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Study of the Iterations of a Mapping Associated to a Spin Glass Model

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Abstract. We study the iterations of the mapping

$$\mathcal{N}[F(s)] = \frac{(F(s))^2 - (F(0))^2}{s} + (F(0))^2,$$

with the constraints $F(1)=1$, $F(s)=\sum a_n s^n$, $a_n \geqq 0$, and find that, except if $F(s) \equiv s$, $\mathcal{N}^k[F(s)]$ approaches either 0 or 1 for $|s|<1$ as $k \to \infty$.

I. Introduction and Summary of the Results

In a simplified version of a spin glass model [1] (CEGM), the probability distribution of the spin-spin interaction is given by a discrete set of coefficients a_n,

$$\sum_{n=0}^{\infty} a_n = 1, \qquad a_n \geqq 0, \tag{1}$$

and after the operation of the renormalization group, this distribution is replaced by a new one. The operation is best described by writing the equation which gives the new generating function of the probabilities, $\mathcal{N}F$, in terms of the old one, F:

$$\mathcal{N}[F(s)] = \frac{(F(s))^2 - (F(0))^2}{s} + (F(0))^2, \tag{2}$$

with $F(s) = \sum a_n s^n$. This mapping preserves conditions (1).

We want to study the iterations of (2) and find out what happens to

$$\mathcal{N}^k[F(s)] = \underbrace{\mathcal{N}[\mathcal{N}[\mathcal{N} \dots \mathcal{N}}_{k \text{ times}}[F(s)]]], \quad \text{for} \quad k \to \infty,$$

and see whether $\mathcal{N}^k F(s)$ approaches a limit or has a chaotic behavior. In the sequel, we use the abbreviation

$$\begin{aligned} \mathcal{N}^k[F(s)] &= F^{(k)}(s), \\ F^{(k)}(s) &= \sum a_n^{(k)} s^n. \end{aligned} \tag{3}$$

Let us summarize the results.

1) There are only two fixed points of the mapping: $F(s)=s$, which is unstable, and $F(s)=1$. A pseudofixed point is $F(s)=$ "s^∞", i.e., $F(s)=0$ for $0\leqq s<1$, and $F(s)=1$ for $s=1$. There is no periodic point, i.e., the equation

$$F^{(k)}(s)=F(s), \quad \forall s,$$

has no new solutions when $k>1$.

2) Only three things happen to the iterations for $k\to\infty$:

i) if $F(s)=s$, $F^{(k)}(s)=s$, $\forall k$;

ii) otherwise, either $F^{(k)}(s)\to 0$ pointwise for $0\leqq s<1$, and in fact for $|s|<1$ in the complex plane, or $F^{(k)}(s)\to 1$ for $|s|\leqq 2$.

We see thus that the attractors are "trivial". However, it will be seen that the approach to these attractors is complicated.

3) $F^{(k)}(s)\to 1$ for $k\to\infty$ if and only if the following conditions are simultaneously fulfilled:

i) $F(s)\neq s$,

ii) $F(s)$ is analytic in $|s|<2$,

iii) $\lim\limits_{s\to 2} F(s)$ exists and $\lim\limits_{s\to 2} F'(2)$ exists,

iv) $F(2)-2F'(2)\geqq 0$.

4) Under conditions 3), $F^{(k)}(s)$ approaches unity in the following way:

i) if $F(2)-2F'(2)>0$ strictly, $\sum|F^{(k)}(s)-1|$ converges for $|s|\leqq 2$,

ii) if $F(2)-2F'(2)=0, \sum|F^{(k)}(s)-1|$ converges for $|s|<2$, while $\sum|F^{(k)}(2)-1|^\gamma$ converges for any $\gamma>1$. Furthermore, $\liminf k(F^{(k)}(2)-1)\leqq 2$.

5) If $F(s)\neq s$ and if any one of suppress conditions 3) is not satisfied, then $F^{(k)}(s)\to 0$ for $|s|<1$.

6) Finally, let us indicate that similar results, using similar methods, can be obtained for the mapping

$$\left.\begin{aligned} &\mathcal{N}[F(s)]=\frac{(F(s))^n-(F(0))^n}{s}+(F(0))^n, \qquad n\in\mathbb{N}, \qquad n\geqq 3, \\ &F(s)=\sum a_n s^n, \qquad a_n\geqq 0, \qquad \sum a_n=1. \end{aligned}\right\} \tag{4}$$

There is no unstable fixed point. $F^{(k)}$ approaches 1 for $k\to\infty$ if F is analytic in $|s|<n$ and $D=F(n)-n(n-1)F'(n)>0$. In the limit case, $D=0$, we only know that $F^{(k)}$ does not approach zero. In the other cases, $F^{(k)}(s)$ approaches zero for $|s|<1$.

7) The mapping,

$$\mathcal{N}[F(s)]=\frac{(F(s))-(F(0))}{s}+F(0), \tag{5}$$

where again $F(s)=\sum a_n s^n$, $a_n\geqq 0$, $\sum a_n=1$, is such that $F^{(k)}(s)\to 1$ for $k\to\infty$, $|s|\leqq 1$.

In the sequel, we shall not discuss separately the problem of fixed points (though direct proofs exist!) because the absence of non-trivial fixed points follows from the rest of the study.

II. Necessary Conditions for $F^{(k)}(s)\nrightarrow 0$: Analyticity

First note that from condition (1), we see that if $F(s)$ is defined in $0\leqq s\leqq 1$ it can be extended to $|s|\leqq 1$.

