The Gurarij spaces are unique

By

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This paper is concerned with a problem mentioned in [2] and [4]. We show that those Banach spaces, which are distinguished by the extension property given below, are all isometrically isomorphic. For this purpose we investigate isometric embeddings from l_∞^n into l_∞^{n+m}; $n, m \in \mathbb{N}$; where l_∞^n denotes the Banach space of all n-tupels with the sup-norm. All Banach spaces in this article are over the reals.

Definition. *A separable Banach space X is called Gurarij space if for an arbitrary positive ε, arbitrary finite dimensional Banach spaces $F \supset E$ and an arbitrary isometric isomorphism (into) $T : E \to X$ there is a linear extension $\tilde{T} : F \to X$ of T with*

$$(1 - \varepsilon) \|x\| \leqq \|\tilde{T}(x)\| \leqq (1 + \varepsilon) \|x\| \quad \textit{for all } x \in F.$$

The dual space X^* of a Gurarij space X is an abstract L space (c.f. [3] and also [2]). Hence the following holds:

If E, $F \subset X$, $E \cong l_\infty^n$, F a finite dimensional subspace, and $\varepsilon > 0$, then there is $\tilde{E} \supset E$, $\tilde{E} \cong l_\infty^{n+m}$ such that $\inf\{\|x - y\| \mid y \in \tilde{E}\} \leqq \varepsilon \|x\|$ *for all* $x \in F$ ([2] Theorem 3.1.).

Thus, since X is separable, X can be represented in the following way: Let $E \subset X$ with $E \cong l_\infty^m$. Then there are $E_n \subset X$, $E_n \subset E_{n+1}$, E_n isometrically isomorphic to l_∞^n, $n \in \mathbb{N}$, such that $X = \overline{\bigcup_{n \in \mathbb{N}} E_n}$ and $E_m = E$ ([2] Theorem 3.2.). Consider the unit vectors $(0, \ldots, 0, 1, 0, \ldots, 0)$ of E_n, denote them by $e_{i,n}$; $i = 1, \ldots, n$; where negative signs and permutations are admitted and call $\{e_{i,n} \mid i = 1, \ldots, n\}$ an *admissible basis* of E_n.

It is an elementary fact that there are real numbers $a_{1,n}, \ldots, a_{n,n}$ with $\sum_{i=1}^{n} |a_{i,n}| \leqq 1$ and an admissible basis $\{e_{i,n+1} \mid i = 1, \ldots, n+1\}$ of E_{n+1} with

$$(*) \qquad e_{i,n} = e_{i,n+1} + a_{i,n} e_{n+1,n+1}; \quad i = 1, \ldots, n \qquad ([6]).$$

Conversely such numbers define by (*) an isometry $T : E_n \to E_{n+1}$. Hence X can be described by an infinite triangular matrix $A = (a_{i,n})$ whose n'th column consists of the numbers $a_{i,n}$; $i = 1, \ldots, n$. Gurarij was the first one to prove the existence of a separable Banach space satisfying the condition of the above definition ([1]). In [2] Lazar and Lindenstrauss gave another proof depending on the techniques previously described:

Consider an infinite triangular matrix A whose columns are dense in the unit ball of l_1 with respect to the l_1-norm. Then the elements $a_{i,n}$; $i = 1, \dots, n$; $n \in \mathbb{N}$; of A define by (*) *a Gurarij space.*

It is easy to show that this Gurarij space has the following property (c.f. [2] concluding remarks):

The set $\operatorname{ex} B(X^*)$ *of all extreme points of the unit ball* $B(X^*)$ *of* X^* *is* w^**-dense in* $B(X^*)$.

Lemma 1. *Let X be a Gurarij space and let $E \subset X$ with $E \cong l_\infty^n$. Furthermore let $\{e_{1,n}, \dots, e_{n,n}\}$ be an admissible basis of E and $r_1, \dots, r_n \in \mathbb{R}$ with $\sum_{i=1}^{n} |r_i| < 1$. Then there is an element $\Phi \in \operatorname{ex} B(X^*)$ with $\Phi(e_{i,n}) = r_i$; $i = 1, \dots, n$.*

Proof. Consider a representation $X = \overline{\bigcup_{m \in \mathbb{N}} E_m}$, $E_m \subset E_{m+1}$, $E_m \cong l_\infty^m$, $m \in \mathbb{N}$, $E_n = E$. We define by induction a sequence of subspaces $F_m \subset X$ with $X = \overline{\bigcup F_m}$ and a linear functional Φ on X.

