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Submodular Subgroups in Finite Groups

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In his Habilitationsschrift [12] published in 1939 H. Wielandt introduced the concept of a subnormal subgroup, generalizing the concept of a normal subgroup in such a way that the resulting embedding property becomes transitive. Since then subnormality has been the object of numerous investigations, see e.g. the recent monograph by J.C. Lennox and S.E. Stonehewer [5]. The concept of a modular subgroup originally comes from the theory of lattices. A subgroup M of a group G is called modular in G if the following equalities are satisfied:

$$(U \cup M) \cap V = U \cup (M \cap V) \quad \text{for all } U, V \leqq G \quad \text{with } U \leqq V$$

and

$$(U \cup M) \cap V = M \cup (U \cap V) \quad \text{for all } U, V \leqq G \quad \text{with } M \leqq V$$

where $X \cup Y = \langle X, Y \rangle$ denotes the subgroup generated by X and Y. Thus M is a modular element of the subgroup lattice of G. Detailed analysis of modular subgroups has been carried out by R. Schmidt (cf. [7], [8]). Like normality, modularity is not a transitive relation, i.e. a subgroup of G is not necessarily modular in G if it is modular in a modular subgroup of G.

Our aim here is to study submodular subgroups in finite groups, defined in the obvious way: we call a subgroup T of a group G submodular in G if there exists a finite series

$$T = T_1 < T_2 < \ldots < T_s < T_{s+1} = G$$

of subgroups T_i of G such that T_i is modular in T_{i+1} for all $i = 1, \ldots, s$. Thus submodularity is just the transitive closure of the embedding relation of modularity and hence it also generalizes the concept of subnormality.

Observe that submodular subgroups naturally arise as images of subnormal subgroups under lattice isomorphisms of groups, because normal subgroups are mapped onto modular (not necessarily normal) subgroups under lattice isomorphisms.

The paper is organized as follows. Basic properties of submodular subgroups are stated in Section 1. Although, in general, the join of two submodular subgroups fails to be submodular, it turns out that a subgroup generated by a submodular subgroup and a modular or a subnormal subgroup is again submodular.

In the second section we study the structure of the factor group of a submodular subgroup T in a finite group G by the (uniquely determined) maximal modular resp. subnormal subgroup of G in T. We show, for instance, that the commutator subgroup of T is subnormal in G.

Submodular subgroups of solvable groups are investigated in Section 3. We prove that every minimal subgroup of a supersolvable group is submodular and characterize finite groups all of which submodular subgroups are subnormal or modular, respectively.

If G is a finite group with trivial Frattini subgroup, then all subgroups of G are submodular provided that all maximal subgroups of G are (sub)modular. Groups having all their subgroups submodular are the object of Section 4.

Finally, in the last section, we study groups with submodular Sylow subgroups. These groups possess a Sylow tower and are characterized by some additional conditions. In contrast to finite groups with modular Sylow subgroups, their Fitting length cannot be bounded.

1. Definition and Basic Properties of Submodular Subgroups

Definition. A subgroup T of a group G is called *submodular* in G if there exists a series

$$(*)\quad T=T_1<T_2<\ldots<T_s<T_{s+1}=G$$

of subgroups T_i of G such that T_i is modular in T_{i+1} for $i=1, \ldots, s$. Obviously, in a finite group the series $(*)$ can be chosen in such a way that T_i is a maximal modular subgroup in T_{i+1} for every $i=1, \ldots, s$.

The following simple lemma does not require the finiteness of the group G and can be proved by using the corresponding properties for modular subgroups (cf. [7]).

Lemma 1. *Let G be an arbitrary group.*

(i) *If T is submodular in G and U is a subgroup of G, then $U \cap T$ is submodular in U.*

(ii) *If T is submodular in G and N is a normal subgroup of G contained in T, then T/N is submodular in G/N.*

(iii) *If T/N is submodular in G/N, then T is submodular in G.*

(iv) *If T is submodular in G, then T^x is submodular in G for every $x \in G$.*

(v) *If T_1 and T_2 are submodular in G, then $T_1 \cap T_2$ is submodular in G.*

If M is a modular subgroup of the group G, then M^φ is modular in G^φ for every homomorphism φ of G. It is therefore clear that submodularity is also preserved under homomorphisms, i.e.:

If T is submodular in G, then TN is submodular in G for every normal subgroup N of G.

