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# Quantum $R$ Matrix for the Generalized Toda System

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**Abstract.** We report the explicit form of the quantum $R$ matrix in the fundamental representation for the generalized Toda system associated with non-exceptional affine Lie algebras.

## 1. Introduction

It has been known for some time that the Yang–Baxter (YB) equations play a crucial rôle in classical and quantum integrable systems (see e.g. [1]). The structure of the classical YB equation is now fairly well understood [2–3]. In ref. [3] a classification of non-degenerate solutions related to simple Lie algebras is given, subject to the unitarity condition. Unfortunately such classification is yet unavailable in the quantum case. One of the consequences of [3] is that the trigonometric solutions, up to certain equivalence, are finite in number, and that they allow a neat description in terms of Dynkin diagrams. An immediate question would be whether it is possible to quantize all these solutions. The most typical ones among them are the classical solutions associated with the generalized Toda system (GTS). In this paper we report on the corresponding quantum solutions for the case of non-exceptional affine Lie algebras.

To be more specific, we consider the solutions $r(x)$ of the classical YB equation

$$[r^{12}(x), r^{13}(xy)] + [r^{12}(x), r^{23}(y)] + [r^{13}(xy), r^{23}(y)] = 0 \tag{1.1}$$

for the GTS of type $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$ and $D_{n+1}^{(2)}$, as given in Eq. (2.3), (3.1–4). Here the notations are standard: $r(x)$ is a $\mathfrak{G} \otimes \mathfrak{G}$-valued rational function, $\mathfrak{G}$ being a finite dimensional simple Lie algebra, and $r^{12}(x) = r(x) \otimes I$, etc. The problem is to find an $R(x) = R(x, \hbar)$ containing an arbitrary parameter $\hbar$, such that (i) it satisfies the quantum YB equation

$$R^{12}(x)R^{13}(xy)R^{23}(y) = R^{23}(y)R^{13}(xy)R^{12}(x), \tag{1.2}$$

and (ii) as $\hbar \to 0$,

$$R(x, \hbar) = \kappa(x, \hbar)(I + \hbar r(x) + \cdots) \tag{1.3}$$

holds with some scalar $\kappa(x, \hbar)$. In contrast to the classical case (1.1), the quantum

Eq. (1.2) is formulated for a function $R(x)$ with values in $\mathfrak{U}(\mathfrak{G})\otimes\mathfrak{U}(\mathfrak{G})$, where $\mathfrak{U}(\mathfrak{G})$ denotes the universal enveloping algebra of $\mathfrak{G}$. Existence of such a solution would imply that for any finite dimensional representation $V_i$ $(i=1,2,3)$ of $\mathfrak{G}$ there correspond matrices $R^{ij}(x)\in\mathrm{End}(V_i\otimes V_j)$ satisfying (1.2)[4]. The main result of the present article is the explicit construction of $R(x)\in\mathrm{End}(V\otimes V)$, taking $V_1=V_2=V_3=V$ to be the fundamental representation. Construction of the "universal" $(=\mathfrak{U}(\mathfrak{G})\otimes\mathfrak{U}(\mathfrak{G})$-valued) solution is an interesting future problem (cf. [4, 5]).

The method of construction is described in Sect. 2. The line of arguments essentially follows that of ref. [5] (except for the examination of the sufficiency part). In Sect. 3 explicit forms of solutions are presented. The solutions for the type $A_n^{(1)}$[6] and $A_2^{(2)}$[7] have been known. The quantum "spin" Hamiltonians obtained as the first log derivative of the transfer matrix are also given.

## 2. The GTS and the YB Equation

First let us recall the formulation of the GTS and the corresponding classical $r$-matrix. Let $\hat{\mathfrak{G}}$ be an affine Lie algebra, and $\mathfrak{h}$ be a Cartan subalgebra thereof. The GTS associated with $\hat{\mathfrak{G}}$ is the following equation for a $\mathfrak{h}$-valued function $q=q(t)$ [8]:

$$q_{tt}=-\nabla_q U,\quad U=\sum_{\alpha\in\pi}e^{2\alpha(q)}.$$

Here $\pi$ denotes the set of simple roots of $\hat{\mathfrak{G}}$. It is known to be representable in the Lax form $L_t=[A,L]$. In terms of the standard Chevalley basis $\{e_\alpha, f_\alpha, h_\alpha\}$, $L$ and $A$ are given by [8]

$$\begin{aligned}L&=p+e^{\mathrm{ad}q}e+e^{-\mathrm{ad}q}f,\\ A&=-e^{\mathrm{ad}q}e+e^{-\mathrm{ad}q}f,\end{aligned}\tag{2.1}$$

where $p=q_t\in\mathfrak{h}$, $e=\sum_{\alpha\in\pi}e_\alpha$ and $f=\sum_{\alpha\in\pi}f_\alpha$.