For this purpose, take any $E_m \subset X$, $E_m \cong l_\infty^m$ and an admissible basis $\{e_{1,m}, \dots, e_{m,m}\}$ of E_m. Suppose that Φ is already defined on E_m with

$$\sum_{i=1}^{m} |\Phi(e_{i,m})| \leqq \frac{1}{1 + \delta_m}$$

for some δ_m with $0 < \delta_m < 1/m$.

Embed E_m into a Banach space $\tilde{F}_{m+1} \cong l_\infty^{m+1}$ by

$$e_{i,m} = f_{i,m+1} + \Phi(e_{i,m})(1 + \delta_m) f_{m+1,m+1}; \quad i = 1, \dots, m; \tag{1}$$

where $\{f_{i,m+1} \mid i = 1, \dots, m+1\}$ is an admissible basis of $\tilde{F}_{m+1}$. Extend Φ to an element Φ_{m+1} of $\tilde{F}_{m+1}^*$ by defining

$$\Phi_{m+1}(f_{m+1,m+1}) = \frac{1}{1 + \delta_m}; \quad \Phi_{m+1}(f_{i,m+1}) = 0. \tag{2}$$

Choose $\varepsilon > 0$ with $\varepsilon < \dfrac{\delta_m}{1 + \delta_m}$ and find a linear extension $T_{m+1} : \tilde{F}_{m+1} \to X$ of $\mathrm{id} : E_m \to X$ with the property:

$$(1 - \varepsilon)\|y\| \leqq \|T_{m+1}(y)\| \leqq (1 + \varepsilon)\|y\| \quad \text{for all } y \in \tilde{F}_{m+1}. \tag{3}$$

One may choose T_{m+1} such that there is in addition $k(m) \in \mathbb{N}$, $k(m) \geqq m + 1$ with $T_{m+1}\tilde{F}_{m+1} \subset E_{k(m)}$. This is possible since $\bigcup_{k \in \mathbb{N}} E_k$ is dense in X. Put $F_{m+1} = T_{m+1}\tilde{F}_{m+1}$ and extend $\Phi_{m+1} \circ T_{m+1}^{-1} \in F_{m+1}^*$ to a linear functional Φ on $E_{k(m)}$ with

$$\|\Phi\| \leqq \frac{1}{1 - \varepsilon}\|\Phi_{m+1}\| \leqq \frac{1}{(1 - \varepsilon)(1 + \delta_m)} < 1 \quad \text{by (2) and (3).}$$

Starting with $m = n$, $\Phi(e_{i,n}) = r_i$, δ_n such that $\sum_{i=1}^{n} |r_i| < \dfrac{1}{1 + \delta_n}$, we then obtain by induction an increasing chain of subspaces $F_m \subset X$ where m runs through a subsequence of $\mathbb{N}$, such that $X = \overline{\bigcup F_m}$ holds. By this construction an element $\Phi \in \operatorname{ex} B(X^*)$ is defined: Indeed, let:

$$\Phi = 1/2\, x^* + 1/2\, y^*, \quad \text{where } x^*, y^* \in B(X^*). \tag{4}$$

Let $\varrho > 0$ and $x \in \bigcup F_m$ with $\|x\| \leqq 1$, say $x = \sum_{i=1}^{m} \lambda_i T_m(f_{i,m})$ for some $m \in \mathbb{N}$ with $\delta_{m-1} < \varrho$, hence for some $0 < \varepsilon < \dfrac{\delta_{m-1}}{1+\delta_{m-1}}$:

$$|\lambda_i| \leqq \frac{1}{1-\varepsilon} < \frac{1}{1-\delta_{m-1}} < \frac{1}{1-\varrho}$$

by (3). Then by (2):

$$\Phi(T_m(f_{i,m})) = (\Phi_m \circ T_m^{-1})(T_m(f_{i,m})) = \Phi_m(f_{i,m}) = \begin{cases} 0 & i \neq m, \\ \dfrac{1}{1+\delta_{m-1}} & i = m. \end{cases}$$

