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CONVEX BODIES FORMING PAIRS OF CONSTANT WIDTH

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1. INTRODUCTION

All sets considered in this note are subsets of the n-dimensional Euclidean space E^n. Let X be a nonempty compact convex set and u be a point in E^n different from the origin o. The width of X in the direction $\vec{ou}$, w(X;u), is the distance between the two supporting hyperplanes of X that are perpendicular to the line ou. In terms of the support function

$$h_X(u) = \sup\{<u,x>; x \in X\}$$

we have

$$w(X;u) = h_X(u/|u|) + h_X(-u/|u|),$$

where | | and < > denote the Euclidean norm and inner product. A <u>set of constant width</u> is a nonempty compact convex set whose width is constant in any direction.

We extend the notion of width to a pair (X,Y) of nonempty compact convex sets. Define the width of (X,Y) in the direction $\vec{ou}$, w(X,Y;u), by

$$w(X,Y;u) = h_X(u/|u|) + h_Y(-u/|u|).$$

A pair (X,Y) of sets is called a <u>pair of constant width</u> if X and Y are nonempty compact convex sets and w(X,Y;u) is constant for all $u \neq o$. It is clear that a set X is of constant width if and only if the pair (X,X) is of constant width.

Pairs of constant width have certain properties in common with sets of constant width (see Theorems 1 and 2). Furthermore, if (X,Y) is a pair of constant width r, then the 'vector sum' $X + Y$ is a set of constant width $2r$.

Pairs of constant width are well described in terms of the following operation Ω_r. For a nonempty set X and a positive number r, $\Omega_r(X)$ is defined to be the intersection of all closed balls of radius r whose centers belong to X. Then it is proved that a pair (X,Y) is of constant width r if and only if $\Omega_r(X)=Y$ and $\Omega_r(Y)=X$. Moreover, for a nonempty set X of circumradius r, the pair $(\Omega_r(X),\Omega_r^2(X))$ turns out to be of constant width r. Figure 1 shows two pairs of constant width constructed in this way from a line segment, and from a triangle, in the plane.

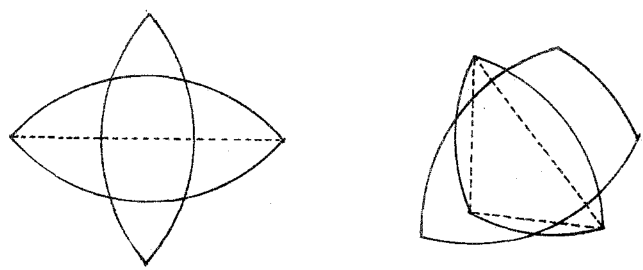

Figure 1.

2. PAIRS OF CONSTANT WIDTH

We first note that if (X,Y) is a pair of constant width r, then $r \geq 0$. For $2r = w(X,Y;u) + w(X,Y;-u) = w(X;u) + w(Y;u) \geq 0$.

A sphere of largest diameter that lies in a compact convex set X is called an <u>insphere</u> of X. The sphere of smallest diameter that encloses X is called the <u>circumsphere</u> of X. The radius of the circumsphere of X is the <u>circumradius</u> of X. In general, a convex set may have many inspheres, but the circumsphere is always unique.

<u>Theorem 1</u>. Let (X,Y) be a pair of constant width r. Then any insphere of X is concentric with the circumsphere of Y (hence X has the unique insphere), and the sum of their radii equals r.

In the case X = Y (then X is a set of constant width), the theorem is well known, see e.g. [1,p.125]. The proof of the theorem for $X \neq Y$ is much the same, and is omitted.

<u>Theorem 2</u>. (The plane case n=2) Let (X,Y) be a pair of constant width r in E^2. Then the sum of the perimeters of X and Y equals $2\pi r$.

<u>Proof</u>. It is well known that for a compact convex set S in E^2, the perimeter of S equals

$$\frac{1}{2}\int_0^{2\pi} w(S;u_\theta)d\theta, \quad u_\theta = (\cos\theta,\sin\theta).$$

Therefore the sum of the perimeters of X and Y equals

$$\frac{1}{2}\int_0^{2\pi}\{w(X;u_\theta)+w(Y;u_\theta)\}d\theta = \frac{1}{2}\int_0^{2\pi}\{h_X(u_\theta)+h_X(-u_\theta)+h_Y(u_\theta)+h_Y(-u_\theta)\}d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 2r\, d\theta = 2\pi r.$$

<u>Theorem 3</u>. If (X,Y) is a pair of constant width r, then the set $X + Y := \{x+y; x \in X, y \in Y\}$ is a set of constant width 2r.

<u>Proof</u>. Since the support function $h_{X+Y}(u)$ of X+Y equals $h_X(u)+h_Y(u)$ as easily verified, we have

$$w(X+Y;u) = h_{X+Y}(u/|u|) + h_{X+Y}(-u/|u|)$$
$$= h_X(u/|u|)+h_Y(u/|u|)+h_X(-u/|u|)+h_Y(-u/|u|)$$
$$= w(X,Y;u) + w(X,Y;-u) = 2r.$$

3. THE OPERATION Ω_r

We denote by $B(x,r)$ the closed ball of radius r centered at x. Then $\Omega_r(X) = \cap\{B(x,r); x \in X\}$. The operation Ω_r also appeared in [1] in connection with sets of constant width, but in rather restricted form and in different notation.

By the definition, $\Omega_r(X)$ is compact and convex, $\Omega_r(X \cup Y)$ equals $\Omega_r(X) \cap \Omega_r(Y)$, and if $X \subset Y$ then $\Omega_r(Y) \subset \Omega_r(X)$.