In order to describe the corresponding classical $r$-matrix, we employ the homogeneous picture of $\hat{\mathfrak{G}}$ (cf. [9]). We find it simpler than the principal picture adopted in [3], for then the degree of the rational function $r(x)$ will become independent of the rank of $\hat{\mathfrak{G}}$. Thus let $\mathfrak{G}$ be a complex finite-dimensional simple Lie algebra, and let $\sigma$ be its diagram automorphism of order $k(=1,2,3)$. Put $\mathfrak{G}_j=\{X\in\mathfrak{G}/\sigma(X)=\omega^jX\}$, where $\omega$ is a primitive $k^{\text{th}}$ root of unity. Let $\mathfrak{G}_j=\bigoplus_{\alpha\in\Delta_j}\mathfrak{G}_{j,\alpha}$ be its root space decomposition with respect to a Cartin subalgebra $\mathscr{h}_0$ of $\mathfrak{G}_0$. Fixing an invariant bilinear form (,) on $\mathfrak{G}$, we choose $X_{j,\alpha}\in\mathfrak{G}_{j,\alpha}$, and normalize them as $(X_{j,\alpha}, X_{-j,-\alpha})=1$. We write $E_\alpha=X_{0,\alpha}$, $F_\alpha=X_{0,-\alpha}$ $(\alpha\in\pi_0)$, $E_0=X_{1,-\theta}$ and $F_0=X_{-1,\theta}$, where $\pi_0$ is the set of simple roots of $\mathfrak{G}_0$ and $\theta$ denotes the highest weight of $\mathfrak{G}_0$ in $\mathfrak{G}_{-1}$. As is well known [10], if $\mathfrak{G}$ is of type $X_N$, then the loop algebra $\mathfrak{G}^{(k)}[\lambda,\lambda^{-1}]=\bigoplus_{j\in\mathbb{Z}}\lambda^j\mathfrak{G}_{j\,\mathrm{mod}\,k}$ gives a realization of the affine Lie algebra of type $X_N^{(k)}$ modulo the center. In this picture the Chevalley basis is given by

$$e_0=\lambda E_0,\ f_0=\lambda^{-1}F_0,\ e_\alpha=E_\alpha,\ f_\alpha=F_\alpha\quad(\alpha\in\pi_0).\tag{2.2}$$

(With the above normalization the diagonal of the Cartan matrix is $(\alpha,\alpha)$.) For an orthonormal basis $\{I_\mu\}$ of $\mathfrak{G}$, we set

$$t = \sum_\mu I_\mu \otimes I_\mu = \sum_{j=0}^{k-1} t_j, \ (\sigma \otimes 1)t_j = \omega^j t_j.$$

Set further

$$r_0 = \sum_{\alpha \in \Delta_0} \operatorname{sgn} \alpha X_{0,\alpha} \otimes X_{0,-\alpha}.$$

The classical $r$-matrix for the GTS of type $X_N^{(k)}$ is then given by the formula

$$r(x) = r_0 - t_0 + \frac{2}{1-x^k} \sum_{j=0}^{k-1} x^j t_j. \tag{2.3}$$

This $r$-matrix is related to the $L$-operator of (2.1) through the fundamental Poisson bracket relation

$$\{L(\lambda) \underset{,}{\otimes} L(\mu)\} = [r(\lambda/\mu), L(\lambda) \otimes 1 + 1 \otimes L(\mu)]. \tag{2.4}$$

Here the $\lambda$-dependence of $L$ is explicitly exhibited, regarding $\mathfrak{G} \otimes 1$ and $1 \otimes \mathfrak{G}$ as realized in $\mathfrak{G}^{(k)}[\lambda, \lambda^{-1}] \otimes 1$ and $1 \otimes \mathfrak{G}^{(k)}[\mu, \mu^{-1}]$, respectively. In the left-hand side of (2.4) the Poisson bracket is introduced by letting $p$ and $q$ be canonically conjugate; namely, writing $p = \sum p_i H_i$, $q = \sum q_i H_i$ for an orthonormal basis $\{H_i\}$ of $\mathfrak{h}_0$, one has $\{p_i, q_j\} = \delta_{ij}$.

To find the corresponding quantum $R$ matrix, we quantize the relation (2.4) following the line of ref. [5]. Let now $p$ and $q$ denote $\mathfrak{h}_0$-valued operators acting on some Hilbert space satisfying the Heisenberg commutation relations $[p_i, q_j] = \hbar\delta_{ij}$, where $\hbar$ is an arbitrary parameter. We introduce further the elements $\hat{E}_\alpha, \hat{F}_\alpha$ of $\mathfrak{U}(\mathfrak{G})$ (or more precisely its completion) with the properties

$$[H, \hat{E}_\alpha] = \alpha(H)\hat{E}_\alpha, \quad [H, \hat{F}_\alpha] = -\alpha(H)\hat{F}_\alpha \quad (H \in \mathfrak{h}_0), \tag{2.5}$$

$$[\hat{E}_\alpha, \hat{F}_\beta] = \delta_{\alpha\beta} \sinh(2\hbar H_\alpha)/\sinh(2\hbar), \tag{2.6}$$

$$\hat{E}_\alpha \to E_\alpha, \quad \hat{F}_\alpha \to F_\alpha \quad \text{as} \quad \hbar \to 0. \tag{2.7}$$

Here $H_\alpha$ denotes the image of $\alpha \in \mathfrak{h}_0^*$ under the identification $\mathfrak{h}_0^* \simeq \mathfrak{h}$ via the bilinear form ( , ). Eventually we shall restrict to the fundamental representation of $\mathfrak{G}$ and identify $\hat{E}_\alpha, \hat{F}_\alpha$ with $E_\alpha, F_\alpha$. However the following arguments go through under (2.5–7). Define $\hat{e}_\alpha, \hat{f}_\alpha$ as in (2.2) and put $\hat{e} = \sum \hat{e}_\alpha$, $\hat{f} = \sum \hat{f}_\alpha$. In place of the classical $L$-operator (2.1) we use (cf. [5])