More generally, $\langle T, H\rangle$ is submodular in G if H is a modular or subnormal subgroup of G. To show this we need

Lemma 2. *Let U be a non-modular maximal subgroup of the finite group G.*
(i) *The maximal subnormal subgroup of G in U is normal in G.*
(ii) *The maximal modular subgroup of G in U is normal in G.*

Here statement (i) is due to Deskins [1] while (ii) was proved by Schmidt in [9].

Proposition 1. *Let T be a submodular subgroup of the finite group G. If H is a modular or a subnormal subgroup of G, then $\langle T, H\rangle$ is again submodular in G.*

Proof. We argue by induction on $|G|$ and may assume that $|G|>1$. Take a maximal subgroup U of G with $\langle T, H\rangle \leqq U$. Obviously, one can assume that U is not modular in G. Since U contains the modular or subnormal subgroup H of G it follows from Lemma 2 that U contains a normal subgroup N of G with $H \leqq N$. Now TN is submodular in G and as $|TN|<|G|$ it follows by induction that $\langle T, H\rangle$ is submodular in TN. Hence $\langle T, H\rangle$ is submodular in G.

The set of all submodular subgroups of a group forms a meet-semilattice by Lemma 1(v). The join of two submodular subgroups, however, is not submodular in general.

Example. Let $G=\langle a, b \mid a^7=b^6=1, ab=ba^3\rangle$ be the holomorph of the cyclic group $A=\langle a\rangle$ of order 7. The subgroups $B_1=\langle b^2\rangle$ and $B_2=\langle b^3\rangle$ are both submodular in G: B_i is modular in the normal subgroup $B_i A$ for $i=1, 2$. (It is easy to see that neither B_1 nor B_2 is modular or subnormal in G.) The join $\langle B_1, B_2\rangle$ is the maximal subgroup $\langle b\rangle$ of order 6 which is not modular in G.

2. The Subnormal and the Modular Kernel of a Submodular Subgroup

The above example shows that submodularity is in fact a generalization of both subnormality and modularity. It seems therefore interesting to know how close the concepts of submodularity and subnormality resp. modularity are. In this section we investigate the structure of the factor groups T/S^* and T/M^* where T is a submodular subgroup of a group G and S^* and M^* denote the subnormal and the modular kernel of G in T, respectively, i.e. the unique maximal subnormal and the unique maximal modular subgroup of G contained in T.

It turns out, for example, that the commutator subgroup of a submodular subgroup is subnormal in the whole group, which implies that T/S^* is always abelian.

For a proof of this we shall make use of the following result due to Schmidt [7] which is frequently needed throughout the paper.

Lemma 3. *The subgroup M of the group G is a maximal modular subgroup in G if and only if M is a maximal normal subgroup of G or G/M_G is a non-abelian group of order pq, p and q prime numbers.*

Lemma 4. *Let T be a submodular subgroup of the group G. If K is the uniquely determined smallest normal subgroup of T such that T/K is abelian of squarefree exponent, then K is subnormal in G.*

Proof. The assertion follows directly from Lemma 3 in case T is a maximal (sub)modular subgroup of G. Suppose T is properly contained in a maximal modular subgroup M of G. Induction on the order of G allows us to assume that K is subnormal in M, M is not normal in G, and T is not contained in M_G. Therefore, by Lemma 3, $T/T \cap M_G \simeq TM_G/M_G = M/M_G$ is of prime order. This implies $K \leqq M_G$, i.e. K is subnormal in M_G and hence in G.

From Lemma 4 we obtain

Proposition 2. *If T is a submodular subgroup of the group G, then T/S^* is abelian of squarefree exponent where S^* denotes the subnormal kernel of G in T. In particular, the commutator subgroup T' of T is subnormal in G.*

In [7] Schmidt proves that a perfect modular subgroup of a finite group is already a normal subgroup. Proposition 2 gives an analogous result for submodular subgroups.

