Journal of Geometry
Vol. 32 (1988)

0047-2468/88/020040-11$1.50+0.20/0

ON CHEN SURFACES IN A MINKOWSKI SPACE TIME

Chorng Shi Houh

$\mathcal{A}$-submanifolds of a pseudo-Euclidean space E_s^{m+1} are considered. A characterization for them is given. A theorem on $\mathcal{A}$-submanifolds contained in a de Sitter space-time S_s^n or an anti-de Sitter space-time H_{s-1}^n is proved. A number of non-trivial examples of $\mathcal{A}$-surfaces in a Minkowski space-time E_1^4 are studied. Some classification theorems are proved for $\mathcal{A}$-surfaces contained in S_1^3 or H^3.

1. PRELIMINARIES.

Let (E_s^{m+1}, g_0) be the flat (m+1)-dimensional pseudo-Euclidean space of signature (s,m+1-s). The metric tensor g_0, if no specified mention is given, is $g_0=-\Sigma_1^s dx_i{}^2+\Sigma_{s+1}^{m+1} dx_j{}^2$ where $(x_1,\dots,x_{m+1})$ is a rectangular coordinate system of E_s^{m+1}. Let $S_s^m=\{x\in E_s^{m+1}/\langle x,x\rangle=1\}$, $H_{s-1}^m=\{x\in E_s^{m+1}/\langle x,x\rangle=-1\}$, where $\langle\ ,\ \rangle$ denotes the inner product on E_s^{m+1}. S_s^m and H_{s-1}^m are called the pseudo-Riemannian sphere and the pseudo-hyperbolic space with their center at the origin of E_s^{m+1}. For s=1 S_1^m is called the de Sitter space-time and H_1^m the anti-de Sitter space-time. Both S_1^m and H_1^m are pseudo-Riemannian manifolds of signature (1,m-1). Let M be an n-dimensional pseudo Riemannian submanifold of E_s^{m+1}. By definition each tangent space $T_x(M)$ is a nondegenerate subspace of $T_x(E_s^{m+1})$ and $T_x(E_s^{m+1})=T_x(M)\oplus T_x^{\perp}(M)$, where the normal space $T_x^{\perp}(M)$ is also nondegenerate. If g_0

induces a Riemannian metric on M then M is called a space-like submanifold. Let $\tilde{\nabla}$ be the metric connection on E_s^{m+1} and ∇ the induced metric connection on M. Let D be the linear connection induced on the normal bundle $T^{\perp}(M)$. Then for any vector fields X,Y tangent to M and any vector field ξ normal to M we have the following Gauss formula and Weingarten formula $\tilde{\nabla}_X Y=\nabla_X Y+h(X,Y)$, $\tilde{\nabla}_X \xi=-A_\xi X+D_X\xi$, where h is the second fundamental form of M in E_s^{m+1} and A_ξ is the Weingarten map with respect to ξ. A_ξ is a self-adjoint endomorphism of the tangent bundle T(M). h and A_ξ are related by

(1.1) $$\langle h(X,Y),\xi\rangle=\langle A_\xi X,Y\rangle.$$

Let $\{e_1,e_2,\ldots,e_{m+1}\}$ be a moving orthonormal frame in E_s^{m+1} along M, with $\{e_1,\ldots,e_n\}$ being tangent to M and $\langle e_j,e_j\rangle=\varepsilon_j=\pm 1$. Let $A_{e_r}=A_r$ for r=n+1,...,m+1 and h_{ij}^r (i,j=1,...,n) be defined by

(1.2) $$A_r e_i=\Sigma\varepsilon_j h_{ij}^r e_j.$$

We also use A_r to denote the matrix (h_{ij}^r):

(1.3) $$A_r=(h_{ij}^r).$$

A_r acts on TM according to (1.2). For A_r and A_s the matrix for the linear transformation A_sA_r

(1.4) $$A_sA_r=(\Sigma\varepsilon_j h_{ij}^r h_{jk}^s).$$

A_ξ can be diagonalized only when M is space-like. For the second fundamental form h the covariant differentiation $\bar{\nabla}_X h$ is defined by $(\bar{\nabla}_X h)(Y,Z)=D_X(h(Y,Z))-h(\nabla_X Y,Z)-h(Y,\nabla_X Z)$, for $X,Y,Z\in TM$. The Codazzi equation of M in E_s^{m+1} is $(\bar{\nabla}_X h)(Y,Z)=(\bar{\nabla}_Y h)(X,Z)$.

A normal vector field ξ is said to be parallel if $D_X\xi=o$ for any $X\varepsilon TM$. If F is an endomorphism of TM, let $F_{ij}=\langle Fe_i,e_j\rangle$ then F is given by the matrix $F=(F_{ij})$. The trace of F is defined by

(1.5) $$TrF=\Sigma\varepsilon_i F_{ii}.$$

By this definition and (1.4) we have

(1.6) $$Tr(A_sA_r)=\Sigma(\varepsilon_j\varepsilon_i h_{ij}^s h_{ji}^r).$$

The mean curvature vector H of M in E_s^{m+1} is defined by

(1.7) $$H=(1/n)Trh=(1/n)\Sigma\varepsilon_i h(e_i,e_i).$$

M is said to be minimal if H=0 and pseudo-umbilical if $\langle H,H\rangle\neq 0$ and $A_H=\lambda I$ for some function λ on M, where I is the identity transformation on TM.