Assume that $F(s_0)/s_0<1$ for $0<s_0<1$, then from (2) we get

$$\frac{F^{(k)}(s_0)}{s_0}<\left(\frac{F(s_0)}{s_0}\right)^{2^k}.$$

Hence, if $F(s_0)/s_0<1$, $F^{(k)}(s_0)\to 0$ as $k\to\infty$; but since, from the positivity properties (1) $|F(s)|\leqq F(s_0)$ for $|s|\leqq s_0$, we find also $F^{(k)}(s)\to 0$ for $k\to\infty$, $|s|\leqq s_0$. However, since $|F^{(k)}(s)|<1$ in $|s|<1$, Vitali's theorem tells us that $|F^{(k)}(s)|\to 0$ for $|s|<1$.

Hence, if $F^{(k)}(s)$ *does not* approach zero, we have $F(s)/s\geqq 1$ for $0\leqq s\leqq 1$. This implies the following necessary conditions:

$$F^{(k)\prime}(1)\leqq 1\,,\quad \text{for any } k\,. \tag{6}$$

Let us now look at the recursion relations for $a_n^{(k)}$ following from the definitions (2) and (3). They are

$$a_n^{(k+1)}=\sum_{p+q=n+1} a_p^{(k)}a_q^{(k)}+\delta_{n0}(a_0^{(k)})^2\,. \tag{7}$$

If

$$I_n^{(k)}=\sum_{p=n}^{\infty} a_p^{(k)}\quad (\text{notice } I_0^{(k)}=1)\,, \tag{8}$$

we get from (7)

$$I_n^{(k+1)}>\left[\sum_{p\leqq n,\,q\geqq n+1}+\sum_{q\leqq n,\,p\geqq n+1}\right]a_p^{(k)}a_q^{(k)}\,, \tag{9}$$

i.e.,

$$I_n^{(k+1)}\geqq 2I_{n+1}^{(k)}[1-I_{n+1}^{(k)}]\,. \tag{10}$$

Assume now that $F^{(k)}$ does not tend to zero for $k\to\infty$. Then from (6), we get $\sum na_n^{(k)}\leqq 1$, and hence

$$nI_n^{(k)}<\sum_{p=n}^{\infty} pa_p^{(k)}\leqq\sum_{p=0}^{\infty} pa_p^{(k)}\leqq 1\,,$$

i.e.,

$$I_n^{(k)}<\frac{1}{n}, \tag{11}$$

and, by inserting in the bracket in (10):

$$\frac{I_{n+1}^{(k)}}{I_n^{(k+1)}}<\frac{1}{2}\frac{n+1}{n}\,.$$

Iterating this inequality, one gets, using $I_q^{(p)}\leqq 1$,

$$a_n^{(k)}<I_n^{(k)}<2n\times 2^{-n}\,. \tag{12}$$

Hence $F^{(k)}(s)=\sum a_n^{(k)}s^n$ is analytic in $|s|<2$. Furthermore, using the explicit bound (12), we get a bound on $F^{(k)}$ which is *independent of* k:

$$|F^{(k)}(s)|<1+|s|+2|s|\frac{1}{\left(1-\frac{|s|}{2}\right)^2}. \tag{13}$$

III. The Condition $F(2)-2F'(2)\geqq 0$

Consider the quantity

$$D^{(k)}(s)=F^{(k)}(s)-sF^{(k)\prime}(s). \tag{14}$$

By differentiating (2), we get

$$D^{(k+1)}(s)=\frac{2F^{(k)}(s)}{s}D^{(k)}(s)-\frac{2-s}{s}(F^{(k)}(0))^2. \tag{15}$$

Assume that$F^{(k)}(s)\not\to 0$ for $k\to\infty$. Then the $F^{(k)}$'s are bounded by (13). Take $1<s_0<2$, and assume $D^{(k)}(s_0)<0$. Then from (15) we get $D^{(k+1)}(s_0)<0$ and

$$\left|\frac{D^{(k+1)}(s_0)}{D^{(k)}(s_0)}\right|>2\frac{F^{(k)}(s_0)}{s_0}>\frac{2}{s_0}>1,$$

since $F^{(k)}(s_0)>1$ for $s_0>1$. Therefore, if $D^{(0)}(s)=F(s)-sF'(s)$ is negative for one particular s_0, $1\leqq s_0<2$, then

$$|D^{(k)}(s_0)|>|D^{(0)}(s_0)|\left(\frac{2}{s_0}\right)^{k-1}.$$

However, $F^{(k)}(s)$ is bounded by (13) and $(2-|s|)F'^{(k)}(s)$ is also bounded by (13). For k large enough, this is a contradiction. Hence a *necessary* condition for $F^{(k)}(s)$ not to approach 0 in $0<s<1$ is

$$D^{(0)}(s)=F(s)-sF'(s)\geqq 0, \quad \forall s, \quad 1\leqq s<2. \tag{16}$$

If (16) holds, we can integrate (16) from 1 to s and get

$$\left.\begin{array}{l}\frac{F(s)}{s}\leqq 1\\ F'(s)\leqq 1\end{array}\right\}, \quad 1\leqq s<2. \tag{17}$$

This means that $F(s)$ and $F'(s)$ which are increasing functions of s have limits for $s=2$ that we designate as $F(2)$ and $F'(2)$. Hence we have shown the