Corollary 1. *A perfect submodular subgroup is subnormal.*

We now want to show that an abelian group H of squarefree exponent occurs as a submodular subgroup of a group G such that H contains no non-trivial subnormal subgroup of G. Let

$$H = C_{p_1} \times \dots \times C_{p_r}$$

where C_{p_i} is a cyclic group of prime order p_i for $i = 1, \dots, r$. Choose primes $q_1, \dots, q_r$ such that $p_i | q_i - 1$ and put

$$K = C_{q_1} \times \dots \times C_{q_r}.$$

If a_i and b_i are generating elements of C_{p_i} and C_{q_i} respectively, define

$$G_1 = K \lambda H$$

by

$$[a_i, b_j] = 1 \quad \text{if } i \neq j$$

and

$$b_i^{a_i} = b_i^t, \quad t \not\equiv 1 (q_i) \quad \text{and} \quad t^{p_i} \equiv 1 (q_i).$$

Then H is submodular in G_1 since every term in the series

$$H < C_{q_1} H < (C_{q_1} \times C_{q_2}) H < \dots < (C_{q_1} \times \dots \times C_{q_{r-1}}) H < KH = G_1$$

is modular in the subsequent.

Clearly, no subgroup $U \neq 1$ of H is subnormal in G_1. Now it might happen that H is modular in G_1. This is the case, for instance, if $(p_i\, q_i,\, p_j\, q_j) = 1$ for all $i \neq j$, $i, j = 1, \dots, r$. We want to embed G_1 in a group G in such a way that H is still submodular but not modular in G.

Lemma 5. *Let $A \wr B = (A_1 \times \ldots \times A_n) \lambda B = C$ be the regular wreath product of A with B, $|B| = n > 1$. Then there does not exist a subgroup $M_i \neq 1$ in $A_i \simeq A$, $i = 1, \ldots, n$, which is modular in C.*

Proof. Take an arbitrary subgroup $M_i \neq 1$ of A_i. For $j \neq i, j \in \{1, \ldots, n\}$, A_j contains a B-conjugate M_j of M_i. We have

$$\langle B, M_i \rangle \cap (M_i \times M_j) = M_i \times M_j > M_i = \langle M_i, B \cap (M_i \times M_j) \rangle,$$

i.e., with $U = B$, $V = M_i \times M_j$ and $M = M_i$ the second modularity equation is not satisfied. Hence M_i cannot be modular in C.

Take G_1 from above and an arbitrary group $B \neq 1$. By Lemma 5, H is not modular in $G = G_1 \wr B$, but H is submodular in the subnormal subgroup G_1 of G, hence submodular in G. This proves

Proposition 3. *An abelian group $H \neq 1$ of squarefree exponent can be embedded in a group G such that the following conditions are satisfied:*
(i) *H is submodular in G.*
(ii) *H is neither modular nor subnormal in G.*
(iii) *The subnormal kernel S^* of G in H is trivial.*

In order to obtain information about the factor group of a submodular subgroup in a group G by its modular kernel in G we consider – in view of Corollary 1 – only non-perfect submodular subgroups. Here we have

Proposition 4. *Let $K \neq 1$ be a non-perfect group. Then K can be embedded in a group G such that the following holds:*
(i) *K is submodular in G.*
(ii) *K is neither modular nor subnormal in G.*
(iii) *The modular kernel M^* of G in K is trivial.*

Proof. Since K is not perfect, there exists a normal subgroup N of K such that $|K/N| = p$, p a prime number. Choose a prime q such that p divides $q-1$. If $C = \langle a \rangle$ is a cyclic group of order q and $\langle x, N \rangle = K$ we put

$$G_1 = C \lambda K;$$
$$a^x = a^r, \quad r \not\equiv 1\,(q) \quad \text{and} \quad r^p \equiv 1\,(q)$$
$$a^n = a \quad \text{for every } n \in N.$$

Obviously, $K_{G_1} = N$. Therefore K is a maximal modular but not normal subgroup of G_1. Take a group $B \neq 1$. Then K is submodular in $G_1 \wr B = G$. Furthermore, K is not subnormal in G and, by Lemma 5, no subgroup $U \neq 1$ of K is modular in G, i.e. $M^* = 1$.