Let M be a pseudo-Riemannian submanifold of S^m_s (or of H^m_{s-1}) in E^{m+1}_s. Let h, h' and $\tilde{h}$ be the second fundamental forms of M in E^{m+1}_s, of M in S^m_s (or in H^m_{s-1}) and of S^m_s (or H^m_{s-1}) in E^{m+1}_s respectively. Let x denote the position vector of M in E^{m+1}_s, H and H' denote the mean curvature vectors of M in E^{m+1}_s and in S^m_s (or in H^m_{s-1}). Then the following relations are known (B-Y.Chen [2], Lemma 1): $h(X,Y)=h'(X,Y)+\tilde{h}(X,Y)$, $H=H'-x$ (or $H=H'+x$), $A_x=\tilde{A}_x=-I$, where $\tilde{A}_x$ denotes the Weigarten map of S^m_s (or H^m_{s-1}) in E^{m+1}_s. By (1.1) and (1.2) we have $h(e_i,e_j)=\Sigma\varepsilon_r h^r_{ij}e_r$, $\langle h(e_i,e_j),e_r\rangle=h^r_{ij}$ and h^r_{ij} are symmetric in i,j.

2. $\mathcal{A}$-SUBMANIFOLDS.

Let M be an n-dimensional pseudo-Riemannian submanifold of E^{m+1}_s. Let ξ be a normal vector field in $T^\perp(M)$ so that $\langle\xi,\xi\rangle\neq 0$. The allied vector field $a(\xi)$ of a normal vector feild ξ is defined by the formula

(2.1) $$a(\xi)=(|\xi|/n)\Sigma\varepsilon_r \mathrm{Tr}(A_{n+1}A_r)e_r$$

where $|\xi|=\langle\xi,\xi\rangle^{1/2}$, $\{e_{n+1}=\xi/|\xi|, e_{n+2},\dots,e_{m+1}\}$ is an orthonormal basis for $T^\perp(M)$.

DEFINITION 1. A pseudo-Riemannian submanifold M in E^{m+1}_s is called an $\mathcal{A}$-submanifold or a Chen submanifold if its mean curvature vector H satisfied that $H=0$ or $\langle H,H\rangle\neq 0$ and $a(H)=0$. The notion of an $\mathcal{A}$-submanifold in a pseudo-Riemannian manifold M is defined similarly. Riemannian $\mathcal{A}$-submanifolds were first considered by B-Y. Chen in [1] and developed by other authors (for example see [3]) and subsequently were called Chen submanifolds. The definition given above is a pseudo-Riemannian version of Chen's definition. The class of Chen submanifolds of a Riemannian manifold contains all minimal and pseudo-umbilical submanifolds which are said to be trivial Chen submanifolds ([3]).

Let M be an n-dimensional pseudo-Riemannian submanifold in S^m_s(or in H^m_{s-1}). Let $\{e_1,\dots,e_n,e_{n+1},\dots,e_{m+1}\}$ be an orthonormal frame along M so that $e_1,\dots,e_n$ are tangent to M and $e_{m+1}=-x$, where x is the position vector of S_s^m (or H_{s-1}^m) in E_s^{m+1}. We denote $\varepsilon_r=\langle e_r,e_r\rangle$, $(r=n+1,\dots,m+1)$, $\varepsilon=\varepsilon_{m+1}$. $\varepsilon=1$ for S_s^m and $\varepsilon=-1$ for H_{s-1}^m. If the mean curvature vector H' of M in S_s^m (or in H_{s-1}^m) satisfies $\langle H',H'\rangle\neq 0$, then we may choose $e_{n+1}=H'/|H'|$ or $H'=\alpha' e_{n+1}$ with $|H'|=\alpha'\neq 0$. Then $H=H'-\varepsilon x=\alpha' e_{n+1}+\varepsilon e_{m+1}$ and $|\langle H,H\rangle|=|\alpha'^2\varepsilon_{n+1}+\varepsilon|=\alpha^2$, $\alpha\geq 0$. Now for points of M at which $\alpha\neq 0$, let

$$e'_{n+1}=(1/\alpha)H=(1/\alpha)(\alpha' e_{n+1}+\varepsilon e_{m+1}),\quad A'_{n+1}=A_{e'_{n+1}},$$
$$e'_{m+1}=(1/\alpha)(\varepsilon_{n+1}e_{n+1}-\alpha' e_{m+1}),\quad A'_{m+1}=A_{e'_{m+1}}.$$

Then

$$a(H)=(|H|/n)(\Sigma\varepsilon_r \mathrm{Tr}(A'_{n+1}A_r)e_r+\varepsilon'_{m+1}\mathrm{Tr}(A'_{n+1}A'_{m+1})e'_{m+1},\ \varepsilon'_{m+1}=\pm\varepsilon\varepsilon_{m+1}.$$

Thus M is a Chen submanifold in E_s^{m+1} if and only if $a(H)=0$, that is $\mathrm{Tr}(A'_{n+1}A_r)=0$ $(r=n+2,\dots,m)$ and $\mathrm{Tr}(A'_{n+1}A'_{m+1})=0$. We define the following operator $\tilde{A}$: $T^\perp(M)\longrightarrow T^\perp(M)$.

