# A Note on Permutable Subgroups

By

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**1. Introduction and statement of results.** Let $H$ and $K$ be subgroups of a group $G$. It is clear that if $H$ and $K$ permute, then their images in the homomorphic images of $G$ permute. Our object in this note is to establish sufficient conditions to ensure that the permutability of $H$ and $K$ follows from the permutability of their images in the finite homomorphic images of $G$. We state our main result as

**Theorem A.** *Suppose that $H$ and $K$ are subgroups of a polycyclic-by-finite group $G$. If $H^{\varphi}K^{\varphi} = K^{\varphi}H^{\varphi}$ for all homomorphisms $\varphi$ from $G$ onto finite groups, then $HK = KH$.*

A routine argument, similar to that given in [2, Section 7], shows that Theorem A implies the solubility of a decision problem for polycyclic-by-finite groups:

**Corollary A1.** *There is an algorithm for deciding whether or not two given subgroups of a polycyclic-by-finite group permute.*

Recalling that a subgroup $H$ of a group $G$ is said to be quasinormal in $G$ if $HK = KH$ for all subgroups $K$ of $G$, we may also deduce

**Corollary A2.** *A subgroup $H$ of a polycyclic-by-finite group $G$ is quasinormal in $G$ if and only if $H^{\varphi}$ is a quasinormal subgroup of $G^{\varphi}$ for all homomorphisms $\varphi$ from $G$ onto finite groups.*

The hypothesis that $G$ be polycyclic-by-finite in Theorem A cannot be replaced either by the condition that $G$ be a finitely generated metabelian group or by the condition that $G$ be a soluble linear group. For let $G$ be the subgroup of GL $(2, \mathbb{Q})$ generated by

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $G$ is metabelian, and for every homomorphism $\varphi$ from $G$ onto a finite group we have $\langle a \rangle^{\varphi} \lhd G^{\varphi}$, and $\langle a \rangle^{\varphi}$ is certainly quasinormal in $G^{\varphi}$. On the other hand, it is easy to verify that $\langle a \rangle \langle b \rangle \neq \langle b \rangle \langle a \rangle$.

In our second theorem, which is proved by combining Theorem A and a theorem of Brewster [1], the condition of Theorem A that $G$ be polycyclic-by-finite is replaced by conditions on the subgroups $H$ and $K$ and on their embedding in $G$:

**Theorem B.** *Suppose that $H$ and $K$ are subnormal subgroups of their join $G$ and that the derived quotient groups $H/H'$ and $K/K'$ of $H$ and $K$ are finitely generated. If $H^\varphi K^\varphi = K^\varphi H^\varphi$ for all homomorphisms $\varphi$ from $G$ onto finite nilpotent groups, then $HK = KH$.*

Brewster's theorem asserts that two arbitrary subnormal subgroups permute if their images in all nilpotent images of their join permute. Thus in the situation of Theorem B we may restrict attention to the case in which $G$ is nilpotent. Since $H/H'$ and $K/K'$ are finitely generated, so is $G/G'$, and therefore, by a result of Baer [3, p.55], so is $G$. Thus $G$ is polycyclic and Theorem B follows, once we have proved Theorem A.

Of course the condition on $H/H'$ and $K/K'$ in Theorem B can be relaxed a little: all that is needed is that $G/G'$ be finitely generated. However the restriction cannot be removed entirely. This may be seen, for example, by considering the group of $3 \times 3$ upper unitriangular matrices over the rationals and two suitable subgroups; of course this group has no non-trivial finite images.

A routine argument shows that Theorem B, like Theorem A, implies the solubility of a decision problem:

**Corollary B.** *There is an algorithm for deciding whether or not two subnormal subgroups with finitely generated derived quotient groups of an arbitrary group permute.*

**2. Proof of Theorem A.** We show first of all that Theorem A follows from

**Theorem A*.** *Suppose that $H$ and $K$ are subgroups of a polycyclic-by-finite group $G$ and that $G = HKN$ for all normal subgroups $N$ of finite index in $G$. Then $G = HK$.*

In order to achieve this reduction we need the

**Lemma.** *Suppose that $J$ is any subgroup of a polycyclic-by-finite group $G$. If $K$ is a subgroup of finite index in $J$, then there exists a subgroup $L$ of finite index in $G$ such that $J \cap L = K$.*

This Lemma is an easy corollary to the theorem of Mal'cev [2, Theorem 6], that if $K$ is a subgroup of a polycyclic-by-finite group $G$ then $K$ is the intersection of all the subgroups of finite index in $G$ which contain $K$.

Assume now that Theorem A* holds, and suppose that $H$, $K$ are subgroups of a polycyclic by finite group $G$ such that $H^\varphi K^\varphi = K^\varphi H^\varphi$ for all homomorphisms from $G$ onto finite groups. It follows at once that $HKN = KHN$ for all normal subgroups $N$ of finite index in $G$. We set $J = \langle H, K \rangle$ and suppose that $M$ is a normal subgroup of finite index in $J$. By the Lemma there exists a subgroup $L$ of finite index in $G$ such that $J \cap L = M$. The normal interior $N$ of $L$ in $G$ has finite index in $G$ and so $KHN = HKN$. Hence

$$J = HK(N \cap J) \leqq HKM,$$

so that $J = HKM$. Therefore $J$ satisfies the hypothesis of Theorem A*, and $J = HK$.

We now proceed with the proof of Theorem A*. Suppose the result is false and let $G$ be a counterexample of least Hirsch number $h = h(G)$. Then $h \geqq 1$.