Theorem. *If* $F^{(k)}(s)$ *does not approach* 0 *in* $0<s<1$, *then*

$$D^{(0)}(2)\equiv F(2)-2F'(2)\geqq 0. \tag{18}$$

From the positivity of the a_n's, it is easy to see that (18) implies (16).
Let us now assume

$$D^{(0)}(2)>0.$$

We can take the limit of Eq. (15):

$$D^{(k+1)}(2)=F^{(k)}(2)D^{(k)}(2)\,, \tag{19}$$

and

$$D^{(k)}(2)=F^{(k-1)}(2)F^{(k-2)}(2)\ldots F(2)D^{(0)}(2)\,. \tag{20}$$

But $D^{(k)}(s)$ is a decreasing function of s, so

$$D^{(k)}(2)\leqq F^{(k)}(0)\equiv a_0^{(k)}\leqq 1\,,$$

and

$$1<\prod_{k=0}^{N} F^{(k)}(2)<\frac{1}{D^{(0)}(2)}\,.$$

We see that the infinite product $\prod_0^\infty F^{(k)}(2)$ is *convergent*. So $F^{(k)}(2)\to 1$ for $k\to\infty$ and, since

$$1\leqq F^{(k)}(s)\leqq F^{(k)}(2)\,,\qquad 1\leqq s\leqq 2\,,$$

$$2-F^{(k)}(2)\leqq F^{(k)}(s)\leqq 1\,,\qquad 0\leqq s\leqq 1\,,$$

(from convexity)

$$F^{(k)}(s)\to 1\,,\qquad 0\leqq s\leqq 2\,.$$

Furthermore, $\forall |s|\leqq 2$ the infinite product $\prod_{k=0}^{\infty} F^{(k)}(s)$ converges. Therefore, we have the

Theorem. *If* $F(2)-2F'(2)>0$, *then* $F^{(N)}(s)\to 1$ *as* $N\to\infty$, *when* $|s|\leqq 2$, *and the infinite product* $\prod_{N=0}^{\infty} F^{(N)}(s)$ *converges for* $|s|\leqq 2$.

This implies, of course, that $\sum_{N=0}^{\infty} (F^{(N)}(2)-1)$ converges.

IV. The Limit Case $F(2)=2F'(2)$

If

$$D^{(0)}(2)=0\,, \tag{21}$$

we get from (19), $D^{(k)}(2)=0$. The only easy result in this case is that $D^{(0)}(2)=0$ is a *sufficient* condition to guarantee that $F^{(k)}(s)$ does not approach 0 in $0<s<1$. Indeed, we still have $F^{(k)}(s)\leqq s$ for $1<s<2$, and by convexity $F^{(k)}(s)\geqq s$ for $0\leqq s<1$. Let us show that if $F(0)>0$, one can, in fact, get a lower bound on $F^{(k)}(0)$ for k large enough. Replacing F and F' by their power expansion, one gets from (21)

$$a_0=\sum_{n=2}^{\infty} (n-1)2^n a_n\,, \tag{22}$$

and hence $4\sum_{2}^{\infty} a_n < a_0$. Combining with $\sum_{0}^{\infty} a_n = 1$, one gets

$$a_1 > \left[1 - \frac{5a_0}{4}\right]_+, \tag{23}$$

where $[x]_+ = x$ for $x \geqq 0$, 0 for $x \leqq 0$. Hence

$$a_0^{(k+1)} > (a_0^{(k)})^2 + 2a_0^{(k)}\left[1 - \frac{5a_0^{(k)}}{4}\right]_+. \tag{24}$$

The graph corresponding to this inequality is presented in Fig. 1:

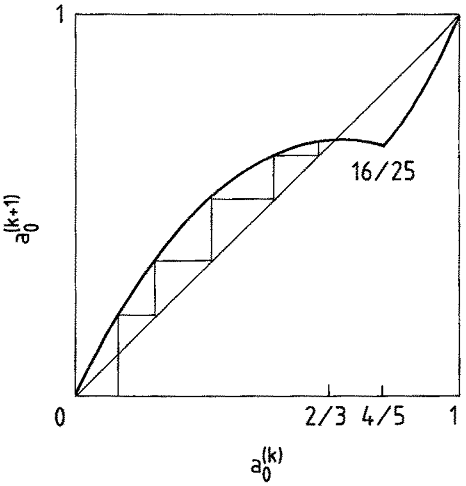

Fig. 1

It is easy to see that after a finite number of iterations, one gets

$$a_0^{(k)} \geqq \tfrac{16}{25}. \tag{25}$$

One can generalize this technique to get more refined lower bounds, but we shall prefer to use a completely different method.

The trouble with Eq. (15), in the case $D(2) = 0$, is that it reduces to $0 = 0$. Differentiating (15) leads to an equation on the second derivatives of $F^{(k)}(2)$ and $F^{(k+1)}(2)$ which do not necessarily exist. An interesting quantity to consider is

$$\Delta = F - sF' - \frac{s^2(2-s)}{2}F''. \tag{26}$$

It is easy to see that Δ is monotonously decreasing. Hence

$$\Delta(s) \leqq F(0) \leqq 1. \tag{27}$$

We have also

$$\Delta(s) > (2-s)^2 F''(1) \quad \text{for} \quad s \geqq 1. \tag{28}$$

The iteration of Δ is given by:

$$\Delta^{(k+1)}(s) = \frac{2F^{(k)}(s)}{s}\Delta^{(k)}(s) - \frac{2-s}{s}(F^{(k)} - sF'^{(k)})^2. \tag{29}$$

In Appendix I we prove the inequality

$$(F - sF')^2 < 2F\Delta. \tag{30}$$

Therefore,

$$\Delta^{(k+1)}(s) > \frac{2(s-1)}{s} F^{(k)}(s)\Delta^{(k)}(s). \tag{31}$$

Iterating, we get

$$\Delta^{(N+1)}(s) > \left(\frac{2(s-1)}{s}\right)^{N+1} \prod_{k=0}^{N} F^{(k)}(s)\Delta(s), \tag{32}$$

and using inequalities (27) and (28) together with $F^{(k)}(s)/s > F^{(k)}(2)/2$, we get, assuming $F''(1) \neq 0$, i.e., excluding $F(s) \equiv s$,

$$\prod_{k=0}^{k=N} F^{(k)}(2) < \frac{1}{F''(1)(2-s)^2(s-1)^{N+1}}. \tag{33}$$

Optimizing with respect to s, we find with $s = 2 - 2/N$,

$$\prod_{k=0}^{k=N} F^{(k)}(2) < \frac{C_N N^2}{F''(1)}, \qquad N \geqq 1. \tag{34}$$