3. Submodular Subgroups in Solvable Groups

The aim of this section is to investigate solvable groups in which every submodular subgroup is already a subnormal or a modular subgroup. We begin to study supersolvable groups since they contain "many" submodular subgroups.

Lemma 6. *Every minimal subgroup of a supersolvable group G is submodular in G.*

Proof. Suppose the assertion is true for all supersolvable groups of order smaller than $|G|$. If U is a minimal subgroup and N a minimal normal subgroup of G, we can assume that UN/N is submodular in G/N, i.e. UN is submodular in G. Now UN is a group of order pq, p and q two (not necessarily distinct) prime numbers (unless $U = N$, a trivial case). This implies that U is modular in UN and hence submodular in G.

We remark that the converse of the above lemma does not hold. Using results of Gaschütz [3] one can show that every finite group is isomorphic to the factor group of a finite group in which all minimal subgroups are contained in the Frattini subgroup.

Proposition 5. *Let G be a supersolvable group. If every submodular subgroup of G is subnormal or modular in G, then every subgroup of G is subnormal or modular in G.*

Proof. We prove the assertion by induction on $|G|$. If U is a maximal and modular subgroup of G, then every subgroup of U is subnormal or modular in U and hence submodular in G. Therefore we may assume that G contains a non-modular maximal subgroup V. By Lemma 6, a minimal subgroup of V is submodular in G and consequently subnormal or modular in G. Applying Lemma 2, we conclude that V contains a normal subgroup N of G. Now V/N is modular in G/N if $|G/N| < |G|$ holds. This implies the modularity of V in G. If $N = 1$, then V itself is a minimal and, by Lemma 6, modular subgroup of G, contradicting our choice of V.

Proposition 5 is not true for arbitrary solvable groups: every submodular subgroup of the alternating group A_4 is subnormal and A_4 contains a maximal non-modular subgroup of order 3. In the following we are interested in the question to what extent one of the two stronger conditions
(1) every submodular subgroup is modular
and
(2) every submodular subgroup is subnormal
restricts the structure of a finite solvable group.

The next corollary is an immediate consequence of Proposition 5. A group is called an M-group if every subgroup is modular in the group.

Corollary 2. *Let G be a supersolvable group.*
(i) *If every submodular subgroup of G is modular in G, then G is an M-group.*
(ii) *If every submodular subgroup of G is subnormal in G, then G is nilpotent.*

Theorem 1. *A solvable group G is an M-group if and only if every submodular subgroup of G is modular.*

Proof. It suffices to show that a solvable group in which every submodular subgroup is modular is supersolvable. Corollary 2 (i) then proves the assertion.

We proceed by induction on $|G|$ and may assume that G contains a unique minimal normal subgroup N and G/N is supersolvable. Now $N = N_1 \times \ldots \times N_r$

where the N_i are cyclic groups of prime order. Each N_i is subnormal in G and consequently, by hypothesis, modular in G. It was shown by Heineken (cf. [8]) that a subgroup which is both subnormal and modular is quasinormal, and a result of Maier and Schmid [6] implies that each N_i is either normal in G or is contained in the hypercenter of G. In the first case, $N=N_1$ is cyclic whence G is supersolvable. If the N_i are contained in the hypercenter of G, then N is actually in the center of G since N is the unique minimal normal subgroup of G. Again we conclude that N must be cyclic of prime order. This proves the theorem.

Remarks. a) The groups in question in the above theorem are just those finite solvable groups in which modularity is a transitive relation. From this point of view Theorem 1 has been also proved by A. Frigerio (see [2]).

b) The property that all submodular subgroups are modular is not inherited by direct products since direct products of M-groups may contain non-modular subgroups. (Take for example the direct product of two symmetric groups of order 3.) However, every subgroup of such a direct product is still submodular.

Theorem 2. *The following conditions are equivalent for a solvable group G.*

(i) *Every submodular subgroup of G is subnormal in G.*

(ii) *Every supersolvable subnormal section is nilpotent.*

(A subnormal section is a factor group of a subnormal subgroup.)

