embedded in E^d as a hyperplane through the origin, having a line Rz with $\|z\| = 1$ as orthogonal supplement. For $1 \leqq i \leqq t$ let $q_{s+1} = 2iz$. Let $(Q_1, \ldots, Q_{s+t})$ denote the Voronoi diagram $V(q_1, \ldots, q_{s+t})$. We prove

$$M_0(d, s+t) \geqq t\gamma(d-1, s-1)$$

by showing for $1 \leqq i \leqq t$ that Q_{s+i} has $\gamma(d-1, s-1)$ vertices in the hyperplane

$$J_i = E^{2r} + (2i-1)Z.$$

All points of J_i are equidistant from q_{s+i-1} and q_{s+i}, and are closer to these than to any other point of the set $\{q_s, \ldots, q_{s+t}\}$. The point $(2i-1)_z$ is closer to q_{s+i-1} and to q_{s+i} than to any other point of the set $\{q_1, \ldots, q_{s+t}\}$. Thus J_i contains a facet F of Q_{s+i}, and in fact

$$F = \bigcap_{1 \leqq k < s} (H_k \cap J_i)$$

where

$$H_k = \{x \in E^d : \|x - q_{s+i}\| \leqq \|x - q_k\|\}.$$

To see that F has the same number of vertices as K, note that there is a point $-\mu z$ such that F is the intersection with J_i of the convex cone formed by all rays that issue from $-\mu z$ and pass through points of K. The vertices of F are the intersections of J_i with the edges of the cone, and these in turn correspond to vertices of K. The existence of $-\mu z$, which can be deduced from the lemma below, depends on all the points $q_1, \ldots, q_{s-1}$ having the same norm, and that was the reason for the special choice of neighborly polytopes in this construction (Thus having $\|q_1\| = \cdots = \|q_{s-1}\|$ appears to be essential here, though having these norms $= 1$ is merely a computational convenience.)

Lemma. *Suppose E is a hyperplane through the origin in a Euclidean space, having a line Rz with $\|z\| = 1$ as orthogonal supplement. Suppose $q \in E$ with $\|q\| = 1$, and suppose $0 < \beta < \eta \leqq 2\beta$. Let*

$$\psi = \frac{\eta(2\beta - n) + 1}{2} \text{ and } \mu = \frac{\beta}{2\psi - 1} = \frac{\beta}{\eta(2\beta - \eta)}.$$

Then for each point x of the hyperplane $E + \beta z$, the following two conditions are equivalent:

(i) $\|x - \eta z\| \leqq \|x - q\|$;

(ii) *if x' is the point at which the segment $[-\mu z, x]$ intersects E, then $\|x'\| \leqq \|x' - q\|$.*

To prove the lemma, consider an arbitrary point $x \in E + \beta z$ — say $z = y + \beta z$ with $y \in E$. Consideration of similar triangles shows that $x' = \varepsilon y$ with $\varepsilon = \mu/(\mu + \beta) = 1/(2\psi)$. Using the facts that $\langle z, y\rangle = \langle z, q\rangle = 0$ and $\langle z, z\rangle = \langle q, q\rangle = 1$, both (i) and (ii) are seen to be equivalent to the inequality $\langle q, y\rangle \leqq \psi$. That settles the lemma and completes the proof of Theorem 4. □

Corollary 2. *For odd $d = 2r - 1$ and for $n > d + 1$,*

$$\frac{n - r - 1}{r + 1}\,\frac{nr - r - 1}{nr - r^2 + 1}\binom{\lceil nr/(r+1)\rceil - r}{r - 1} < M_0(d, n) < \frac{n}{n - r}\binom{n - r}{r}.$$

Proof. Use Theorem 3 for the upper bound. For the lower bound, apply Theorem 4 with $s = \lceil nr/(r + 1)\rceil$ and $t = \lfloor n/(r + 1\rfloor$, obtaining

$$M_0(d, n) \geqq t\gamma(2r - 2, s - 1) = t\,\frac{s - 1}{s - r}\binom{s - r}{r - 1}$$

and hence the stated lower bound. From the latter it follows that

$$\liminf_{n \to \infty} M_0(d, n) \geqq \frac{1}{r}\left(\frac{r}{r + 1}\right)^r n^r > \frac{1}{re}\, n^r$$

thus settling (3). □

Comments. For applications of Voronoi diagrams to problems of packing and covering in E^d, and for references to the earlier literature, see Rogers [10]. In recent years, Voronoi diagrams in E^2 have been of interest because of their use by Shamos [12] and Shamos and Hoey [13] in providing efficient algorithms for a number of computational problems. For n points $p_1, \ldots, p_n$ of E^2, the diagram $(P_1, \ldots, P_n)$ can be computed in time $O(n \log n)$, each P_i being output as its sequence of successive vertices. The same computation in E^d would in worst cases require time $\Omega(n^{\lceil d/2\rceil})$ because of the possible number of vertices. However, it is unknown whether, in time bounded by some polynomial in d and n, one can compute the facets of the P_i's. For input $p_1, \ldots, p_n \in E^d$, the output would consist of n subsets $I_1, \ldots, I_n$ of $\{1, \ldots, n\}$ such that $i \in I_j$ if and only if the hyperplane $\{x : \|x - p_i\| = \|x - p_j\|\}$ contains a facet of P_j. By results of Reiss and Dobkin [9], this can be accomplished in polynomial time if and only if linear programming problems with d variables and n constraints can be solved in polynomial time.

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Anschrift des Autors:

Victor Klee
Department of Mathematics
University of Washington
Seattle, Washington 98195