$$\begin{aligned} L(\lambda) &= e^p(1 + \varepsilon(e^{\operatorname{ad} q}\hat{e} + e^{-\operatorname{ad} q}\hat{f}))e^p \\ &= \left(1 + \varepsilon \sum_{\alpha \in \pi} e^{\alpha(q)}(e^{\alpha(p)} K_\alpha \hat{e}_\alpha + e^{-\alpha(p)} K_\alpha \hat{f}_\alpha)\right) e^{2p}, \end{aligned}$$

where $K_\alpha = \exp(\hbar H_\alpha)$. In the second line the operators are normal-ordered ($q$ to the left, $p$ to the right). For the quantum $R$ matrix we require the relation

$$\begin{aligned} &R(\lambda/\mu) L_1(\lambda) L_2(\mu) \equiv L_2(\mu) L_1(\lambda) R(\lambda/\mu) \mod \varepsilon^2, \\ &L_1(\lambda) = L(\lambda) \otimes 1, \quad L_2(\mu) = 1 \otimes L(\mu). \end{aligned} \tag{2.8}$$

Reducing the expressions $L_1(\lambda)L_2(\mu)$, $L_2(\mu)L_1(\lambda)$ into the normal-ordered form and comparing the coefficients of $e^{\alpha(q)}e^{\pm\alpha(p)}$, we find that (2.8) is equivalent to

$$[R(x), H\otimes 1 + 1\otimes H] = 0 \quad (H\in\mathfrak{h}_0), \tag{2.9}$$

$$R(x)(\hat{e}_\alpha\otimes K_\alpha^{-1} + K_\alpha\otimes\hat{e}_\alpha) = (\hat{e}_\alpha\otimes K_\alpha + K_\alpha^{-1}\otimes\hat{e}_\alpha)R(x), \tag{2.10}$$

$$R(x)(\hat{f}_\alpha\otimes K_\alpha^{-1} + K_\alpha\otimes\hat{f}_\alpha) = (\hat{f}_\alpha\otimes K_\alpha + K_\alpha^{-1}\otimes\hat{f}_\alpha)R(x). \tag{2.11}$$

Here $x = \lambda/\mu$ (recall that the $\lambda$ or $\mu$ dependence enters through $\hat{e}_0\otimes 1 = \lambda\hat{E}_0\otimes 1$, $1\otimes\hat{e}_0 = 1\otimes\mu\hat{E}_0$, etc.). Below we shall discuss the uniqueness of solutions of the system (2.9–11) and its sufficiency for the validity of the YB Eq. (1.2). In the sequel we fix finite-dimensional irreducible representation spaces $V_i (i = 1,2)$ of $\mathfrak{G}$ and consider (2.9–11) in $\mathrm{End}(V_1\otimes V_2)$.

**Proposition 1.** *For a general value of $\hbar$, the dimension of the solution space of the linear system* (2.10) *is at most 1.*

*Proof.* It suffices to show that the dimension is 1 for the special value $\hbar = 0$. In this case the proof reduces to the following lemma, which we formulate in a slightly more general way. Let $V_i (i = 1,\dots,N)$ be finite dimensional irreducible $\mathfrak{G}$-modules. For $X\in\mathfrak{G}$ we write $X^{(i)} = 1\otimes\cdots 1\otimes \overset{i}{X}\otimes 1\cdots\otimes 1$. Consider the linear equations for $R\in\mathrm{End}(V_1\otimes\cdots\otimes V_N)$,

$$[R, E_\alpha^{(1)} + \cdots + E_\alpha^{(N)}] = 0 \quad (\alpha\in\pi), \quad [R, \lambda_1 E_0^{(1)} + \cdots + \lambda_N E_0^{(N)}] = 0. \tag{2.12}$$

**Lemma.** *For general values of $\lambda_i$, the only solution of* (2.12) *is $R = \mathrm{const}\, I$.*

*Proof of Lemma.* First we note that $[\sum\lambda_i^m X^{(i)}, \sum\lambda_i^n Y^{(i)}] = \sum\lambda_i^{m+n}[X,Y]^{(i)}$. Since $E_0$ is the lowest weight vector of the ad irreducible $\mathfrak{G}_0$-module $\mathfrak{G}_1$ [10], (2.12) implies that $[R, \lambda_1 X^{(1)} + \cdots + \lambda_N X^{(N)}] = 0$ for any $X\in\mathfrak{G}_1$. Hence we have

$$[R, \lambda_1^j X^{(1)} + \cdots + \lambda_N^j X^{(N)}] = 0 \tag{2.13}$$

for any $X\in[\mathscr{L}_1,[\mathscr{L}_2,\dots,[\mathscr{L}_{r-1},\mathscr{L}_r]\dots]]$, where $\mathscr{L}_s$ denotes either $\mathfrak{G}_1$ or $\bigoplus_{\alpha\in\Pi_0}\mathbb{C}E_\alpha$, and $\mathfrak{G}_1$ appears $j$ times in the sequence $\{\mathscr{L}_s\}$. It can be checked that such elements generate $\mathfrak{G}_{j\,\mathrm{mod}\,k}$. Taking $j$ to be $j, j+k, j+2k,\dots$ in (2.13), we conclude that $[R, X^{(i)}] = 0$ holds for any $i$ and $X\in\mathfrak{G}_j$. In other words $R$ commutes with $\mathfrak{U}(\mathfrak{G})\otimes\cdots\otimes\mathfrak{U}(\mathfrak{G})$. The lemma now follows from the fact that, for an irreducible $V_i$, $\mathfrak{U}(\mathfrak{G})$ spans $\mathrm{End}(V_i)$.

**Corollary.** *If* (2.10) *admits a non-trivial solution, it has the form $R = I + \hbar R_1 + \cdots$ up to constant multiple. In particular* $\det R \not\equiv 0$, $\mathrm{tr}\, R \not\equiv 0$.