Proof. Since condition (i) is inherited by subnormal subgroups as well as by factor groups, it follows from Corollary 2 that (i) implies (ii).

To prove the converse, suppose that every subnormal supersolvable section of G is nilpotent and let T be an arbitrary submodular subgroup of G. There exists a subgroup chain

$$T=T_1<T_2<\ldots<T_s<T_{s+1}=G$$

such that each T_i is a maximal modular subgroup in T_{i+1} for every $i=1, \ldots, s$. Assume T is not subnormal in G. Then T_2 is not subnormal in G as well. This follows from Lemma 3 because otherwise $T_2/(T)_{T_2}$ were a supersolvable non-nilpotent subnormal section of G. Repeating this argument we conclude that none of the T_i is subnormal in G which is certainly a contradiction for $i=s+1$.

The property of solvable groups to contain only submodular subgroups which are subnormal is preserved by taking direct products.

Proposition 6. *Let G_1 and G_2 be solvable groups in which every submodular subgroup is subnormal. Then all submodular subgroups in $G_1 \times G_2$ are subnormal.*

Proof. Suppose the result is false and let $G=G_1 \times G_2$ provide a counterexample of least possible order. By Theorem 2 there exists a subnormal section H/K of G such that H/K is supersolvable and not nilpotent. We may assume that H/K is non-abelian of order pq, i.e. $p|q-1$. (A finite supersolvable group is nilpotent if every subnormal section of order pq is abelian!)

We may further assume that $\pi_1(H)=G_1$, $\pi_2(H)=G_2$, and $K\cap G_1=K\cap G_2=1$, where $\pi_i (i=1, 2)$ denotes the projection on G_i. Since K is subnormal in G (of defect d, say), we have

$$[\underbrace{\pi_1(K), \ldots, \pi_1(K)}_{d+1}]=[\pi_1(K), \underbrace{K, \ldots, K}_{d}]\leqq K\cap G_1=1,$$

whence $\pi_1(K)$ and (using the same argument for the second factor) $\pi_2(K)$ are nilpotent. Now $\pi_1(H)/\pi_1(K)$ and $\pi_2(H)/\pi_2(K)$ are isomorphic to a (proper) factor group of H/K, i.e. $|\pi_i(H)/\pi_i(K)|=1$ or p. In particular, if S denotes the normal subgroup of H such that $K<S<H$ and $|S/K|=q$, then $\pi_i(S)=\pi_i(K)$ which implies that $\pi_i(S)$ is nilpotent.

Assume that $q\,||\pi_1(S)|$ and put $V=O_q(\pi_1(S))/\phi(O_q(\pi_1(S)))$. If $(G_1)_p\in\mathrm{Syl}_p(G_1)$, then V, regarded as a vector space, is the direct sum of 1-dimensional $(G_1)_p$-invariant subspaces. Suppose $(G_1)_p$ does not centralize V. Then $V=V_1\oplus V_2$ where V_1 is a 1-dimensional subspace of V on which $(G_1)_p$ does not act trivially. If W_2 denotes the complete preimage of V_2 in $O_q(\pi_1(S))$ and $L_1=W_2\,O_{q'}(\pi_1(S))$, then G_1/L_1 is a non-abelian group of order pq, which is impossible. We conclude that $(G_1)_p$ centralizes V. Therefore $(G_1)_p$ centralizes $O_q(\pi_1(S))$. Since $O_q(\pi_1(S))\in\mathrm{Syl}_q(G_1)$, we have shown that a $\{p, q\}$-Hall subgroup of G_1 is nilpotent. Analogously, a $\{p, q\}$-Hall subgroup of G_2 is nilpotent. But this implies that a $\{p, q\}$-Hall subgroup of $G=G_1\times G_2$ is nilpotent, which in particular means that H/K is abelian. This contradiction proves the proposition.

It is well known that a group generated by solvable subnormal subgroups is itself solvable. Next we show that this is also true if the generating subgroups are submodular. More generally we have

Proposition 7. *Let $G=\langle U, T\rangle$ be a group where U is a solvable and T is a solvable submodular subgroup. Then G itself is solvable.*

Essential for the proof of Proposition 7 is the following property of solvable submodular subgroups.