We first establish

(1) *if $N$ is an infinite normal subgroup of $G$, then $HN$ has finite index in $G$.*

For let $J = HN$, and suppose $L$ is a normal subgroup of finite index in $J$. Then $N^m \leqq L$ for some integer $m$, and $N^m \neq 1$ since $N$ is infinite. Also $h(G/N^m) < h$, and since the hypotheses pass to homomorphic images of $G$ the minimality of $h$ yields $G = HN^mK$. Therefore

$$J = HN^m(K \cap J) = H(K \cap J)L.$$

If $h(J) < h$ we conclude that $J = H(K \cap J)$ and that $G = JK = HK$, a contradiction. Thus $h(J) = h$, and $J$ has finite index in $G$.

The above argument also shows that

(2) *if $N$ is an infinite normal subgroup of $G$, then the group $HN$ and subgroups $H$ and $(HN) \cap K$ provide a counterexample to Theorem A*.*

Writing $F$ for the Fitting subgroup of $G$, we next prove

(3) *$H \cap F$ is finite.*

Suppose that this is not the case; then $F$ is infinite and nilpotent, and its centre $A$ is an infinite normal subgroup of $G$. Furthermore

$$M = (H \cap F)^J = (H \cap F)^H$$

is infinite and normal in the subgroup $J = HA$. By (2), $J$ and its subgroups $H$ and $K \cap J$ inherit the conditions of Theorem A*, and these conditions pass to the quotient group $J/M$. Because $h(J/M) < h$, it follows that $J/M$ is the product of its subgroups $H/M$ and $(K \cap J)\,M/M$, and that $J = H(K \cap J)$. Thus $G = JK = HK$, a contradiction.

Since polycyclic groups are nilpotent by Abelian by finite ([3, 3.25]), it follows from (3) that $H$ is finite by Abelian by finite, and therefore Abelian by finite. Similarly $K \cap F$ is finite and $K$ is Abelian by finite.

Because $G$ is infinite, there is a free Abelian normal subgroup $M \neq 1$ of $G$. Using (2) with the roles of $H$ and $K$ interchanged, we may suppose that $G = MK$. We assert that $H$ acts rationally irreducibly on $M$ by conjugation; it then follows *a fortiori* that $K$ acts rationally irreducibly on $M$. Let

$$1 \neq B = B^H \leqq M;$$

then $B$ is an infinite normal subgroup of $MH$, and (1), applied to $MH$ and its subgroups $H$ and $K \cap (MH)$, implies that $|MH : BH|$ is finite.Thus $|M : B(M \cap H)|$ is finite, and since obviously $M \leqq F$, we conclude from (3) that $|M : B|$ is finite.

None of our hypotheses or conclusions above are altered if we pass to the quotient group of $G$ by the normal interior of $K$; thus we may suppose that $K$ acts faithfully by conjugation on $M$.

Let $H_0$ and $K_0$ be Abelian normal subgroups of finite index in $H$, $K$ respectively, and set $n = |H : H_0|\ |K : K_0|$. Let $p$ be a prime not dividing $n$, and let $M_1$ be a normal subgroup of $G$ maximal subject to

$$M^p \leqq M_1 < M .$$

Writing $C = c_K(M/M_1)$, we have $M_1C \lhd G$. We denote by bars factor groups modulo $M_1C$. Clearly $\bar{G}$ is a split extension of $\bar{M}$ by $\bar{K}$.

Now $\bar{K}_0$ is Abelian and acts faithfully on the irreducible (or trivial) $K$-module $\bar{M}$, so that $\bar{K}_0$ is a $p'$-group. Hence $\bar{K}$ is a Hall $p'$-subgroup of $\bar{G}$. Further $\bar{G} = \bar{H}\bar{K}$, so that $| \bar{H}\bar{M} : \bar{H} |$ divides $| \bar{H}\bar{K} : \bar{H} |$. But $| \bar{H}\bar{M} : \bar{H} |$ is equal to $| \bar{M} : \bar{M} \cap \bar{H} |$, a power of $p$, while $| \bar{H}\bar{K} : \bar{H} |$ is equal to $| \bar{K} : \bar{K} \cap \bar{H} |$, a $p'$-number; therefore $\bar{H}\bar{M} = \bar{H}$ and $\bar{M} \leqq \bar{H}$. Because $| \bar{H}/\bar{H}_0 |$ divides $n$, it follows further that $| \bar{M} |$ divides $| \bar{H}_0 |$ and that $\bar{M} \leqq \bar{H}_0$. Hence

$$[\bar{M}, \bar{H}_0] \leqq [\bar{H}_0, \bar{H}_0] = 1,$$

and $[M, H_0] \leqq M_1$.

Because we may perform this argument for infinitely many primes $p$, the subgroup $[M, H_0]$ must have infinite index in $M$. But $[M, H_0]$ is $H$-invariant, and therefore is trivial. Thus, since $M$ coincides with its centralizer in $G$, we must have $H_0 \leqq M \cap H$, which is trivial by (3). Therefore $H$ is finite. By (1), $| G : MH |$ is finite, and so $| G : M |$ is finite. Since $M \cap K = 1$, it follows that $K$ also is finite. We conclude that the set $HK$ is finite. Because $G$ is infinite and residually finite, there is a finite image of $G$ of order greater than $| HK |$, and this certainly cannot be the product of images of $H$ and $K$. This contradiction completes the proof of Theorem A* and therefore of Theorem A.

## References

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[3] D. J. S. ROBINSON, Finiteness conditions and generalised soluble groups, Part 1. Berlin-Heidelberg-New York 1972.

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