DEFINITION 2. $\tilde{A}(\xi)=\Sigma\langle h(e_i,e_j),\xi\rangle\varepsilon_i\varepsilon_j h(e_i,e_j)$, $\xi\in T^\perp(M)$.

It is easy to show that $\tilde{A}$ is defined independently of the choices of the orthonormal frames $\{e_1,\dots,e_n\}$ in TM. By the above definition for a normal vector $\xi=\Sigma\varepsilon_r\xi_r e_r$,

$$\begin{aligned}(2.2)\qquad \tilde{A}(\xi)&=\Sigma\varepsilon_r\varepsilon_i\varepsilon_j\xi_r h_{ij}^r h(e_i,e_j)=\Sigma\varepsilon_r\varepsilon_s\varepsilon_i\varepsilon_j\xi_r h_{ij}^r h_{ij}^s e_s\\ &=\Sigma\varepsilon_r\varepsilon_s\xi_r \mathrm{Tr}(A_rA_s)e_s.\end{aligned}$$

Especially if $\xi=H$, the mean curvature vector of M in E_s^{m+1} is $e_{n+1}=H/|H|$, then $\tilde{A}(H)=\varepsilon_{n+1}|H|\Sigma\varepsilon_s \mathrm{Tr}(A_{n+1}A_s)e_s$.

Since M is a Chen submanifold in E_s^{m+1} if and only if $\mathrm{Tr}(A_{n+1}A_s)=0$ for $s>n+1$, we have the following Lemma which is proved in [3] for Riemannian case.

LEMMA. M, with $\langle H,H\rangle\neq 0$, is a Chen submanifold in E_s^{m+1} if and only if $\tilde{A}(H)$ is parallel to H.

The definition of the operator $\tilde{A}$ and the Lemma remain valid when we consider M as a submanifold of a pseudo-Riemannian manifold M instead of E_s^{m+1}. Now let M be a submanifold in S_s^m (or H_{s-1}^m) in E_s^{m+1}. We have an operator $\tilde{A}$ for M in E_s^{m+1}. Let the operator of M in S_s^m (or H_{s-1}^m) corresponding to $\tilde{A}$ of M in E_s^{m+1} be $\tilde{A}'$. Let h, h' and $\tilde{h}$ be those considered in section 1. Let $\{e_1,\dots,e_n,e_{n+1},\dots,e_m,e_{m+1}=-x\}$ be the orthonormal frame along M considered above. Then we have $h(e_i,e_j)=h'(e_i,e_j)+\tilde{h}(e_i,e_j)$, where $\tilde{h}(e_i,e_j)$

$=-\varepsilon\langle e_i, e_j\rangle x$. From this relation it is easy to see that the Weingarten's map $A'_r = A'_{e_r}$ of M in S^m_s (or in H^m_{s-1}) and A_r of M in E^{m+1}_s satisfy