Since

$$1 - \varrho \leqq \left\| \sum_{i=1}^{m} \theta_i T_m(f_{i,m}) \right\| \leqq 1 + \varrho$$

for all $\theta_i \in \{1, -1, 0\}$; $i = 1, \ldots, m-1$; $\theta_m = \pm 1$, and

$$1 - \varrho \leqq \frac{1}{1+\varrho} \leqq \Phi(T_m(f_{m,m})) \leqq 1,$$

it follows by (4) that

$$1 - 3\varrho \leqq x^*(T_m(f_{m,m})) \leqq 1 + \varrho \quad \text{and} \quad \sum_{i=1}^{m-1} |x^*(T_m(f_{i,m}))| \leqq 4\varrho.$$

Thus

$$|\Phi(x) - x^*(x)| \leqq \frac{7\varrho}{1-\varrho}.$$

Since ϱ and x were arbitrarily chosen, $\Phi = x^*$ holds, hence $\Phi \in \operatorname{ex} B(X^*)$ and our assertion follows. ▮

Lemma 2. *Under the assumptions of Lemma 1 there is an admissible basis*

$$\{e_{i,n+1} \in X \mid i = 1, \ldots, n+1\}$$

of l_∞^{n+1} *with* $e_{i,n} = e_{i,n+1} + r_i e_{n+1,n+1}$; $i = 1, \ldots, n$.

Proof. Consider $\Phi_1, \ldots, \Phi_n \in \operatorname{ex} B(X^*)$ with

$$\Phi_i(e_{j,n}) = \begin{cases} 0 & i \neq j, \\ 1 & i = j \end{cases} \quad i, j = 1, \ldots, n.$$

Such elements exist by the theorems of Hahn-Banach and Krein-Milman. Lemma 1 yields a $\Phi \in \operatorname{ex} B(X^*)$ with $\Phi(e_{i,n}) = r_i$; $i = 1, \ldots, n$. Let H be the absolutely convex hull of the Φ_i, $1 \leqq i \leqq n$ and Φ. Define $g : B(X^*) \to [0, \infty)$ by

$$g(x^*) = \min \left\{ \frac{1 - \sum_{i=1}^{n} \theta_i x^*(e_{i,n})}{1 - \sum_{i=1}^{n} \theta_i r_i} \;\middle|\; \theta_i = \pm 1,\ i = 1, \ldots, n \right\}.$$

Then g is w^*-continuous, concave and $f(x^*) \leqq g(x^*)$ for all $x^* \in H$, where $f : H \to \mathbb{R}$ is the affine function with $f(\pm \Phi_i) = 0$; $i = 1, \ldots, n$; $f(\pm \Phi) = \pm 1$.

Thus, by [2] Theorem 2.1, there is an element $e \in X$ with

$$x^*(e) \leqq g(x^*) \quad \text{for all } x^* \in B(X)^*,$$
$$x^*(e) = f(x^*) \quad \text{for all } x^* \in H.$$

An elementary computation shows that $e_{i,n} - r_i e$; $i = 1, \ldots, n$; and e are the elements of an admissible basis satisfying the desired condition. ∎

Corollary. *Let X be a Gurarij space, $E \simeq l_\infty^n$, $F \simeq l_\infty^{n+1}$ such that $E \subset F$ and $\operatorname{ex} B(E) \cap \cap \operatorname{ex} B(F) = \emptyset$. Then any linear isometric operator $T : E \to X$ can be extended to an isometric isomorphism (into) $\tilde{T} : F \to X$.*

Proof. We may E identify with a subspace of X and regard T as the identity $\mathrm{id} : E \to X$. Consider admissible bases $\{e_{i,n} \mid i \leqq n\}$ and $\{f_{i,n+1} \mid i \leqq n+1\}$ of E and F respectively with:

$$e_{i,n} = f_{i,n+1} + r_i f_{n+1,n+1}; \qquad i = 1, \ldots, n.$$

From our assumption on E and F we infer $\sum_{i=1}^{n} |r_i| < 1$. Hence there is an admissible basis $\{e_{i,n+1} \in X \mid i \leqq n+1\}$ of l_∞^{n+1} with $e_{i,n} = e_{i,n+1} + r_i e_{n+1,n+1}$, $i \leqq n$, by Lemma 2. Then we obtain our extension by setting $\tilde{T}(f_{n+1,n+1}) = e_{n+1,n+1}$. ∎