**Proposition 2.** *A solution of* (2.10) *satisfies both* (2.9) *and* (2.11).

*Proof.* It is enough to consider the case of a non-trivial solution $R(x)$. Using (2.5) and (2.6), one checks that $R_1 = [R, H\otimes 1 + 1\otimes H]$ $(H\in\mathfrak{h}_0)$, $R_2 = R(\hat{f}_\alpha\otimes K_\alpha^{-1} + K_\alpha\otimes\hat{f}_\alpha) - (\hat{f}_\alpha\otimes K_\alpha + K_\alpha^{-1}\otimes\hat{f}_\alpha)R$ both solve (2.10). It follows that $R_i = \kappa_i R$ with some scalar $\kappa_i$. Taking the trace of $R_i R^{-1}$, we find $\kappa_i = 0$.

**Proposition 3.** *Assume that* (2.10) *admits non-trivial solutions $R^{ij}(x)\in\mathrm{End}(V_i\otimes V_j)$ for $(i,j) = (1,2), (1,3), (2,3)$. Then the* YB *Eq.* (1.2) *is satisfied.*

*Proof.* Put $Q_1 = R^{12}(x)R^{13}(xy)R^{23}(y)$, $Q_2 = R^{23}(y)R^{13}(xy)R^{12}(x)$, where $x = \lambda/\mu$ and $y = \mu/\nu$. The relation (2.8) implies that both $Q_i (i = 1,2)$ have the intertwining property,

$$Q_i L_1(\lambda)L_2(\mu)L_3(\nu) \equiv L_3(\nu)L_2(\mu)L_1(\lambda)Q_i \mod \varepsilon^2.$$

Hence their ratio $Q = Q_1^{-1}Q_2$ should satisfy

$$\begin{aligned}&[Q, H^{(1)} + H^{(2)} + H^{(3)}] = 0 \quad (H \in \mathfrak{h}_0), \\ &[Q, K_\alpha^{\pm 1} \otimes K_\alpha^{\pm 1} \otimes \hat{e}_\alpha + K_\alpha^{\pm 1} \otimes \hat{e}_\alpha \otimes K_\alpha^{\mp 1} + \hat{e}_\alpha \otimes K_\alpha^{\mp 1} \otimes K_\alpha^{\mp 1}] = 0,\end{aligned} \tag{2.14}$$

and those obtained by replacing $\hat{e}_\alpha \leftrightarrow \hat{f}_\alpha$. Arguing similarly as above, one can show that (2.14) has the only solution $Q = \text{const}\, I$ for a general $\hbar$.

Comparing the determinant we have

$$R^{12}(x)R^{13}(xy)R^{23}(y) = \zeta R^{23}(y)R^{13}(xy)R^{12}(x),$$

where $\zeta$ is a root of unity. Letting $\hbar \to 0$ we find that $\zeta = 1$.

Thus the YB equation is reduced to solving the homogeneous linear Eqs. (2.10) for $R(x)$. In the next section we give the result by taking $V_1 = V_2 = V$ to be the fundamental representation of $\mathfrak{G}$ and $\hat{e}_\alpha = e_\alpha$, $\hat{f}_\alpha = f_\alpha$.

## 3. Quantum $R$ Matrix (Main Results)

In the sequel we adopt the following realization of classical Lie algebras: $\mathrm{sl}(n) = \{X \in \mathrm{Mat}(n) | \mathrm{tr}\, X = 0\}$, $o(n) = \{X \in \mathrm{sl}(n) | X = -S^t X S\}$, $\mathrm{sp}(2n) = \{X \in \mathrm{sl}(2n) | X = -\tilde{S}^{-1t} X \tilde{S}\}$, where $S = (\delta_{\alpha, n+1-\beta})_{1 \leqq \alpha, \beta \leqq n}$ and $\tilde{S} = \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix}$. Diagram automorphisms of order 2 are given by $\sigma(X) = -S^t X S$ for $\mathrm{sl}(n)$ and $\sigma(X) = TXT^{-1}$ for $o(2n)$ with

$$T = \left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & 1 & & & \\ \hline & & & 1 & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array}\right].$$

By convention the indices $\alpha, \beta$ run over $1, 2, \dots, N$, where $N$ is the size of the matrix: $N = n+1, 2n+1, 2n, 2n, 2n+1, 2n, 2n+2$ for $\mathfrak{G} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n}^{(2)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}$. We put $\alpha' = N + 1 - \alpha$. $E_{\alpha\beta}$ will denote the matrix $(\delta_{i\alpha}\delta_{j\beta})$. Let further $\varepsilon_\alpha = 1$ $(1 \leqq \alpha \leqq n)$, $= -1$ $(n+1 \leqq \alpha \leqq 2n)$ for $\mathfrak{G} = C_n^{(1)}$ and $\varepsilon_\alpha = 1$ in the remaining cases. Under these notations the classical $r$-matrix (2.3) reads as follows:

$\mathfrak{G} = A_n^{(1)}$:

$$(1-x)r(x) = (1+x)\left(\sum E_{\alpha\alpha} \otimes E_{\alpha\alpha} - \frac{1}{N}I\right) + 2\left(\sum_{\alpha<\beta} + x\sum_{\alpha>\beta}\right)E_{\alpha\beta} \otimes E_{\beta\alpha}, \tag{3.1}$$