(2.3) $$A_r = A'_r \ (n+1 \le r \le m), \quad A_{m+1} = I = \text{identity}.$$

Now we can come up with a relation between $\tilde{A}(H)$ and $\tilde{A}(H')$. In fact taking $H' = |H'| e_{n+1} = \alpha' e_{n+1}$, $H = H' - \varepsilon x = H' + \varepsilon e_{m+1}$, $\tilde{A}(H) = \tilde{A}(\alpha' e_{n+1} + \varepsilon e_{m+1}) = \Sigma^{m+1}_{n+1} \varepsilon_{n+1} \varepsilon_s \alpha' \mathrm{Tr}(A_{n+1} A_s) e_s + \Sigma^{m+1}_{n+1} \varepsilon_s \mathrm{Tr}(A_{m+1} A_s) e_s = \alpha' \mathrm{Tr}(A^2_{n+1}) e_{n+1} + \Sigma^{m+1}_{n+2} \alpha' \varepsilon_{n+1} \varepsilon_s \mathrm{Tr}(A_{n+1} A_s) e_s + \varepsilon_{n+1} \varepsilon \alpha' \mathrm{Tr}(A_{n+1}) e_{m+1} + \Sigma^{m+1}_{n+1} \varepsilon_s \mathrm{Tr}(A_s) e_s$. Since $H' = (1/n) \Sigma \varepsilon_i h'(e_i, e_i) = (1/n) \Sigma^m_{n+1} \varepsilon_r \mathrm{Tr}(A'_r) e_r$, we have $\mathrm{Tr}(A'_r) = 0$ for $r = n+2, \ldots, m$ and $\mathrm{Tr}(A_{n+1}) = \mathrm{Tr}(A'_{n+1}) = n\varepsilon_{n+1} \alpha'$. Thus $\tilde{A}(H) = \alpha' \mathrm{Tr}(A^2_{n+1}) e_{n+1} + \Sigma^m_{n+2} \varepsilon_{n+1} \varepsilon_s \alpha' \mathrm{Tr}(A_{n+1} A_s) e_s + n\varepsilon \alpha'^2 e_{m+1} + nH' + n\varepsilon e_{m+1}$. On the other hand $\tilde{A}(H') = \Sigma^m_{n+1} \varepsilon_{n+1} \varepsilon_s \alpha' \mathrm{Tr}(A'_{n+1} A'_s) e_s = \alpha' \mathrm{Tr}(A'^2_{n+1}) e_{n+1} + \varepsilon_{n+1} \Sigma^m_{n+2} \alpha' \varepsilon_s \mathrm{Tr}(A'_{n+1} A'_s) e_s$. We then have

(2.4) $$\tilde{A}(H) = (\tilde{A}'(H') + nH') - n\varepsilon(\alpha'^2 + 1)x.$$

Thus if $\tilde{A}(H)$ is parallel to $H = H' - \varepsilon x$ then $\tilde{A}'(H')$ is parallel to H'. We have the following theorem.

THEOREM 1. Let M be a (pseudo-Riemannian) submanifold of S^m_s (or H^m_{s-1}) in E^{m+1}_s. If M is a Chen submanifold in E^{m+1}_s then M is a Chen submanifold in S^m_s (or H^m_{s-1}).

3. EXAMPLES OF $\mathcal{A}$-SURFACES IN E^4_1.

In Gheysens, Verheyen and Verstraelen [3], a series of examples of $\mathcal{A}$-surfaces in E^4 is given. Here we consider some examples of their pseudo-Riemannian version.

Example 1. In E^4_1 with $g_0 = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$, let $f(u)$, $g(u)$ be differentiable functions satisfying $f'^2 - g'^2 > 0$ and $\alpha(v)$, $\beta(v)$ are differentiable functions. Let M be the surface in E^4_1 given by

(3.1) $x(u,v) = (f(u)\mathrm{ch}\alpha(v), f(u)\mathrm{sh}\alpha(v), g(u)\cos\beta(v), g(u)\sin\beta(v))$.

Consider the following orthonormal frame $\{e_1, e_2, e_3, e_4\}$ along M so that $e_1, e_2 \in TM$:

$e_1 = (1/(f'^2 - g'^2)^{1/2})(f'\mathrm{ch}\alpha,\ f'\mathrm{sh}\alpha,\ g'\cos\beta,\ g'\sin\beta)$, $\langle e_1, e_1\rangle = -1$,

$e_2 = (1/(f^2\alpha'^2 + g^2\beta'^2)^{1/2})(f\alpha'\mathrm{sh}\alpha,\ f\alpha'\mathrm{ch}\alpha,\ -g\beta'\sin\beta,\ g\beta'\cos\beta)$,

$\langle e_2, e_2\rangle=1$,
$e_3=(1/(f'^2-g'^2)^{1/2})(g'\mathrm{ch}\alpha,\ g'\mathrm{sh}\alpha,\ f'\cos\beta,\ f'\sin\beta)$, $\langle e_3, e_3\rangle=1$,
$e_4=(1/(f^2\alpha'^2+g^2\beta'^2)^{1/2})(g\beta'\mathrm{sh}\alpha,\ g\beta'\mathrm{ch}\alpha,\ f\alpha'\sin\beta,\ -f\alpha'\cos\beta)$,
$\langle e_4, e_4\rangle=1$.

By straightforward computation we found that h^r_{ij} are given by the following expressions:

$h^3_{11}=(f'g''-g'f'')/(f'^2-g'^2)^{3/2}$, $h^3_{12}=0$,
$h^3_{22}=(-fg'\alpha'^2-f'g\beta'^2)/(f'^2-g'^2)^{1/2}(f^2\alpha'^2+g^2\beta'^2)$, $h^4_{11}=0$,
$h^4_{12}=\alpha'\beta'(gf'-fg')/(f'^2-g'^2)^{1/2}(f^2\alpha'^2+g^2\beta'^2)$,
$h^4_{22}=fg(\alpha''\beta'-\alpha'\beta'')/(f^2\alpha'^2+g^2\beta'^2)^{3/2}$.

If α, β, satisfy $\alpha''\beta'-\alpha'\beta''=0$ then $h^4_{22}=0$, $H=(1/2)(-h(e_1,e_1)+h(e_2,e_2))=(1/2)(-h^3_{11}+h^3_{22})e_3$. Hence e_3 is the direction of H. For this situation $\mathrm{Tr}(A_3A_4)=0$, so M is a Chen surface.