In the limit of large N, we have

$$C_N \to \frac{e^2}{4}. \tag{35}$$

If we define ε_N by

$$F^{(N)}(2) = 1 + \varepsilon_N, \tag{36}$$

we can rewrite (34) as

$$\sum_{k=0}^{N} \varepsilon_k \leqq 2\ln N + \text{const}. \tag{37}$$

This means that, in an *average* sense, ε_N goes to zero. However, we cannot exclude from (37) the existence of an infinite sequence of ε_N's not tending to zero. The only safe thing we can say is

$$\liminf_{N\to\infty} N\varepsilon_N \leqq 2. \tag{38}$$

To prove that ε_N goes to zero, we will use the fact that successive ε's are correlated. Specifically, from

$$2F^{(N+1)}(2) = (F^{(N)}(2))^2 + (F^{(N)}(0))^2 < (F^{(N)}(2))^2 + 1, \tag{39}$$

we get

$$\varepsilon_{N+1} - \varepsilon_N < \tfrac{1}{2}\varepsilon_N^2 . \tag{40}$$

In Appendix II we exploit this inequality to obtain

$$\varepsilon_{N-p} > \frac{\varepsilon_N}{1 + \frac{P\varepsilon_N}{2}} , \tag{41}$$

and

$$\prod_{k=M}^{N} F^{(k)}(2) > \frac{1}{q} (\varepsilon_N)^2 (N - M + 2)^2 . \tag{42}$$

If we combine the inequality (34) with the set of inequalities (42), we get

$$CN_p^2 > \prod_{k=1}^{p} \left(\frac{\varepsilon_{N_k}}{3}\right)^2 \times (N_1+1)^2 (N_2 - N_1 + 1)^2 \dots (N_p - N_{p-1} + 1)^2 . \tag{43}$$

In Appendix II we also show that this set of inequalities is very constraining and allows us to get an upper limit on $v(x)$, the number of ε_k^2 larger or equal to x:

$$v(x) < n^{\frac{n}{n-1}} \left(\frac{9}{x}\right)^{\frac{n}{2n-2}} + n , \tag{44}$$

for any integer $n \geqq 1$.

Noticing that $v(1) = 0$, if $F(s) \neq s$, we get the Stieltjes integral:

$$\sum_{k=0}^{\infty} (\varepsilon_k)^{2\alpha} = \int_0^1 x^\alpha [-dv(x)] = \int_0^1 \alpha x^{\alpha-1} v(x) dx . \tag{45}$$

For any $\alpha > \frac{1}{2}$ we can find an n such that from (44) we get a convergent upper bound to the last integral in (45). In particular, to prove that $\sum_{k=0}^{\infty} (\varepsilon_k)^2$ is convergent, it is sufficient to take $n = 3$. Hence we have shown

Theorem. $\sum (\varepsilon_k)^{2\alpha}$ *converges for* $\alpha > \frac{1}{2}$.

On the other hand, we have the

Theorem. *If* $D(2) = F(2) - 2F'(2) = 0$, $\sum \varepsilon_k$ *must diverge.*

We begin the proof by summing Eqs. (15) to get

$$D(s) = \frac{2-s}{s} \sum_{N=0}^{\infty} \frac{(F^{(N)}(0))^2}{\prod_N(s) \left(\frac{2}{s}\right)^{N+1}} , \tag{46}$$

where

$$\prod_N(s) = \prod_{k=0}^{N} F^{(k)}(s) . \tag{47}$$

Assume that $\sum\varepsilon_k$ converges, then $\prod_N(2)$ tends to a limit L for $N\to\infty$, and $\prod_N(s)<L$. We can also start after a finite number of iterations so that $F^{(N)}(0) > 16/25$, according to (25).

Then we get, summing the geometric series in (46):

$$D(s)>\frac{1}{L}\left(\frac{16}{25}\right)^2, \tag{48}$$

a contradiction.

The convergence of $\sum_0^\infty \varepsilon_k^2$ can now be used to prove the convergence of $\sum_{k=0}^\infty |1-F^{(k)}(s)|$, or equivalently $\prod_{k=0}^\infty F^{(k)}(s)$ for $|s|<2$.

From (39) we have

$$1-(F^{(N)}(0))^2=2(\varepsilon_N-\varepsilon_{N+1})+\varepsilon_N^2.$$

Hence

$$\sum_{k=0}^N (1-(F^{(k)}(0))^2)=2\varepsilon_0-2\varepsilon_{N+1}+\sum_0^N \varepsilon_k^2. \tag{49}$$

The right-hand side is bounded for arbitrary N and this therefore convergent. It is also obvious from the monotonicity of F that $\sum_{k=0}^\infty (1-(F^{(k)}(s))^2)$ is convergent for $0\leqq s\leqq 1$.

For $1\leqq s<2$ we have to use a more complicated argument. If we write $F^{(k)}(s) =1+\varepsilon_k(s)$, we get from the mapping equation

$$\varepsilon_{k+1}(s)>\frac{2}{s}\varepsilon_k(s)-\frac{s-1}{s}(1-(F^{(k)}(0))^2). \tag{50}$$

Introducing

$$x_k=\left(\frac{s}{2}\right)^k \varepsilon_k(s), \tag{51}$$

we notice that $x_k\to 0$ for $k\to\infty$, and we get

$$x_N<\frac{s-1}{s}\sum_{k=N}^\infty\left(\frac{s}{2}\right)^{k+1}[1-(F^{(k)}(0))^2]. \tag{52}$$

Thus we get a bound on $\varepsilon_k(s)$ and can establish the inequality

$$\begin{aligned}\sum_{k=N}^\infty \varepsilon_k(s)&<\frac{s-1}{2-s}\sum_{k=N}^\infty (1-(F^{(k)}(0))^2)\left(1-\left(\frac{s}{2}\right)^{k+1-N}\right)\\ &<\frac{s-1}{2-s}\sum_{k=N}^\infty (1-(F^{(k)}(0))^2).\end{aligned} \tag{53}$$

Therefore, the infinite product $\prod_{k=0}^\infty F^{(k)}(s)$ *converges* for $0\leqq s<2$ and this can be extended, using positivity, to $|s|<2$.