Lemma 7. *If T is a solvable submodular subgroup of the group G and N is a non-abelian minimal normal subgroup of G, then $T\leqq C_G(N)$.*

Proof. Let G be a minimal counterexample to the assertion. It is not hard to see that N cannot contain a solvable submodular subgroup. Therefore $T\cap N=1$.

Suppose $TN<G$. By assumption, $N=N_1\times\ldots\times N_r$ is the direct product of isomorphic non-abelian simple groups. Take $i\in\{1, \ldots r\}$ and $t\in T$. Then $(N_i^t)^n=(N_i^{n'})^t$ for all $n\in N$, i.e. $N_i^t\lhd N$ resp. $N_i^t=N_k$ for some $k\in\{1, \ldots, r\}$. If $N_i, N_i^{t_1}, \ldots, N_i^{t_s}$ are the distinct conjugates of N_i under T, then $N_i^T=N_i\times N_i^{t_1}\times\ldots\times N_i^{t_s}$ is a minimal normal subgroup in TN. Since $|TN|<|G|$, we now conclude that $T\leqq C_{TN}(N_i^T)$ for all $i=1, \ldots, r$ whence $T\leqq C_{TN}(N)\leqq C_G(N)$. This implies $G=TN$. Now T is contained in a maximal modular subgroup M of G. In case M is a maximal normal subgroup of G, we get $G=M\times N$ and $T\leqq M=C_G(N)$. Therefore $|G:M|=q$ and G/M_G is non-abelian of order pq. Since $N\cap M_G=1$, it follows that $|N|=|NM_G/M_G|=p$. This final contradiction proves the lemma.

Proof of Proposition 7. We argue by induction on the order of G where $G=\langle U, T\rangle$ with $T\neq 1\neq U$. Since G contains a non-trivial submodular subgroup, it cannot be a simple non-abelian group. Assume G contains two different minimal normal subgroups N_1 and N_2. By induction and the fact that submodularity is inherited by homomorphic images, both $G/N_1=\langle UN_1/N_1, TN_1/N_1\rangle$ and $G/N_2=\langle UN_2/N_2, TN_2/N_2\rangle$ are solvable, which implies the solvability of G.

Therefore G contains a unique minimal normal subgroup N. If N is abelian, we again conclude that G is solvable. The other possibility that N is non-abelian must be excluded in view of Lemma 7. Otherwise there exists a second minimal normal subgroup of G in $C_G(N)$ because $1\neq T\leqq C_G(N)$ and $C_G(N)\cap N=1$.

The following corollary is immediate from Proposition 7.

Corollary 3. *Let $G=\langle T_1, \dots, T_r\rangle$ be a group. If each T_i is a solvable submodular subgroup of G, then G is solvable.*

A finite group generated by nilpotent subnormal subgroups is nilpotent. A similar result does not hold for submodular generating subgroups: the finite solvable group

$$G=\langle x, y, a, b \mid x^2=y^3=a^7=b^7=1, [a, b]=1, xy=y^2x, a^x=b, a^y=a^2, b^y=b^4\rangle$$

is generated by a Sylow-2-subgroup of order 2 and a suitable Sylow-3-subgroup of order 3. Both are submodular in G, and G is not even supersolvable.

4. Groups in Which Every Subgroup is Submodular

In this section we consider finite groups in which every subgroup is submodular. Since in these groups all maximal subgroups are modular, they form a (proper) subclass of the class of supersolvable groups.

In [10], Schmidt studies, among other things, finite groups all of which maximal subgroups are modular, so-called $M(1)$-groups. Obviously, every subgroup of a group is submodular if and only if every subgroup is an $M(1)$-group. We show that a subgroup of an $M(1)$-group is submodular if it contains the Frattini subgroup, i.e. if G is an $M(1)$-group, then every subgroup of $G/\phi(G)$ is submodular in G.

A group G is called an LM-group if the subgroup lattice of G is lower semimodular, i.e. if for arbitrary subgroups U and V of G the intersection $U\cap V$ is maximal in V whenever U is maximal in $\langle U, V\rangle$.