The following are special cases of example 1.

Example 1A. In example 1 let $f(u)=u, g(u)=1, \alpha(v)=\beta(v)=v$. Then $f'^2-g'^2=1>0$, $x(u,v)=(u\mathrm{ch}v, u\mathrm{sh}v, \cos v, \sin v)$ and $h^3_{11}=0, h^3_{12}=0, h^3_{22}=-1/(u^2+1)$, $h^4_{11}=0, h^4_{12}=1/(u^2+1), h^4_{22}=0$. $H=(1/2)\Sigma\varepsilon_i h(e_i,e_i)=\{-1/2(u^2+1)\}e_3$. This Chen surface in E^4_1 is neither minimal nor pseudo-umbilical.

Example 1B. In example 1 let $f(u)=\mathrm{sh}u$, $g(u)=\mathrm{ch}u$, $\alpha(v)=\beta(v)=v$. Then $f'^2-g'^2=1>0$, $x(u,v)=(\mathrm{sh}u\,\mathrm{ch}v, \mathrm{sh}u\,\mathrm{sh}v, \mathrm{ch}u\cos v, \mathrm{ch}u\sin v)$ and

$$h^3_{11}=1,\ h^3_{12}=0,\ h^3_{22}=-1,\ h^4_{11}=0,\ h^4_{12}=\mathrm{sech}2u,\ h^4_{22}=0,$$
$$H=(1/2)\Sigma\varepsilon_i h(e_i,e_i)=-e_3.$$

Since $A_3=-1$, this Chen surface is pseudo-umbilical but not minimal. Furthermore, $\langle x,x\rangle=1$, so $M\subset S^3_1\subset E^4_1$. The matrix A_3 has double eigenvalue with repect to the induced metric in M.

Example 2. In E^4_1 with $g_0=dx_1^2+dx_2^2+dx_3^2-dx_4^2$, let $f(u), g(u)$ be differentiable functions satisfying $f'^2-g'^2>0$ and $\alpha(v), \beta(v)$ are differentiable functions. Let M be a surface in E^4_1 given by

(3.2) $x(u,v)=(f(u)\cos\alpha(v), f(u)\sin\alpha(v), g(u)\mathrm{sh}\beta(v), g(u)\mathrm{ch}\beta(v))$.

Consider the following orthonormal frames $\{e_1,e_2,e_3,e_4\}$ along M so that $e_1, e_2\in TM$:

$e_1=(1/(f'^2-g'^2)^{1/2})(f'\cos\alpha, f'\sin\alpha, g'\mathrm{sh}\beta, g'\mathrm{ch}\beta)$, $\langle e_1, e_1\rangle=1$,
$e_2=(1/(f^2\alpha'^2+g^2\beta'^2)^{1/2})(-f\alpha'\sin\alpha, f\alpha'\cos\alpha, g\beta'\mathrm{ch}\beta, g\beta'\mathrm{sh}\beta)$,
$\langle e_2, e_2\rangle=1$,
$e_3=(1/(f'^2-g'^2)^{1/2})(g'\cos\alpha, g'\sin\alpha, f'\mathrm{sh}\beta, f'\mathrm{ch}\beta)$, $\langle e_3, e_3\rangle=-1$,

$e_4=(1/(f^2\alpha'^2+g^2\beta'^2)^{1/2})(g\beta'\sin\alpha,-g\beta'\cos\alpha,f\alpha'\mathrm{ch}\beta,f\alpha'\mathrm{sh}\beta)$,

$<e_4,e_4>=1$.

By straightforward computation we found that h^r_{ij} are given by the following expressions:

$$h^3_{11}=(f'g''-g'f'')/(f'^2-g'^2)^{3/2},\quad h^3_{12}=0,$$
$$h^3_{22}=(fg'\alpha'^2+gf'\beta'^2)/(f'^2-g'^2)^{1/2}(f^2\alpha'^2+g^2\beta'^2),\quad h^4_{11}=0,$$
$$h^4_{12}=\alpha'\beta'(fg'-gf')/(f'^2-g'^2)^{1/2}(f^2\alpha'^2+g^2\beta'^2),$$
$$h^4_{22}=fg(\alpha'\beta''-\alpha''\beta')/(f^2\alpha'^2+g^2\beta'^2)^{3/2}.$$

If α, β satisfy $\alpha'\beta''-\alpha''\beta'=0$ then $h^4_{22}=0$ and $H=(1/2)\times(h(e_1,e_1)+h(e_2,e_2))=(1/2)(h^3_{11}+h^3_{22})e_3$, thus $\mathrm{Tr}(A_3A_4)=0$ and M is a spacelike Chen surface in E^4_1.

The following are special cases of example 2.