We have already said that when $D(2)=0$, the product $\prod_{k=0}^{\infty} F^{(k)}(2)$ *must diverge*. One could ask how fast. This depends on details of the initial $F(s)$.

If $F''(2)$ and $F'''(2)$ *exist* (as limits from $s<2$), one can obtain the equation

$$3F^{(N+1)\prime\prime}(2)+2F^{(N+1)\prime\prime\prime}(2)=F^{(N)}(2)[3F^{(N)\prime\prime}(2)+2F^{(N)\prime\prime\prime}(2)]. \tag{54}$$

By exploiting this equation, we prove in Appendix III that

$$C_1N^2<\textstyle\prod_N(2)<C_2N^2, \qquad C_1>0. \tag{55}$$

Then, assuming that the $F^{(N)}$'s are *sufficiently smooth* functions of N, one can obtain their asymptotic behaviour:

$$F^{(N)}(s)\cong 1+\frac{s-1}{N}\frac{8}{N(2-s)+4}, \tag{56}$$

where the *relative* error on $F^{(N)}-1$ is uniformly in s of the order of $1/N$. This is again explained in Appendix III.

If $F'''(2)$ or $F'''(2)$ and $F''(2)$ do not exist, the situation is more complex. If F'' exists but $F'''(s)\sim(2-s)^{-\alpha}$, $\prod_N(2)\sim N^{2-\alpha}$, if $F''\sim(2-s)^{-\alpha}$, $\prod_N(2)\sim N^{1-\alpha}$. However, this is far from covering all possibilities! F'' can be singular at $s=2$ without behaving like a definite power of $2-s$. This is described in Appendix IV.

V. Generalization of the Results to Other Mappings

First, for completeness, we treat the simple case

$$\mathcal{N}F(s)=\frac{F(s)-F(0)}{s}+F(0), \qquad F=\sum a_n s^n, \qquad a_n\geqq 0, \qquad \sum a_n=1. \tag{57}$$

If $\mathcal{N}^kF=F^{(k)}=\sum a_n^{(k)}s^n$, the recursion relation reduces to

$$a_n^{(k+1)}=a_{n+1}^{(k)}+\delta_{n,0}a_0^{(k)}, \tag{58}$$

and by iterating this relation from $k=0$, we find

$$a_0^{(k)}=\sum_{n=0}^{k} a_n\to 1 \quad \text{for} \quad k\to\infty. \tag{59}$$

Hence

$$F^{(k)}(s)\to 1 \quad \text{for} \quad k\to\infty, \qquad 0\leqq s\leqq 1.$$

Now, consider the mapping

$$\mathcal{N}F(s)=\frac{(F(s))^n-(F(0))^n}{s}+(F(0))^n, \qquad n\geqq 3. \tag{60}$$

Here things are very similar to the case $n=2$, except that there is no unstable fixed point analogue to $F(s)=s$. The results are as follows:

i) If $F(s)$ is analytic in $|s|<n$, and if $F(n)$ and $F'(n)$ exist, and if $F(n)-n(n-1)F'(n)>0$, the iterates $F^{(k)}(s)$ approach 1 for $k\to\infty$, $|s|\leqq n$.

ii) In the limit case $F(n)=n(n-1)F'(n)$, $F^{(k)}(s)$ remains finite and $F^{(k)}(0)$ is lower bounded by $1-(1/n-1)$, but we have not carried out a detailed analysis to see if $F^{(k)}$ approaches unity.

iii) Otherwise, $F^{(k)}(s)$ approaches zero for $|s|<1$.

The methods being essentially the same as in the case $n=2$, we feel that we only have to give a few indications. First, one shows that if $F(s_0)/s_0^{1/n-1}<1$ for a given $0<s_0<1$, necessarily $F^{(k)}(s)$ goes to zero for $k\to\infty$. Taking the limit $s_0\to 1$ one deduces that if $F^{(k)}$ does not approach zero,

$$F'^{(k)}(1)\leqq\frac{1}{n-1}, \tag{61}$$

which is the analogue of (6). Then, using the recursion relations

$$a_q^{(k+1)}=\sum_{r_1+r_2+\ldots+r_n=q+1} a_{r_1}^{(k)}a_{r_2}^{(k)}\ldots a_{r_n}^{(k)}+\delta_{q0}(a_0^{(k)})^n. \tag{62}$$

One proves the analyticity of $F^{(k)}$ inside $|s|<n$ and one obtains inside this domain a bound independent of k. Next, one introduces the analogue of (14):

$$D^{(k)}(s)=F^{(k)}(s)-(n-1)sF'^{(k)}(s), \tag{63}$$

which satisfies the recursion relation

$$D^{(k+1)}(s)=\frac{n(F^{(k)}(s))^{n-1}}{s}D^{(k)}(s)-\frac{n-s}{s}(F^{(k)}(0))^n. \tag{64}$$

One proves that $D^{(k)}$ has to be positive for $1<s<n$ and one deduces that if $F^{(k)}$ does not go to zero, then

$$D^{(k)}(n)\geqq 0, \tag{65}$$

and, then, if $D^{(k)}(n)\neq 0$, one proves that $\left[\prod_{k=0}^{N}F^{(k)}(n)\right]^{n-1}$ has an upper bound independent of N. This leads to the desired result. In the limiting case, one cannot exactly carbon-copy the reasonings because more terms appear when one differentiates equations more than once. Our guess is that nothing changes, but we leave it as an exercise for the reader.