$\mathfrak{G} = B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$:

$$(1-x)r(x) = (1+x)\sum(E_{\alpha\alpha}\otimes E_{\alpha\alpha} - E_{\alpha\alpha}\otimes E_{\alpha'\alpha'}) + 2\left(\sum_{\alpha<\beta} + x\sum_{\alpha>\beta}\right)(E_{\alpha\beta}\otimes E_{\beta\alpha} - \varepsilon_\alpha\varepsilon_\beta E_{\alpha\beta}\otimes E_{\alpha'\beta'}), \tag{3.2}$$

$\mathfrak{G} = A_{2n}^{(2)}, A_{2n-1}^{(2)}$:

$$\begin{aligned}(1-x^2)r(x) = {}&(1+x)^2\sum E_{\alpha\alpha}\otimes E_{\alpha\alpha} - (1-x)^2\sum E_{\alpha\alpha}\otimes E_{\alpha'\alpha'} - \frac{4x}{N}I \\ &+ 2(1+x)\left(\sum_{\alpha<\beta} + x\sum_{\alpha>\beta}\right)E_{\alpha\beta}\otimes E_{\beta\alpha} \\ &+ 2(1-x)\left(-\sum_{\alpha<\beta} + x\sum_{\alpha>\beta}\right)E_{\alpha\beta}\otimes E_{\alpha'\beta'},\end{aligned} \tag{3.3}$$

$\mathfrak{G} = D_{n+1}^{(2)}$:

$$\begin{aligned}(1-x^2)r(x) = {}&(1+x^2)\sum_{\alpha\neq n+1,n+2}(E_{\alpha\alpha}\otimes E_{\alpha\alpha} - E_{\alpha\alpha}\otimes E_{\alpha'\alpha'}) \\ &+ 2x(E_{n+1,n+1} - E_{n+2,n+2})\otimes(E_{n+1,n+1} - E_{n+2,n+2}) \\ &+ 2\left(\sum_{\alpha<\beta,\alpha,\beta\neq n+1,n+2} + x^2\sum_{\alpha>\beta,\alpha,\beta\neq n+1,n+2}\right)(E_{\alpha\beta}\otimes E_{\beta\alpha} - E_{\alpha\beta}\otimes E_{\alpha'\beta'}) \\ &+ (1+x)\left(\sum_{\alpha<n+1,\beta=n+1,n+2} + x\sum_{\alpha>n+2,\beta=n+1,n+2}\right)(E_{\alpha\beta}\otimes E_{\beta\alpha} \\ &- E_{\alpha\beta}\otimes E_{\alpha'\beta'} + E_{\beta'\alpha'}\otimes E_{\alpha'\beta'} - E_{\beta'\alpha'}\otimes E_{\beta\alpha}) \\ &+ (1-x)\left(\sum_{\alpha<n+1,\beta=n+1,n+2} - x\sum_{\alpha>n+2,\beta=n+1,n+2}\right)(E_{\alpha\beta}\otimes E_{\beta'\alpha} \\ &- E_{\alpha\beta}\otimes E_{\alpha'\beta} - E_{\beta'\alpha'}\otimes E_{\beta'\alpha} + E_{\beta'\alpha'}\otimes E_{\alpha'\beta}).\end{aligned} \tag{3.4}$$

Corresponding quantum $R$-matrices are given by the following formulas ($k = e^{-2h}$ denotes an arbitrary parameter).

$\mathfrak{G} = A_n^{(1)}$:

$$\begin{aligned}R(x) = {}&(x-k^2)\sum E_{\alpha\alpha}\otimes E_{\alpha\alpha} + k(x-1)\sum_{\alpha\neq\beta}E_{\alpha\alpha}\otimes E_{\beta\beta} \\ &-(k^2-1)\left(\sum_{\alpha<\beta} + x\sum_{\alpha>\beta}\right)E_{\alpha\beta}\otimes E_{\beta\alpha}.\end{aligned} \tag{3.5}$$

$\mathfrak{G} = B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n}^{(2)}, A_{2n-1}^{(2)}$:

$$\begin{aligned}R(x) = {}&(x-k^2)(x-\xi)\sum_{\alpha\neq\alpha'}E_{\alpha\alpha}\otimes E_{\alpha\alpha} + k(x-1)(x-\xi)\sum_{\alpha\neq\beta,\beta'}E_{\alpha\alpha}\otimes E_{\beta\beta} \\ &-(k^2-1)(x-\xi)\left(\sum_{\alpha<\beta,\alpha\neq\beta'} + x\sum_{\alpha>\beta,\alpha\neq\beta'}\right)E_{\alpha\beta}\otimes E_{\beta\alpha} \\ &+\sum a_{\alpha\beta}(x)E_{\alpha\beta}\otimes E_{\alpha'\beta'},\end{aligned} \tag{3.6}$$

where

$$a_{\alpha\beta}(x)=\begin{cases}(k^2x-\xi)(x-1) & (\alpha=\beta,\alpha\neq\alpha')\\ k(x-\xi)(x-1)+(\xi-1)(k^2-1)x & (\alpha=\beta,\alpha=\alpha')\\ (k^2-1)(\varepsilon_\alpha\varepsilon_\beta\xi k^{\bar\alpha-\bar\beta}(x-1)-\delta_{\alpha\beta'}(x-\xi)) & (\alpha<\beta)\\ (k^2-1)x(\varepsilon_\alpha\varepsilon_\beta k^{\bar\alpha-\bar\beta}(x-1)-\delta_{\alpha\beta'}(x-\xi)) & (\alpha>\beta).\end{cases}$$