Example 2A. In example 2 let $f(u)=u$, $g(u)=1$, $\alpha(v)=\beta(v)=v$. Then $f'^2-g'^2=1>0$. $x(u,v)=(u\cos v,u\sin v,\mathrm{sh}v,\mathrm{ch}v)$, $h^3_{11}=0$, $h^3_{12}=0$, $h^3_{22}=1/(u^2+1)$, $h^4_{11}=0$, $h^4_{12}=-1/(u^2+1)$, $h^4_{22}=0$. This is a spacelike Chen surface in E^4_1, neither minimal nor pseudo-umbilical in E^4_1.

Example 2B. In example 2 let $f(u)=\mathrm{sh}u$, $g(u)=\mathrm{ch}u$, $\alpha(v)=\beta(v)=v$, then $f'^2-g'^2=1>0$. $x(u,v)=(\mathrm{sh}u\cos v,\mathrm{sh}u\sin v,\mathrm{ch}u\,\mathrm{sh}v,\mathrm{ch}u\,\mathrm{ch}v)$, $h^3_{11}=0$, $h^3_{12}=0$, $h^3_{22}=1$, $h^4_{11}=0$, $h^4_{12}=-\mathrm{sech}2u$, $h^4_{22}=0$, $H=(1/2)(h(e_1,e_1)+h(e_2,e_2))=-e_3$. Since $A_3=I$ this spacelike Chen surface is pseudo-umbilical but not minimal in E^4_1. Furthermore since $<x,x>=-1$, $M\subset H^3_0\subset E^4_1$. The matrix A_3 has double eigenvalues with respect to the induced metric in M. Let H' be the mean curvature vector of M in H^3_0. For this example $x=e_3$ and $H=H'-x=-e_3$, thus $H'=0$ and M is minimal in H^3_0.

4. $\mathcal{A}$-SURFACES IN S^3_1 (OR IN H^3_0) IN E^4_1.

In this section we consider a Chen surface M in S^3_1 (or in H^3_0) in E^4_1. First let M be a surface in $H^3_0\subset E^4_1$. Then M is spacelike and the mean curvature vector H' of M in H^3_0 is also spacelike. Let $H'=\alpha' e_3$ and $e_4=-x$. Then there is an orthonormal frame $\{e_1,e_2\}$ along M with $<e_1,e_1>=<e_2,e_2>=1$ so that A_{e_3} is diagonalized:

$$A_3=A_{e_3}=\begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{bmatrix},\quad A_4=A_{e_4}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad <e_3,e_3>=-<e_4,e_4>=1$$

The mean curvature vector H of M in E_1^4 is $H=H'-e_4=\alpha'e_3-e_4=(1/2)(\alpha_1+\alpha_2)e_3-e_4$. Since our attention is on non-minimal Chen surfaces in E_1^4, we may assume that $<H,H>\neq 0$. Then $|H|=\{(1/4)(\alpha_1+\alpha_2)^2-1\}^{1/2}\neq 0$. Let $H^\perp=e_3-(1/2)(\alpha_1+\alpha_2)e_4$. Then $H^\perp$ is perpendicular to H and $|H|=|H^\perp|$. Let e_3', e_4' be the unit vectors in the directions of H and $H^\perp$. $H=|H|e_3'$, $H^\perp=|H^\perp|e_4'$. Then $\{e_3',e_4'\}$ is an orthonormal frame in $T^\perp(M)$ with $<e_3',e_3'>=-<e_4',e_4'>$ and

$$A_3'=A_{e_3'}=(1/|H|)\begin{bmatrix}(1/2)\alpha_1(\alpha_1+\alpha_2)-1 & 0\\ 0 & (1/2)\alpha_2(\alpha_1+\alpha_2)-1\end{bmatrix},$$

$$A_4'=A_{e_4'}=(1/|H|)\begin{bmatrix}(1/2)(\alpha_1-\alpha_2) & 0\\ 0 & (1/2)(\alpha_2-\alpha_1)\end{bmatrix}.$$

The condition for M being a Chen surface is $Tr(A_3'A_4')=(1/4)\times(\alpha_1-\alpha_2)^2(\alpha_1+\alpha_2)=0$. That is $\alpha_1=\pm\alpha_2$. If $\alpha_1=-\alpha_2$, $H'=0$. If $\alpha_1=\alpha_2$ then M is umbilical in H_0^3. For this case let $\alpha=\alpha_1=\alpha_2$. Using the Codazzi equation $(\bar\nabla_{e_1}h)(e_1,e_2)=(\bar\nabla_{e_2}h)(e_1,e_1)$ for the frame $\{e_1, e_2,e_3,e_4\}$ we obtain that the connection form $\omega_3^2(e_2)=0$, $e_2(\alpha)=0$. Using the Codazziequation $(\bar\nabla_{e_1}h)(e_2,e_2)=(\bar\nabla_{e_2}h)(e_1,e_2)$ we obtain that $\omega_3^4(e_1)=0$ and $e_1(\alpha)=0$. Thus α is a constant and e_3, e_4 are parallel in the normal bundle. Hence H is parallel in the normal bundle. Furthermore $A_3'=(1/|H|)(\alpha^2-1)I$ shows that M is pseudo-umbilical in E_1^4. By the Limma 2 in [2] we conclude that M is minimal in H_0^3.Next let M be a spacelike surface in $S_1^3\subset E_1^4$. Let $H'=\alpha'e_3$ be the mean curvature vector of M in S_1^3 and $e_4=-x$. Then H' is timelike and $A_3=A_{e_3}$ is diagonalized:

$$A_3=\begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{bmatrix},\quad A_4=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad <e_3,e_3>=-1,\quad <e_4,e_4>=1.$$