VI. Concluding Remarks

Returning to the case $n=2$, we see that if we forget the origin of the problem and think of the equation as describing a dynamical system, we see that there is no room for a chaotic behaviour, irrespective of the choice of the initial F. In a naïve way, one could say that the behaviour $F^{(k)}\to 0$ for $0<s<1$ is infinitely more likely than $F^{(k)}\to 1$. However, we should remember that in this initial problem a_n represents the probability for the variable x to be in the interval $2^{-n}, 2^{-n-1}$. So if x has a bounded probability distribution near $x=0$, the analyticity of $F(s)$ in $|s|<2$ is automatic. What is not automatic is $D(2)\geqq 0$. $F^{(k)}\to 0$ corresponds to a free system in the limit. $F^{(k)}\to 1$ is more difficult to interpret since a_0 corresponds to a large slice $\frac{1}{2}<|x|<1$. This means that the interactions are either strongly ferromagnetic or strongly antiferromagnetic.

Appendix I

Proof of the inequality $(F-sF')^2<2F\Delta$. From $F(2)=2F'(2)$ we get

$$a_0=\sum_{n=2}^{\infty}(n-1)2^n a_n\,.$$

We have

$$F-sF'=\sum_{n=2}^{\infty}a_n(n-1)[2^n-s^n]\,,$$

$$F=\sum_{n=1}^{\infty}a_n[s^n+(n-1)2^n]\,,$$

$$\Delta=\sum_{n=2}^{\infty}a_n(n-1)\left[2^n-(n+1)s^n+\frac{ns^{n+1}}{2}\right].$$

To prove $(\sum u_i)^2<\sum v_i\sum w_i$, for $u_i, v_i, w_i>0$, it is sufficient to prove $u_i^2<v_iw_i$. If we call $x=s/2$ it is sufficient to prove

$$P(x)=2[1-(n+1)x^n+nx^{n+1}][n-1+x^n]-(n-1)(1-x^n)^2\geqq 0$$

for $0<x<1$, $n\geqq 2$. One has to study the roots of $P(x)$, which, fortunately, has at most four positive roots, since the polynomial has only five terms different from zero. For $n\geqq 6$ one proves that

$$\left(\frac{P'}{x^{n-1}}\right)''<0\,,$$

and, with the boundary conditions,

$$\left.\left(\frac{P'}{x^{n-1}}\right)'\right|_{x=1}>0\,,\qquad P'(1)=0\,,\qquad P(1)=0\,,$$

one succeeds to prove that F is positive. The cases $n=2,3,4,5$ are more delicate, but the positivity can still be established.

Appendix II

From

$$2F^{(N+1)}(2)\leqq(F^{(N)}(2))^2+1\,, \tag{AII.1}$$

we get, with

$$F^{(N)}(2)=1+\varepsilon_N\,, \tag{AII.2}$$

$$\varepsilon_{N+1}<\varepsilon_N+\tfrac{1}{2}\varepsilon_N^2\,, \tag{AII.3}$$

and hence

$$\varepsilon_{N+1}-\varepsilon_N<\tfrac{1}{2}\varepsilon_N\varepsilon_{N+1}\,. \tag{AII.4}$$

Indeed, either $\varepsilon_{N+1} < \varepsilon_N$ and the latter inequality is obvious, or $\varepsilon_{N+1} > \varepsilon_N$ and then $\varepsilon_{N+1}\varepsilon_N > \varepsilon_N^2$. From (AII.4) we get

$$\frac{1}{\varepsilon_N} - \frac{1}{\varepsilon_{N+1}} < \frac{1}{2}. \tag{AII.5}$$

By merely adding (AII.5) for successive N's, we get

$$\frac{1}{\varepsilon_{N-P}} - \frac{1}{\varepsilon_N} < \frac{P}{2}, \tag{AII.6}$$

and hence the inequality (41) follows. Now, we want to get a lower bound on $\prod_{k=N}^{M} F^{(k)}(2)$, i.e., on $\sum_{k=N}^{M} \log(1+\varepsilon_k)$. We use

$$\log(1+x) > \frac{x}{1+\frac{x}{2}} \quad \text{for} \quad x>0, \tag{AII.7}$$

and

$$\sum_{k=0}^{N} f(k) > \int_0^{N+1} f(t)dt \quad \text{if} \quad f'(x) \leqq 0, \tag{AII.8}$$

and get, from (AII.6) or (41):

$$\sum_{k=N-P+1}^{N} \ln(1+\varepsilon_k) > \int_0^P \frac{2dq}{\frac{2}{\varepsilon_N}+q+1} \geqq 2\ln\left(\frac{1+\frac{\varepsilon_N(P+1)}{2}}{1+\frac{\varepsilon_N}{2}}\right). \tag{AII.9}$$

Hence, using $\varepsilon_N < 1$ in the denominator of the argument of the logarithm, we get

$$\prod_{k=N-P+1}^{N} F^{(k)}(2) > \frac{1}{9}\varepsilon_N^2(P+1)^2. \tag{AII.10}$$