Remark. The above Corollary is not true in general without the assumption $\operatorname{ex} B(E) \cap \operatorname{ex} B(F) = \emptyset$. The following example may illustrate this: Consider a smooth point $e \in X$, $\|e\| = 1$ (i.e. there is only one $\Phi \in \operatorname{ex} B(X^*)$ with $\Phi(e) = \|e\| = 1$, such an e exists by [7] Proposition 8.4). Embed the linear span E of e into $F \simeq l_\infty^2$ by setting $e = e_{1,2} + e_{2,2}$ where $\{e_{1,2}, e_{2,2}\}$ is an admissible basis of F. Then the identity from E into X cannot be extended to an isometric isomorphism T from F into X. Indeed, otherwise two different elements $\Phi_{1/2} \in \operatorname{ex} B(X^*)$ would exist with:

$$\Phi_i(T(e_{j,2})) = \begin{cases} 0 & i \neq j, \\ 1 & i = j \end{cases} \quad i, j = 1, 2.$$

Hence $\Phi_1(e) = \Phi_2(e) = 1$, a contradiction.

Theorem 3. *Let X and Y be Gurarij spaces. Then there is an isometric isomorphism from X onto Y.*

Proof. We construct a sequence of admissible bases of l_∞^n, $\{e_{i,n}^{(j)} \in X \mid i = 1, \ldots, n\}$ and $\{f_{i,n}^{(j)} \in Y \mid i = 1, \ldots, n\}$, $j \geqq n$, $n \in \mathbb{N}$, with the following properties:

There are $a_{i,n} \in \mathbb{R}$, $i \leqq n$, $n \in \mathbb{N}$ such that

$$\sum_{i=1}^{n} |a_{i,n}| < 1\,, \tag{5}$$

$$e_{i,n}^{(j)} = e_{i,n+1}^{(j)} + a_{i,n}\, e_{n+1,n+1}^{(j)} \qquad \text{and} \tag{6}$$

$$f_{i,n}^{(j)} = f_{i,n+1}^{(j)} + a_{i,n} f_{n+1,n+1}^{(j)} \qquad i = 1, \ldots, n;\; j \geqq n+1;\; n \in \mathbb{N}\,, \tag{6'}$$

$$\| e_{i,n}^{(j)} - e_{i,n}^{(j+1)} \| \leqq 1/2^j\,, \tag{7}$$

$$\| f_{i,n}^{(j)} - f_{i,n}^{(j+1)} \| \leqq 1/2^j\,, \qquad i = 1, \ldots, n;\; j \geqq n;\; n \in \mathbb{N}\,. \tag{7'}$$

Let $\{x_n \in X \mid n \in \mathbb{N}\}$ and $\{y_n \in Y \mid n \in \mathbb{N}\}$ be dense in X and Y respectively. Take some $e_{1,1}^{(1)} \in X$, $\| e_{1,1}^{(1)} \| = 1$, and $f_{1,1}^{(1)} \in Y$, $\| f_{1,1}^{(1)} \| = 1$.

Assume, that $\{e_{i,k}^{(j)} \mid i \leqq k\}$, $\{f_{i,k}^{(j)} \mid i \leqq k\}$ are already defined for $k = 1, \ldots, m$; $k \leqq j \leqq m$ such that (5)—(7′) hold.

Let E_m and F_m be the linear span of $\{e_{i,m}^{(m)} \mid i = 1, \ldots, m\}$ and $\{f_{i,m}^{(m)} \mid i = 1, \ldots, m\}$ respectively.

(I): Consider $E_{m+p} \cong l_\infty^{m+p}$ with $E_m \subset E_{m+p} \subset X$ and

$$\inf\{\| x_k - x \| \mid x \in E_{m+p}\} \leqq 1/m\, \| x_k \| \quad \text{for all } k = 1, \ldots, m\,. \tag{8}$$

Hence there are $E_{m+1} \subset \cdots \subset E_{m+p}$ with $E_{m+k} \cong l_\infty^{m+k}$; $k = 1, \ldots, p$; $E_m \subset E_{m+1}$ ([6] Lemma 3.2).