The same argument as above yields that M is minimal in S_1^3. We now have the following theorem.

THEOREM 2. M is a spacelike surface in S_1^3 (or in H_0^3) in E_1^4. Then M is minimal in S_1^3 (or in H_0^3) if and only if M is a

Chen surface.

Now consider a surface M which is pseudo-Riemannian with signature (1,1) so that $M\subset S_1^3\subset E_1^4$. The mean curvature vector H' of M in S_1^3 is spacelike. Let $H'=\alpha' e_3$, e_3 is a unit vector. $A_3=A_{e_3}$ may not be diagonalizable. However according to Petrov [4], A_3 can be put into one of the following three forms with repect to an orthonormal frame $\{e_1, e_2\}$ on M with the given inner product.

Case 1. $A_3=\begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{bmatrix}$, $\alpha_1\neq\alpha_2$, $\langle e_1, e_1\rangle=-\langle e_2, e_2\rangle=-1$,

Case 2. $A_3=\begin{bmatrix}0 & \alpha\\ \alpha & 0\end{bmatrix}$, $\langle e_1, e_1\rangle=\langle e_2, e_2\rangle=0$, $\langle e_1, e_2\rangle=1$,

Case 3. $A_3=\begin{bmatrix}\alpha & \beta\\ \beta & -\alpha\end{bmatrix}$, $\beta\neq 0$, $\langle e_1, e_1\rangle=-\langle e_2, e_2\rangle=1$.

These cases are devided according to the eigenvalues of A_3 with respect to the induced pseudo-Riemannian metric in M. The case 1 is for A_3 having two different real eigenvalues α_1 and α_2. The case 2 is for A_3 having a realdouble eigenvalue α and the case 3 is for A_3 having complex eigenvalues $\alpha+\beta i$.

When case 1 takes place we have

$$A_3=\begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{bmatrix},\quad A_4=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},\quad \alpha_1\neq\alpha_2,\quad \langle e_1, e_1\rangle=-\langle e_2, e_2\rangle=-1.$$

$H'=(1/2)(-\alpha_1+\alpha_2)e_3$, $\alpha'=(-\alpha_1+\alpha_2)/2$, $H=\alpha' e_3+e_4$. Now let $e_3'=H/|H|=(\alpha' e_3+e_4)/(\alpha'^2+1)^{1/2}$, $e_4'=(e_3-\alpha' e_4)/(\alpha'^2+1)^{1/2}$. Then

$$A_{e_3'}=1/(\alpha'^2+1)^{1/2}\begin{bmatrix}(1/2)\alpha_1(-\alpha_1+\alpha_2)-1 & 0\\ 0 & (1/2)\alpha_2(-\alpha_1+\alpha_2)+1\end{bmatrix},$$

$$A_{e_4'}=1/(\alpha'^2+1)^{1/2}\begin{bmatrix}(1/2)(\alpha_1+\alpha_2) & 0\\ 0 & (1/2)(\alpha_1+\alpha_2)\end{bmatrix}.$$

Suppose M is a Chen surface; then $Tr(A_{e_3'}A_{e_4'})=0$. This implies that $(\alpha_2-\alpha_1)(\alpha_1+\alpha_2)^2=0$. Thus we have $\alpha_1=-\alpha_2=-\alpha'$. Again we apply the Codazzi equation $(\bar\nabla_{e_2}h)(e_1, e_1)=(\bar\nabla_{e_1}h)(e_2, e_1)$. Noticing that

$\omega_2^1=\omega_1^2$ for this case where ω_1^2 is the connection form for M, we obtain that $e_2(\alpha_1)=0$ and $\omega_3^4(e_2)=0$. Similarly from $(\bar{\nabla}_{e_1}h)(e_2,e_2)=(\bar{\nabla}_{e_2}h)(e_1,e_2)$ we obtain $e_1(\alpha_1)=0$ and $\omega_3^4(e_1)=0$. Hence α_1, α' are constants and $H=\alpha' e_3+e_4$ is parallel in the normal bundle. $A_{e_3'}=(\alpha'^2+1)^{1/2}I$ implies that M is pseudo-umbilical in E_1^4. Again by Lemma 2 of [2] we conclude that M is minimal in S_1^3.