Combining this inequality with (34), which says that $\prod_0^N F^{(k)}(2)$ grows at most like N^2, we get (43), which we repeat here in a slightly weakened form

$$C > \left(\frac{1}{9}\right)^n (\varepsilon_{N_1}\varepsilon_{N_2}\dots\varepsilon_{N_n})^2 \left[\frac{N_1(N_2-N_1)\dots(N_n-N_{n-1})}{N_1+N_2-N_1+\dots+N_n-N_{n-1}}\right]^2,$$

$$N_n > N_{n-1} > \dots > N_1 \geqq 1.$$

The bracket is a monotonously increasing function of the $N_{i+1}-N_i$ considered as independent variables (including $N_1 = N_1 - 0$). Suppose now that $\nu(x)$ numbers ε_p^2 are larger than x. It is always possible to pick n of them, $\varepsilon_{N_1}^2, \varepsilon_{N_2}^2, \dots, \varepsilon_{N_n}^2$, in such a way that

$$N_1 \geqq \left[\frac{\nu(x)}{n}\right], \quad N_{p+1}-N_p \geqq \left[\frac{\nu(x)}{n}\right], \quad \forall p,$$

where $[x]$ = integral part of x. Hence

$$C > \left(\frac{x}{9}\right)^n \left[\frac{v(x)}{n}\right]^{2(n-1)} \frac{1}{n^2}, \quad \forall n \geqq 1.$$

Hence

$$\left[\frac{v(x)}{n}\right] < n^{\frac{1}{n-1}} \left(\frac{9}{x}\right)^{\frac{n}{2(n-1)}},$$

and

$$v(x) < n^{\frac{n}{n-1}} \left(\frac{9}{x}\right)^{\frac{n}{2(n-1)}} + n. \tag{AII.11}$$

Appendix III

There is no need to prove in detail the equation

$$3F^{(N+1)''}(2) + 2F^{(N+1)'''}(2) = F^{(N)}(2)[3F^{(N)''}(2) + 2F^{(N)'''}(2)], \tag{AIII.1}$$

with, naturally, $F^{(N)}(2) = 2F^{(N)'}(2)$. It is straightforward algebra. This equation shows that if $F''(2)$ and $F'''(2)$ are finite, we get

$$F^{(N)'''}(2) < C \prod_N(2), \tag{AIII.2}$$

and, from (34)

$$F^{(N)'''}(2) < CN^2. \tag{AIII.3}$$

A similar bound holds for $F^{(N)''}(2)$ but the latter one can be greatly improved by using the following trick: from the Cauchy inequality we have

$$F^{(N)''}(s) < \frac{1}{2-|s|}, \quad \text{since} \quad |F^{(N)'}(s)| < 1 \quad \text{in} \quad |s| < 2,$$

and hence

$$F^{(N)''}(2) < F^{(N)''}(s) + (2-s)F^{(N)'''}(2) < \frac{1}{2-s} + C(2-s)\prod_{N-1}(2).$$

Optimizing with respect to s, we get

$$F^{(N)''}(2) < C\sqrt{\prod_{N-1}(2)} < C'N. \tag{AIII.4}$$

On the other hand, we have, in the case $F(2) = 2F'(2)$

$$F''(2)(F(2)-1) > \tfrac{1}{4}(F(0))^2. \tag{AIII.5}$$

This inequality can be proved by substituting the expansions

$$F''(2)=\sum_{2}^{\infty} n(n-1)2^{n-2}a_n\,,$$

$$F(0)=\sum_{2}^{\infty}(n-1)2^n a_n\,,$$

$$F(2)-1=\sum_{1}^{\infty}(n-1)2^n a_n\,,$$

and using the Schwarz inequality.

So from (AIII.4), (AIII.5), and $F^N(0)>16/25$, we get

$$\frac{\prod_{N+1}(2)}{\prod_N(2)}>1+\frac{C}{\sqrt{\prod_N(2)}}, \tag{AIII.6}$$

and since $\prod_N\to\infty$ for $N\to\infty$

$$\sqrt{\frac{\prod_{N+1}(2)}{\prod_N(2)}}>1+\frac{C}{2}\frac{1}{\sqrt{\prod_N(2)}}(1-\varepsilon),$$

ε arbitrarily small for N big enough, and hence

$$\sqrt{\prod_{N+1}}-\sqrt{\prod_N}\geqq C'>0\,. \tag{AIII.7}$$

Therefore,

$$C_1N^2<\prod_{k=0}^{N}F^{(k)}(2)<C_2N^2\,. \tag{AIII.8}$$

At this point it is tempting to assume that $\prod_N(2)$ behaves like N^2 and to assume that $\varepsilon_N\equiv\varepsilon_N(2)$ has a smooth behaviour in N:

$$\varepsilon_N(2)=\frac{2}{N}+\frac{C}{N^2}+\dots\,. \tag{AIII.9}$$

Substituting into the recursion equation for $F^{(N)}(2)$ [Eq. (39)], we get

$$1-(F^{(N)}(0))^2=\frac{8}{N^2}+O\left(\frac{1}{N^3}\right). \tag{AIII.10}$$

Substituting this in turn into inequality (52) we get, for $s<2$

$$\varepsilon_N(s)\lesssim\frac{s-1}{2-s}\times\frac{8}{N^2},$$

and, in fact, since $(\varepsilon_N(s))^2=O(1/N^4)$, we find

$$\varepsilon_N(s)\simeq\frac{s-1}{2-s}\frac{8}{N^2}. \tag{AIII.11}$$

It is clear that this asymptotic behaviour holds for fixed $s<2$. However, we can also investigate the neighbourhood of $s=2$ for N large. We remark that $\varepsilon_N(s)$ and $\varepsilon_N(2)$

will be of the same order of magnitude for $s > 2 - \frac{2}{N}$ because $F'^{(N)}(2) < 1$. Therefore, we use a scaling variable,

$$z = (2-s)N\,, \tag{AIII.12}$$

and assume

$$\varepsilon_N(s) = \frac{\phi(z)}{N} + \frac{\psi(z)}{N^2} + \dots\,, \tag{AIII.13}$$

and substitute into the recursion equation, using (AIII.10). We get a Riccati equation for ϕ:

$$2z\phi'(z) = (z+2)\phi(z) + \phi^2(z) - 8\,. \tag{AIII.14}$$

This equation [2] has a unique regular solution at $z=0$:

$$\phi(z) = \frac{8}{4+z}\,. \tag{AIII.15}$$

This solution has all the right properties: $\phi(0)=2$, hence $F^{(N)}(2) = 1 + (2/N) + O(1/N^2)$, $\phi'(0) = -\frac{1}{2}$,

$$F^{(N)\prime}(2) = \frac{1}{N}\phi'(0)\frac{dz}{dN} = \frac{1}{2} + O\left(\frac{1}{N}\right),$$

$$F^{(N)\prime\prime}(2) = \frac{N}{4} + O(1)\,.$$

We can find an interpolating formula for (AIII.11) and (AIII.15)

$$\varepsilon_N(s) = \frac{8}{N}\frac{s-1}{N(2-s)+4}\,. \tag{AIII.16}$$

Appendix IV

The cases $F'''(2) = \infty$, $F''(2) = \infty$. Here $\prod_N(2)$ *does not* behave like N^2. Consider first $F''(s) \to \infty$ for $s \to 2$. We have the representation (46)

$$F - sF' = \frac{2-s}{s}\sum_{N=0}^{\infty}\frac{(F^{(N)}(0))^2}{\prod_N(s)\left(\frac{2}{s}\right)^{N+1}}\,, \tag{AIV.1}$$

with

$$\left.\begin{aligned} \prod_N(s) &< \prod_N(2) \\ \prod_N(s)\left(\frac{2}{s}\right)^N &> \prod_N(2) \end{aligned}\right\}. \tag{AIV.2}$$

Using

$$\left(\frac{16}{25}\right)^2 < (F^{(N)}(0))^2 \leqq 1\,,$$

it is not difficult to get

$$\left.\frac{F-sF'}{2-s}\right|_{s=2-\frac{1}{N}} > C_1 \sum_{p=0}^{N} \frac{1}{\prod_p(2)}, \tag{AIV.3}$$

$$\left.\frac{F-sF'}{2-s}\right|_{s=2-\frac{1}{N}} < C_2 \sum_{p=0}^{N} \frac{1}{\prod_p(2)}, \tag{AIV.4}$$

and if $F'' \sim C/(2-s)^\alpha$

$$F-sF' \sim C(2-s)^{1-\alpha},$$

and, using the above inequalities, one gets

$$C_1 N^{1-\alpha} < \prod_N(2) < C_2 N^{1-\alpha}. \tag{AIV.5}$$

If $F''(2)$ is finite, $F'''(s) \to \infty$, the situation is more complicated. We shall only give a very succinct account.

From the recursion relation

$$\begin{aligned} 3F''^{(N+1)}(s) + sF'''^{(N+1)}(s) = {} & \frac{2F^{(N)}(s)}{s}\left[3F''^{(N)}(s) + sF'''^{(N)}(s)\right] \\ & - \frac{6}{s} F''^{(N)}\left(F^{(N)} - sF'^{(N)}\right), \end{aligned} \tag{AIV.6}$$

we get, by summation

$$3F'' + sF''' = \frac{6}{s} \sum_{N=0}^{\infty} \frac{F''^{(N)}(s)\left[F^{(N)} - sF'^{(N)}\right]}{\prod_N(s)\left(\frac{2}{s}\right)^{N+1}}, \tag{AIV.7}$$

as well as the inequality

$$3F''^{(N)} + sF'''^{(N)} < \prod_{N-1}(s)\left(\frac{2}{s}\right)^N \left[3F'' + sF'''\right]. \tag{AIV.8}$$

If we assume $F'''(s) \sim C/(2-s)^\alpha$, we can use (AIV.8) together with $F''^{(N)}(s) < 1/2-s$ to get a bound on

$$F''^{(N)}(2) = F''^{(N)}(s) + \int_s^2 F'''^{(N)}(s')\,ds'.$$

After optimization, one gets

$$\left.\begin{aligned} & F''^{(N)}(2) < C\sqrt{\prod_{N-1}(2) N^\alpha} + C\frac{\prod_{N-1}(2)}{N^{1-\alpha}}, \\ & F''^{(N)}(2) < \varepsilon N + C(\varepsilon)\frac{\prod_{N-1}(2)}{N^{1-\alpha}}, \end{aligned}\right\} \tag{AIV.9}$$

where ε can be chosen arbitrarily small.

Inserting in the right-hand side of (AIV.7), up to a certain N, using $F-sF'<1$, and adjusting s, one can manage to get

$$N^\alpha < \sum_{k=0}^{N} \frac{k}{\prod_k(2)}. \tag{AIV.10}$$

The converse is obtained by using (AIV.9) together with

$$F''^{(N)}(2)(F^{(N)}(2)-1) > \tfrac{1}{4}(F^{(N)}(0))^2.$$

This leads to the inequality

$$\frac{\prod_{N+1}(2)}{\prod_N(2)} - 1 > \frac{C}{N^{\alpha-1}\prod_N(2) + N^{\alpha/2}(\prod_N(2))^{1/2}},$$

which can be shown to imply

$$\prod_N(2) > CN^{2-\alpha}, \tag{AIV.11}$$

which, in turn, can be reinjected into (AIV.10) to get

$$\prod_N(2) < C'N^{2-\alpha}. \tag{AIV.12}$$

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In memoriam

This work, to which Jurko Glaser contributed more than his share, will be his last one. Jurko died on 22 January 1984. Although he knew he had an incurable illness, he continued to collaborate very actively with us, showing great courage and unfailing enthusiasm. We shall always remember him as a remarkable physicist, a man of great culture, and a wonderful friend.
P.C., J.-P.E., A.M.

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