STEP$(m+1)$: Take an admissible basis $\{e_{i,m+1}^{(m+1)} \mid i \leqq m+1\}$ of E_{m+1} with

$$e_{i,m}^{(m)} = e_{i,m+1}^{(m+1)} + r_i\, e_{m+1,m+1}^{(m+1)}\,, \qquad i = 1, \ldots, m; \tag{9}$$

$$\text{where } \sum_{i=1}^{m} |r_i| \leqq 1\,.$$

If $r_i = 0$ for all $i \leqq m$ then put

$$e_{i,m}^{(m+1)} = e_{i,m}^{(m)}\,, \qquad i = 1, \ldots, m\,.$$

Otherwise, assume w.l.g. $r_m \neq 0$ and set

$$e_{m,m}^{(m+1)} = e_{m,m}^{(m)} - r_m/2^{2m}\, e_{m+1,m+1}^{(m+1)}\,, \qquad e_{i,m}^{(m+1)} = e_{i,m}^{(m)}\,, \qquad 1 \leqq i \leqq m-1\,. \tag{10}$$

Of course, by (9), $\{e_{i,m}^{(m+1)} \mid i \leqq m\}$ is an admissible basis and

$$\| e_{i,m}^{(m+1)} - e_{i,m}^{(m)} \| \leqq 1/2^{2m} \leqq 1/2^m\,, \qquad 1 \leqq i \leqq m\,.$$

Furthermore, (6) holds for $n = m$, $j = m+1$ and

$$a_{i,m} = r_i; \qquad 1 \leqq i \leqq m-1; \qquad a_{m,m} = r_m(1 - 1/2^{2m})$$

by (9), (10). Put

$$e_{i,m-1}^{(m+1)} = e_{i,m}^{(m+1)} + a_{i,m-1}\, e_{m,m}^{(m+1)}\,, \qquad 1 \leqq i \leqq m-1\,,$$

$$\vdots$$

$$e_{1,1}^{(m+1)} = e_{1,2}^{(m+1)} + a_{1,1}\, e_{2,2}^{(m+1)}\,.$$

Now continue with STEP$(m+2)$ — that means, proceed in analogy to STEP$(m+1)$ with E_{m+2} instead of E_{m+1} —, then with STEP$(m+3),\dots,$STEP$(m+p)$.

This procedure yields $a_{i,m+j} \in \mathbb{R}$; $0 \leqq j \leqq p-1$; with $\sum_{i=1}^{m+j} |a_{i,m+j}| < 1$ and admissible bases

$$\{e_{i,r}^{(j)} \mid i = 1, \dots, r\}; \quad r = 1, \dots, m+p; \quad m+1 \leqq j \leqq m+p$$

such that (6) and (7) hold.

(II): Now consider F_m and set

$$f_{i,k}^{(j)} = f_{i,k}^{(m)}; \quad 1 \leqq i \leqq k; \quad 1 \leqq k \leqq m; \quad m+1 \leqq j \leqq m+p.$$

Extend the linear injection, which maps $e_{i,m}^{(m+p)}$ onto $f_{i,m}^{(m+p)}$, $1 \leqq i \leqq m$, to an isometric isomorphism T from E_{m+p} into Y. This is possible by (5), the above Corollary and induction. Set

$$f_{i,m+k}^{(j)} = T(e_{i,m+k}^{(m+p)}), \quad 1 \leqq k \leqq p, \quad m+k \leqq j \leqq m+p.$$

Hence (6′) and (7′) are established for all

$$f_{i,k}^{(j)}, \quad 1 \leqq i \leqq k; \quad 1 \leqq k \leqq m+p; \quad k \leqq j \leqq m+p.$$

Then proceed in analogy to (I):

Consider $T E_{m+p} \subset F_{m+p+1} \subset \cdots \subset F_{m+p+q} \subset Y$, $F_{m+p+j} \cong l_\infty^{m+p+j}$; $1 \leqq j \leqq q$; with

$$\inf\{\|y_k - y\| \mid y \in F_{m+p+q}\} \leqq \frac{1}{m+p} \|y_k\|, \quad k = 1, \dots, m+p. \tag{11}$$

The same method as in STEP$(m+1),\dots,$STEP$(m+p)$ is applicable for $F_{m+p+1},\dots,$ F_{m+p+q} instead of $E_{m+1},\dots,E_{m+p}$, which yields

$$f_{i,k}^{(j)}, \quad 1 \leqq i \leqq k;\ 1 \leqq k \leqq m+p+q;\ m+p+1 \leqq j \leqq m+p+q$$

such that (6′) and (7′) holds.