When case 2 takes place we have

$$A_3=\begin{bmatrix}0 & \alpha\\ \alpha & 0\end{bmatrix},\quad \langle e_1,e_1\rangle=0,\quad \langle e_1,e_2\rangle=1,\quad \langle e_2,e_2\rangle=0.$$

We use a new orthonormal frame $\{e_1',e_2'\}$ along M so that $e_1'=(1/2^{1/2})(e_1+e_2)$, $e_2'=(1/2^{1/2})(e_1-e_2)$. Then $\langle e_1',e_1'\rangle=1, \langle e_2',e_2'\rangle=-1$. With respect to this frame let the Weingarten map in the direction e_3 (the direction of H') and e_4 be $\bar{A}_3$ and $\bar{A}_4$. Then

$$\bar{A}_3=\begin{bmatrix}\alpha+1/2 & -1/2\\ -1/2 & -\alpha+1/2\end{bmatrix},\quad \bar{A}_4=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad \langle e_1',e_1'\rangle=-\langle e_2',e_2'\rangle=1.$$

With respect to the frame $\{e_1',e_2'\}$ we find that $H=\alpha e_3+e_4$. Thus α is the mean curvature of M in S_1^3. Let $e_3'=H/|H|$ and $e_4'=(e_3-\alpha e_4)/|H|$, $|H|=(\alpha^2+1)^{1/2}$. Let $\bar{A}_3'$ and $\bar{A}_4'$ be the Weingarten maps in the directions of e_3' and e_4' with respect to the frame $\{e_1',e_2'\}$. Then

$$\bar{A}_3'=1/(\alpha^2+1)^{1/2}\begin{bmatrix}\alpha(\alpha+1/2)+1 & -(1/2)\alpha\\ -(1/2)\alpha & \alpha(-\alpha+1/2)-1\end{bmatrix},\quad \bar{A}_4'=1/(\alpha^2+1)^{1/2}\begin{bmatrix}1/2 & -1/2\\ -1/2 & 1/2\end{bmatrix}.$$

By (1.6) we obtain $Tr(\bar{A}_3'\bar{A}_4')=0$. This shows that M is a Chen surface. That is if $M\subset S_1^3\subset E_1^4$, M is a pseudo-Riemannian surface with signature (1,1) and A_3 has real double eigenvalue, then M is a Chen surface.

When case 3 takes place we have

$$A_3=\begin{bmatrix}\alpha & \beta\\ \beta & -\alpha\end{bmatrix},\ \beta\neq 0,\quad A_4=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad \langle e_1,e_1\rangle=-\langle e_2,e_2\rangle=1.$$

$H=(1/2)(h(e_1,e_1)-h(e_2,e_2))=\alpha e_3+e_4$. So α is the mean curvature of M in S_1^3. Let $H^{\perp}=e_3-\alpha e_4$, $e_3'=H/|H|=\{1/(\alpha^2+1)^{1/2}\}(\alpha e_3+e_4)$, $e_4'=H^{\perp}/|H^{\perp}|=\{1/(\alpha^2+1)^{1/2}\}(e_3-\alpha e_4)$. Then

$$A_{e_3'}=1/(\alpha^2+1)^{1/2}\begin{bmatrix}\alpha^2+1 & \alpha\beta\\ \alpha\beta & -\alpha^2-1\end{bmatrix},\quad A_{e_4'}=1/(\alpha^2+1)^{1/2}\begin{bmatrix}0 & \beta\\ \beta & 0\end{bmatrix}.$$

By (1.6) $Tr(A_{e_3'}A_{e_4'})=-2\alpha\beta^2/(\alpha^2+1)$. Hence if M is a Chen surface then $\alpha=0$. M is minimal in S_1^3.

Combining the above results we have the following theorem.

THEOREM 3. Let M be a pseudo-umbilical surface of signature (1,1) in $S_1^3 \subset E_1^4$, e_3 be the direction of the mean curvature vector of M in S_1^3. If A_{e_3} has a double eigenvalue then M is a Chen surface. If A_{e_3} has two different eigenvalues and M is a Chen surface then M is minimal in S_1^3.

Example 1B in section 3 is a Chen surface in $S_1^3 \subset E_1^4$ which is pseudo-umbilical in E_1^4 but not minimal in S_1^3. A_{e_3} has duble eigenvalue.

REFERENCES

[1] B-Y. Chen, Geometry of submanifolds, Marcel Dekker, New York, 1973.

[2] B-Y. Chen, Finite type submanilolds in pseudo-Euclidean spaces and applictions, Kodai Math. J. 8(1985), 358-374.

[3] L. Gheysens, P. Verheyen and L. Verstraelen, Characterization and examples of Chen submanifolds, Jour. of Geometry, 20(1983), 47-62.

[4] A. Z. Petrov, Einstein spaces, Pergamon Press, 1969.

Department of Mathematics
Wayne State University
Detroit, Michigan, 48202
U. S. A.

(Eingegangen am 8. August 1986)