<u>Theorem 4</u>. If X is a nonempty set of circumradius $\le r$, then $\Omega_r^{\,2}(X)$ is the intersection of all closed balls of radius r that contain X, and $\Omega_r^{\,3}(X) = \Omega_r(X)$.

<u>Proof</u>. Since

$$X \subset B(y,r) \leftrightarrow |y-x| \le r \text{ for all } x \in X \leftrightarrow y \in \Omega_r(X),$$

we have

$$\Omega_r^{\,2}(X) = \cap\{B(y,r); y \in \Omega_r(X)\} = \cap\{B(y,r); X \subset B(y,r)\}.$$

And

$$y \in \Omega_r(X) \leftrightarrow X \subset B(y,r) \leftrightarrow \Omega_r^{\,2}(X) \subset B(y,r)$$

$$\leftrightarrow |y-z| \le r \text{ for all } z \in \Omega_r^{\,2}(X) \leftrightarrow y \in \Omega_r^{\,3}(X).$$

<u>Remark 1</u>. The set $\Omega_r^{\,2}(X)$ is called the <u>r-convex hull</u> of X, see [2,p.99].

<u>Remark 2</u>. If we restrict operands of Ω_r to nonempty compact sets of circumradius $\le r$, then Ω_r is continuous with respect to the

'Hausdorff distance'.

4. PAIRS OF CONSTANT WIDTH AND THE OPERATION Ω_r

Lemma. Suppose $\Omega_r(X) \neq \emptyset$ and let H be a hyperplane supporting $\Omega_r^2(X)$ at x. Let B be the closed ball of radius r which touches H at x and lies on the same side of H as X. Then B containes X.

Proof. First we note the following fact: If the two end points of a minor circular arc (or a semicircle) of radius r belong to a ball of radius r, then the arc is contained in the ball.

Now suppose B does not contain X. Let y be a point of X not in B and let z be the center of B. Since $|x-y| \leq 2r$ and y is not in B, the points x,y,z are not collinear. Let P be the plane determined by x,y,z. Then the line $L = P \cap H$ is tangent to the circle C bounding the disk $P \cap B$. So the line L cuts any circle of radius r on P passing through x and different from the circle C and its reflection in L. Hence, in the plane P, there is a minor circular arc (or semicircle) A of radius r with end points x,y which crosses the line L. Then the arc A intersects with both sides of the hyperplane H. Now, by the fact noted above, every closed ball of radius r that contains X also contains the arc A. Hence $\Omega_r^2(X)$ contains A. But since H is a supporting hyperplane of $\Omega_r^2(X)$, this is a contradiction.

Theorem 5. A pair (X,Y) is of constant width r if and only if $\Omega_r(X) = Y$ and $\Omega_r(Y) = X$.

Proof. First suppose (X,Y) is a pair of constant width r. Then $|x-y| \leq r$ for all $x \in X$ and all $y \in Y$. Hence $Y \subset \Omega_r(X)$. Assume $\Omega_r(X) \not\subset Y$ and take a point u of $\Omega_r(X)$ not in Y. Let v be the point of Y nearest to u. Then the hyperplane H through v and perpendicular to the line uv supports Y at v. Since the width of

(X,Y) in the direction $\vec{uv}$ is r, there is a supporting hyperplane H' of X parallel to H at distance r apart and lying on the side of H opposite to u. Let w be the contact point of H' with X. Then $|u-w| > |v-w| \geq r$. Since $w \in X$, this implies $u \notin \Omega_r(X)$, a contradiction. Hence $\Omega_r(X) = Y$ and similarly $\Omega_r(Y) = X$.

Now suppose $\Omega_r(X) = Y$ and $\Omega_r(Y) = X$. Then $|x-y| \leq r$ for all $x \in X$ and all $y \in Y$. So the width of (X,Y) in any direction is $\leq r$. Therefore it is enough to show that for any direction $\vec{ou}$, there are two points $x \in X$ and $y \in Y$ such that $|x-y| = r$ and $\vec{yx}$ has the same direction as $\vec{ou}$. Let H be the supporting hyperplane of X with exterior normal $\vec{ou}$, and x_0 be the contact point of H with X. Let B be the closed ball of radius r that touches H at x_0 and lies on the same side of H as X. Then by the lemma, B contains X. Hence the center of B, say y_0, belongs to $\Omega_r(X) = Y$. Clearly $|x_0-y_0| = r$ and $\vec{y_0x_0}$ has the same direction as $\vec{ou}$.

Corollary 1. A set X is of constant width r if and only if $\Omega_r(X) = X$.

The 'only if' part of this corollary is also proved in [1,p.123]. But no mention of the converse is made there.

Corollary 2. For any set X of circumradius $\leq r$, $(\Omega_r(X), \Omega_r^2(X))$ is a pair of constant width r, and $\frac{1}{2}(\Omega_r(X) + \Omega_r^2(X))$ is a set of constant width r.

Use Theorem 5 and the last assertion of Theorem 4.

The diameter of a nonempty bounded set is the least upper bound of the distance between any two points of the set. The following is also proved in [1,p.126].

Corollary 3. Every set of diameter r is contained in a set of constant width r.

Proof. Let X be a set of diameter r. Then since $X \subset \Omega_r(X)$ and $X \subset \Omega_r{}^2(X)$, we have $X \subset \frac{1}{2}(\Omega_r(X) + \Omega_r{}^2(X))$.

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