Finally put

$$e_{i,k}^{(j)} = e_{i,k}^{(m+p)}, \quad 1 \leqq i \leqq k;\ 1 \leqq k \leqq m+p;\ m+p+1 \leqq j \leqq m+p+q$$

and extend the linear operator which maps $f_{i,m+p}^{(m+p+q)}$ onto $e_{i,m+p}^{(m+p+q)}$, $1 \leqq i \leqq m+p$, to an isometric isomorphism S from F_{m+p+q} into X. Define

$$e_{i,m+p+k}^{(j)} = S(f_{i,m+p+k}^{(m+p+q)}), \quad 1 \leqq i \leqq m+p+k; \quad 1 \leqq k \leqq q;$$
$$m+p+k \leqq j \leqq m+p+q.$$

Replace E_m by $E_{m+p+q} = S F_{m+p+q}$ and begin with (I). Put

$$e_{i,n} = \lim_{j\to\infty} e_{i,n}^{(j)}, \quad f_{i,n} = \lim_{j\to\infty} f_{i,n}^{(j)}; \quad 1 \leqq i \leqq n;\ n \in \mathbb{N}.$$

It follows by (6), (6′), (7), (7′) that $\{e_{i,n} \mid i \leqq n\}$, $\{f_{i,n} \mid i \leqq n\}$ are admissible bases and that

$$e_{i,n} = e_{i,n+1} + a_{i,n} e_{n+1,n+1},$$
$$f_{i,n} = f_{i,n+1} + a_{i,n} f_{n+1,n+1}; \quad 1 \leqq i \leqq n; \quad n \in \mathbb{N}.$$

(7), (7'), (8), (11) imply that $\{e_{i,n} \mid i \leqq n;\, n \in \mathbb{N}\}$ and $\{f_{i,n} \mid i \leqq n;\, n \in \mathbb{N}\}$ span a dense subspace of X and Y resp. Thus the linear operator $R : X \to Y$ with $R(e_{i,n}) = f_{i,n}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, is bijective and isometric. ∎

Corollary. *Let X be a separable Banach space such that X^* is an abstract L-space. Let G be the Gurarij space. Then there is an isometry $T : X \to G$ and a contractive projection $P : G \to TX$ such that the following hold:*

(i) *$P^*(B((TX)^*)) = \operatorname{conv}(F \cup -F)$ where F is a face of $B(G^*)$ and P^* is the adjoint mapping.*

(ii) *$(\mathrm{id} - P)(G)$ is isometrically isomorphic to G.*

Proof. In [5] and [8] it was shown that there is a Gurarij space G, an isometry $T : X \to G$ and a contractive projection $P : G \to TX$ such that (i) holds and $(\mathrm{id} - P)(G)$ is a Gurarij space too. Our Corollary follows then from the preceeding Theorem. ∎

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Addendum, 10. 1. 1976.

The purpose of this addendum is to relate the concept of the Gurarij space to Mazur's problem of rotations in separable Banachspaces.

Mazur's Problem. *Let X be a separable Banachspace with the following property: For any $x, y \in X$, $\|x\| = \|y\| = 1$, there is an isometric automorphism $T : X \to X$ with $T(x) = y$. Is X then a Hilbertspace?*

We show:

Theorem. *Let G be the Gurarij space and let $x, y \in G$ be smooth points of the unit sphere of G. Then there is an isometric automorphism $T : G \to G$ with $T(x) = y$.*

Remarks. (i) The above Theorem includes a weaker property of G shown by Gurarij ([1]).

(ii) Notice, that the set of smooth points is dense in the unit sphere of G, but G is not reflexive. Hence G cannot be a Hilbertspace.

(iii) The assumption, x, y being smooth points, cannot be omitted since the unit sphere of a separable Banachspace X with Mazur's property clearly consists only of smooth points whereas the unit sphere of G has no smooth points.