Here $\xi$ and $\bar\alpha$ are given respectively by

$$\xi=k^{2n-1},\quad k^{2n+2},\quad k^{2n-2},\quad -k^{2n+1},\quad -k^{2n},$$

for $\hat{\mathfrak{G}}=B_n^{(1)},\ C_n^{(1)},\ D_n^{(1)},\ A_{2n}^{(2)},\ A_{2n-1}^{(2)}$;

$$\bar\alpha=\begin{cases}\alpha-\frac12 & (1\leqq\alpha\leqq n)\\ \alpha+\frac12 & (n+1\leqq\alpha\leqq 2n)\end{cases}$$

for $\hat{\mathfrak{G}}=C_n^{(1)}$, and

$$\bar\alpha=\begin{cases}\alpha+\frac12 & \left(1\leqq\alpha<\dfrac{N+1}{2}\right)\\ \alpha & \left(\alpha=\dfrac{N+1}{2}\right)\\ \alpha-\frac12 & \left(\dfrac{N+1}{2}<\alpha\leqq N\right)\end{cases}$$

in the remaining cases.

$\mathfrak{G}=D_{n+1}^{(2)}$:

$$\begin{aligned}R(x)={}&(x^2-k^2)(x^2-\xi^2)\sum_{\alpha\neq n+1,n+2}E_{\alpha\alpha}\otimes E_{\alpha\alpha}+k(x^2-1)(x^2-\xi^2)\sum_{\substack{\alpha\neq\beta,\beta'\\ \alpha\,\mathrm{or}\,\beta\neq n+1,n+2}}\\ &\cdot E_{\alpha\alpha}\otimes E_{\beta\beta}-(k^2-1)(x^2-\xi^2)\Bigg(\sum_{\substack{\alpha<\beta,\alpha\neq\beta'\\ \alpha,\beta\neq n+1,n+2}}+x^2\sum_{\substack{\alpha>\beta,\alpha\neq\beta'\\ \alpha,\beta\neq n+1,n+2}}\Bigg)E_{\alpha\beta}\otimes E_{\beta\alpha}\\ &-\tfrac12(k^2-1)(x^2-\xi^2)\Bigg((x+1)\Bigg(\sum_{\alpha<n+1,\beta=n+1,n+2}+x\sum_{\alpha>n+2,\beta=n+1,n+2}\Bigg)\\ &\cdot(E_{\alpha\beta}\otimes E_{\beta\alpha}+E_{\beta'\alpha'}\otimes E_{\alpha'\beta'})+(x-1)\Bigg(-\sum_{\alpha<n+1,\beta=n+1,n+2}+x\sum_{\alpha>n+2,\beta=n+1,n+2}\Bigg)\\ &\cdot(E_{\alpha\beta}\otimes E_{\beta'\alpha}+E_{\beta'\alpha'}\otimes E_{\alpha'\beta})\Bigg)+\sum_{\alpha,\beta\neq n+1,n+2}a_{\alpha\beta}(x)E_{\alpha\beta}\otimes E_{\alpha'\beta'}+\tfrac12\sum_{\alpha\neq n+1,n+2,\beta=n+1,n+2}\\ &\cdot(b_\alpha^+(x)(E_{\alpha\beta}\otimes E_{\alpha'\beta'}+E_{\beta'\alpha'}\otimes E_{\beta\alpha})+b_\alpha^-(x)(E_{\alpha\beta}\otimes E_{\alpha'\beta}+E_{\beta\alpha'}\otimes E_{\beta\alpha}))\\ &+\sum_{\alpha=n+1,n+2}(c^+(x)E_{\alpha\alpha}\otimes E_{\alpha'\alpha'}+c^-(x)E_{\alpha\alpha}\otimes E_{\alpha\alpha}\\ &+d^+(x)E_{\alpha\alpha'}\otimes E_{\alpha'\alpha}+d^-(x)E_{\alpha\alpha'}\otimes E_{\alpha\alpha'}),\end{aligned}\tag{3.7}$$

where for $\alpha, \beta \neq n+1, n+2$

$$a_{\alpha\beta}(x) = \begin{cases} (k^2x^2 - \xi^2)(x^2-1) & (\alpha = \beta) \\ (k^2-1)(\xi^2 k^{\bar\alpha - \bar\beta}(x^2-1) - \delta_{\alpha\beta'}(x^2-\xi^2)) & (\alpha < \beta) \\ (k^2-1)x^2(k^{\bar\alpha-\bar\beta}(x^2-1) - \delta_{\alpha\beta'}(x^2-\xi^2)) & (\alpha > \beta), \end{cases}$$

$$b_\alpha^\pm(x) = \begin{cases} \pm k^{\alpha-1/2}(k^2-1)(x^2-1)(x \pm \xi) & (\alpha < n+1) \\ k^{\alpha-n-5/2}(k^2-1)(x^2-1)x(x\pm\xi) & (\alpha > n+2)' \end{cases}$$

$$c^\pm(x) = \pm\tfrac{1}{2}(k^2-1)(\xi+1)x(x\mp 1)(x\pm\xi) + k(x^2-1)(x^2-\xi^2),$$
$$d^\pm(x) = \pm\tfrac{1}{2}(k^2-1)(\xi-1)x(x\pm 1)(x\pm\xi),$$

and $\xi = k^n, \bar\alpha = \alpha + 1 (\alpha < n+1), = n + 3/2 (\alpha = n+1, n+2), = \alpha - 1 (\alpha > n+2)$.