Proof of the Theorem. The proof of the above Theorem is a modification of the proof of Theorem 3. We retain the numeration of this proof. Again, we construct a sequence of admissible bases of l_∞^n,

$$\{e_{i,n}^{(j)} \in G \mid i = 1, \ldots, n\} \quad \text{and} \quad \{f_{i,n}^{(j)} \in G \mid i = 1, \ldots, n\}, \quad j \geqq n,\ n \in \mathbb{N},$$

such that (5), (6), (6'), (7), (7') hold.

Now, we require in addition:

$$e_{1,1}^{(j)} = x\,, \quad f_{1,1}^{(j)} = y \quad \text{for all } j\,.$$

We proceed with (I) and STEP $(m+1)$: We assume that $E_m \cong F_m \cong l_\infty^{m'}$ already have been defined and find suitable $E_{m+1} \subset \cdots \subset E_{m+p}$ with $E_{m+k} \cong l_\infty^{m+k}$, $k = 1, \ldots, p$, and $E_m \subset E_{m+1}$. Firstly, we consider E_{m+1} (STEP$(m+1)$): We take an admissible basis $\{e_{i,m+1}^{(m+1)} \mid i \leqq m+1\}$ of E_{m+1} such that (9) holds.

Now, in the case that $\sum\limits_{i=1}^{m} |r_i| = 1$, our perturbation differs slightly from that above: Induction yields

$$\begin{aligned} x = e_{1,1}^{(m)} &= e_{1,m}^{(m)} + \sum_{j=2}^{m} k_j\, e_{j,m}^{(m)} \\ &= e_{1,m+1}^{(m+1)} + \sum_{j=2}^{m} k_j\, e_{j,m+1}^{(m+1)} + \left(r_1 + \sum_{j=2}^{m} k_j\, r_j\right) e_{m+1,m+1}^{(m+1)} \end{aligned}$$

where $|k_j| < 1$, $2 \leqq j \leqq m$, since x is a smooth point. Similarly, $|r_1| < 1$, hence there is an $r_k \neq 0$, $2 \leqq k \leqq m$. Assume w.l.g. that $r_m \neq 0$ and replace (10) by

$$\begin{aligned} &e_{k,m}^{(m+1)} = e_{k,m}^{(m)} - a_{k,m}\, e_{m+1,m+1}^{(m+1)} \quad \text{where} \\ &a_{1,m} = r_1 + 2^{-2m} k_m r_m\,, \quad a_{i,m} = r_i\,, \quad 2 \leqq i \leqq m-1\,, \\ &a_{m,m} = (1 - 2^{-2m})\, r_m\,. \end{aligned}$$

Hence $$\sum_{i=1}^{m} |a_{i,m}| < 1\,.$$

Our definition of the $a_{i,m}$ yields

$$\begin{aligned} e_{1,1}^{(m+1)} &= e_{1,m}^{(m+1)} + \sum_{j=2}^{m} k_j\, e_{j,m}^{(m+1)} \\ &= e_{1,m+1}^{(m+1)} + \sum_{j=2}^{m} k_j\, e_{j,m+1}^{(m+1)} + \\ &+ \left(r_1 + 2^{-2m} k_m r_m + \sum_{j=2}^{m} k_j r_j - 2^{-2m} k_m r_m\right) e_{m+1,m+1}^{(m+1)} = e_{1,1}^{(m)} = x\,, \end{aligned}$$

since the k_j depend only on $a_{i,k}$, $i \leqq k \leqq m-1$.

Now, the rest of this proof is a mere adoption of the corresponding proof of Theorem 3, which we do not repeat here. So we obtain admissible bases $\{e_{i,n} \in G \mid i \leqq n\}$, $\{f_{i,n} \in G \mid i \leqq n\}$ such that in addition to the required properties

$$e_{1,1} = \lim_{j \to \infty} e_{1,1}^{(j)} = x\,, \quad f_{1,1} = \lim_{j \to \infty} f_{1,1}^{(j)} = y$$

hold. The linear operator $T : G \to G$ defined by

$$T(e_{i,n}) = f_{i,n}\,, \quad i = 1, \ldots, n,\ n \in \mathbb{N}\,,$$

proves our Theorem. ∎

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