Among these, the solutions for $A_n^{(1)}$ [6] and $A_2^{(2)}$ [7] have been known. In the case $A_n^{(1)}, R(x)$ splits into a direct sum of copies of $1 \times 1$ and $2 \times 2$ elementary blocks,

$$(x-k^2), \quad \begin{pmatrix} k(x-1) & -(k^2-1) \\ -(k^2-1)x & k(x-1) \end{pmatrix}.$$

Likewise (3.6) consists of blocks

$$((x-k^2)(x-\xi)), \quad \begin{pmatrix} k(x-1) & -(k^2-1) \\ -(k^2-1)x & k(x-1) \end{pmatrix} \times (x-\xi) \tag{3.8}$$

and an $N \times N$ piece $(a_{\alpha\beta}(x))$. In the case $D_{n+1}^{(2)}$ the elementary blocks are (3.8) (with $x, \xi$ replaced by $x^2$ and $\xi^2$), $4 \times 4$ pieces

$$\begin{bmatrix} k(x^2-1) & 0 & \tfrac{1}{2}(k^2-1)(x-1) & -\tfrac{1}{2}(k^2-1)(x+1) \\ 0 & k(x^2-1) & -\tfrac{1}{2}(k^2-1)(x+1) & \tfrac{1}{2}(k^2-1)(x-1) \\ -\tfrac{1}{2}(k^2-1)x(x-1) & -\tfrac{1}{2}(k^2-1)x(x+1) & k(x^2-1) & 0 \\ -\tfrac{1}{2}(k^2-1)x(x+1) & -\tfrac{1}{2}(k^2-1)x(x-1) & 0 & k(x^2-1) \end{bmatrix} \times (x^2-\xi^2)$$

and an $(N+2) \times (N+2)$ piece. They are all subject to the symmetry

$$\begin{aligned} &[R(x), H \otimes 1 + 1 \otimes H] = 0 \quad (H \in \mathfrak{h}_0), \\ &PR(x)P = (S \otimes S)R(x)(S \otimes S) = {}^tR(x), \\ &R(x^{-1}, k^{-1}) = \gamma(x,k)^{-1}\,{}^tR(x,k), \end{aligned} \tag{3.9}$$

with $\gamma(x,k) = -k^2x(\hat{\mathfrak{G}} = A_n^{(1)}), = k^2\xi x^2(\hat{\mathfrak{G}} = B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n}^{(2)}, A_{2n-1}^{(2)}), = k^2\xi^2x^4(\hat{\mathfrak{G}} = D_{n+1}^{(2)})$. Aside from these symmetries, they have the following properties.

(i) inversion relation

$$\check{R}(x)\check{R}(x^{-1}) = \rho(x)I, \tag{3.10}$$

$$\rho(x) = \begin{cases} (x-k^2)(x^{-1}-k^2) & (\hat{\mathfrak{G}} = A_n^{(1)}) \\ (x-k^2)(x-\xi)(x^{-1}-k^2)(x^{-1}-\xi) & (\hat{\mathfrak{G}} = B_n^{(1)}, \ldots, A_{2n-1}^{(2)}), \\ (x^2-k^2)(x^2-\xi^2)(x^{-2}-k^2)(x^{-2}-\xi^2) & (\hat{\mathfrak{G}} = D_{n+1}^{(2)}) \end{cases}$$

where we have set $\check{R}(x) = PR(x)$.

(ii) As $k \to 1$,

$$R(x,k) = \kappa_1(x)(I + (k-1)(r(x) + \kappa_2(x)I) + \cdots)$$

for appropriate scalars $\kappa_i(x)$.
(iii) As $x \to 1$,

$$\check{R}(x,k) = \check{\kappa}_1(k)(I + (x-1)(s(k) + \check{\kappa}_2(k)I) + \cdots),$$

with some scalars $\check{\kappa}_i(k)$. Here the tensor $s(k)$ is given by

$\mathfrak{G} = A_n^{(1)}$:

$$s(k) = \tfrac{1}{2}(1-k^2)\sum_{\alpha \neq \beta} \operatorname{sgn}(\beta - \alpha) E_{\alpha\alpha} \otimes E_{\beta\beta} + \tfrac{1}{2}(1+k^2)\sum_{\alpha} E_{\alpha\alpha} \otimes E_{\alpha\alpha} + k \sum_{\alpha \neq \beta} E_{\alpha\beta} \otimes E_{\beta\alpha}, \tag{3.11}$$

$\hat{\mathfrak{G}} = B_n^{(1)}, \ldots, A_{2n-1}^{(2)}$:

$$\begin{aligned} s(k) = {} & \tfrac{1}{2}(1-k^2)(1-\xi)\sum_{\alpha \neq \beta} \operatorname{sgn}(\beta - \alpha) E_{\alpha\alpha} \otimes E_{\beta\beta} \\ & + \tfrac{1}{2}(1+k^2)(1-\xi)\sum_{\alpha \neq \alpha'} E_{\alpha\alpha} \otimes E_{\alpha\alpha} \\ & + (k(1-\xi) - \tfrac{1}{2}(1-k^2)(1+\xi))\sum \delta_{\alpha\alpha'} E_{\alpha\alpha} \otimes E_{\alpha\alpha} \\ & + k(1-\xi)\sum_{\alpha \neq \beta, \beta'} E_{\alpha\beta} \otimes E_{\beta\alpha} + (k^2 - \xi)\sum_{\alpha \neq \alpha'} E_{\alpha\alpha'} \otimes E_{\alpha'\alpha} \\ & - (1-k^2)\left(\sum_{\alpha < \beta} \varepsilon_\alpha \varepsilon_\beta k^{\bar{\alpha} - \bar{\beta}} \xi + \sum_{\alpha > \beta} \varepsilon_\alpha \varepsilon_\beta k^{\bar{\alpha} - \bar{\beta}}\right) E_{\alpha'\beta} \otimes E_{\alpha\beta'}, \end{aligned} \tag{3.12}$$

$\hat{\mathfrak{G}} = D_{n+1}^{(2)}$:

$$\begin{aligned} s(k) = {} & (1-k^2)(1-\xi^2)\sum_{\alpha \neq \beta, \alpha, \beta \neq n+1, n+2} \operatorname{sgn}(\beta - \alpha) E_{\alpha\alpha} \otimes E_{\beta\beta} \\ & + \tfrac{1}{2}(1-k^2)(1-\xi^2)\sum_{\alpha = n+1, n+2, \beta \neq n+1, n+2} \operatorname{sgn}(\beta - \alpha)((E_{\alpha\alpha} + E_{\alpha'\alpha}) \otimes E_{\beta\beta} \\ & + E_{\beta'\beta'} \otimes (E_{\alpha'\alpha'} + E_{\alpha\alpha'})) \\ & + (1+k^2)(1-\xi^2)\sum_{\alpha \neq n+1, n+2} E_{\alpha\alpha} \otimes E_{\alpha\alpha} \\ & + 2k(1-\xi^2)\sum_{\alpha = n+1, n+2} (E_{\alpha\alpha} \otimes E_{\alpha\alpha} + E_{\alpha\alpha'} \otimes E_{\alpha'\alpha}) \\ & - \tfrac{1}{2}(1-k^2)\sum_{\alpha = n+1, n+2} ((1-\xi)^2(E_{\alpha\alpha} \otimes E_{\alpha\alpha} + E_{\alpha\alpha'} \otimes E_{\alpha\alpha'}) \\ & + (1+\xi)^2(E_{\alpha\alpha'} \otimes E_{\alpha'\alpha} + E_{\alpha\alpha} \otimes E_{\alpha'\alpha'})) \\ & + 2k(1-\xi^2)\sum_{\substack{\alpha \neq \beta, \beta' \\ \alpha \text{ or } \beta \neq n+1, n+2}} E_{\alpha\beta} \otimes E_{\beta\alpha} + 2(k^2 - \xi^2)\sum_{\alpha \neq n+1, n+2} E_{\alpha\alpha'} \otimes E_{\alpha'\alpha} \\ & - 2(1-k^2)\left(\sum_{\alpha < \beta, \alpha, \beta \neq n+1, n+2} \xi^2 k^{\bar{\alpha} - \bar{\beta}} + \sum_{\alpha > \beta, \alpha, \beta \neq n+1, n+2} k^{\bar{\alpha} - \bar{\beta}}\right) \end{aligned}$$

$$\cdot E_{\alpha'\beta} \otimes E_{\alpha\beta'} - (1-k^2)\Bigg( (1+\xi)\Bigg( \sum_{\alpha < n+1, \beta = n+1, n+2} \xi k^{\bar{\alpha}-\bar{\beta}}$$

$$+ \sum_{\alpha > n+2, \beta = n+1, n+2} k^{\bar{\alpha}-\bar{\beta}} \Bigg) (E_{\alpha'\beta} \otimes E_{\alpha\beta'} + E_{\beta\alpha'} \otimes E_{\beta'\alpha})$$

$$+ (1-\xi)\Bigg( - \sum_{\alpha < n+1, \beta = n+1, n+2} \xi k^{\bar{\alpha}-\bar{\beta}} + \sum_{\alpha > n+2, \beta = n+1, n+2} k^{\bar{\alpha}-\bar{\beta}} \Bigg)$$

$$\cdot (E_{\alpha'\beta} \otimes E_{\alpha\beta} + E_{\beta\alpha'} \otimes E_{\beta\alpha}) \Bigg).$$

*Remark 1.* It can be shown that under the first condition of (3.9), $R'(x) = (D(x) \otimes I) R(x)(D(x) \otimes I)^{-1}$, where $D(x) \in \exp \mathfrak{h}_0$, $D(xy) = D(x)D(y)$, is again a solution to the YB equation. Choosing an appropriate $D(x)$ one obtains the $R$ matrix in the principal picture.

*Remark 2.* Except for the case $\mathfrak{G} = D_{n+1}^{(2)}$, our $R$ matrix satisfies $[\check{R}(x), \check{R}(y)] = 0$ so that $\check{R}(x)$ is diagonalizable independently of $x$. This is a consequence of (3.10), $\check{R}(1) \propto I$ and that $\deg R(x) \leqq 2$.

It is well known that a quantum $R$ matrix gives rise to an integrable vertex model in statistical mechanics, whose transfer matrices

$$T(x) = \mathrm{tr}_{V_0}(R^{01}(x) R^{02}(x) \dots R^{0N}(x)) \in \mathrm{End}(V_1 \otimes \cdots \otimes V_N)$$

commute among themselves: $[T(x), T(y)] = 0$. Hence their log derivatives provide a mutually commuting family of "spin" Hamiltonians. The first one is given by

$$T(1)^{-1} \frac{dT}{dx}(1) = \sum_{j=1}^{N} \frac{d\check{R}^{ii+1}}{dx}(1) = \sum_{j=1}^{N} s^{jj+1}(k) + \text{const } I.$$

In our case the tensors $s(k) \in \mathrm{End}(V \otimes V)$ are given by (3.11–13).

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**Note added in proof.** After submitting the manuscript, the author learned that similar results have been obtained independently by Bazhanov (IHEP preprint 85-18, Serpukhov 1985)