Theorem 3. *Let G be a group. Every subgroup of G is submodular in G if and only if G is an LM-group.*

Proof. Suppose every subgroup of G is submodular and choose $U, V\leqq G$ such that U is maximal in $\langle U, V\rangle$. Then U is modular in $\langle U, V\rangle$. Let R be a subgroup of G with $U\cap V<R\leqq V$. From the first modularity equality we get

$$\langle U, R\rangle\cap V=\langle R, U\cap V\rangle=R.$$

Since R is not contained in U, we have $\langle U, R\rangle = \langle U, V\rangle$ which implies $R = V$. Hence $U \cap V$ is maximal in V and G is an LM-group.

To prove the converse, it suffices to show that every maximal subgroup M of an LM-group G is modular. We verify the validity of the modularity equalities.

For the first one choose $U, V \leqq G$ such that $U \leqq V$ and $U \nleqq M$. It is clear that $G = \langle M, V\rangle$ and since G is an LM-group, $M \cap V$ must be maximal in V. This implies

$$\langle M, U\rangle \cap V = V = \langle U, M \cap V\rangle,$$

as required.

The second modularity equality follows directly from the maximality of M.

Finite LM-groups are characterized by Itô and Jones (cf. [11]). In particular every M-group and every direct product of M-groups is an LM-group.

Let G be a finite $M(1)$-group. If $\phi(G)$ denotes the Frattini subgroup of G, then $\phi(G) = \bigcap M_G$, where the intersection is taken over all maximal subgroups M of G. Therefore $G/\phi(G)$ is isomorphic to a subgroup of the direct product $\times G/M_G$. By Lemma 3, each factor G/M_G is either of prime order or is a non-abelian group of order pq, p and q prime numbers. Hence $G/\phi(G)$ itself is a direct product of such groups. (For this result see also: P. Venzke: Finite groups whose maximal subgroups are modular. Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. fis. mat. natur. 58, 828–832 (1975).) Now $G/\phi(G)$ is a direct product of M-groups and therefore an LM-group. We obtain the following characterization of $M(1)$-groups:

Proposition 8. *The group G is an $M(1)$-group if and only if every subgroup of $G/\phi(G)$ is submodular.*

5. Groups with Submodular Sylow Subgroups

It is a well-known fact that nilpotency in finite groups is equivalent to subnormality of the Sylow subgroups, and it is not difficult to prove that finite groups are supersolvable if all their Sylow subgroups are modular. The main purpose of this section is to obtain a characterization of finite groups in which all Sylow subgroups are submodular.

An arbitrary subgroup U of a finite group G has submodular Sylow subgroups if this is true for G: every Sylow subgroup of U is subnormal in some Sylow subgroup of G and hence submodular in G. Also every homomorphic image of a group with submodular Sylow subgroups has submodular Sylow subgroups. Since a group containing a proper non-trivial submodular subgroup is not simple, we have

Lemma 8. *Groups with submodular Sylow subgroups are solvable.*

Our next step is to show that a finite group G with submodular Sylow subgroups possesses a Sylow tower. Here we say that G has a Sylow tower if every homomorphic image of G has a normal Sylow subgroup with respect to the largest prime divisor of its order, i.e. if $p_1 > p_2 > \dots > p_r$ are the distinct prime divisors of $|G|$, then there exist Sylow-p_i-subgroups $G_{p_1}, G_{p_2}, \ldots, G_{p_r}$ of G such that $G_{p_1} G_{p_2} \ldots G_{p_k}$ is normal in G for every $k=1, \ldots, r$.

Proposition 9. *If the Sylow subgroups of the group G are submodular, then G possesses a Sylow tower.*

Proof. We argue by induction on $|G|$. Since the imposed condition is inherited by subgroups and homomorphic images, we may assume that every proper subgroup and every factor group of G has a Sylow tower. Furthermore, using the fact that Sylow tower groups form a saturated formation and a Fitting class (cf. [4]), one may assume the following:

(1) G contains a unique minimal normal subgroup N which, by Lemma 9, is an elementary abelian p-group, p a prime.

(2) G contains a unique maximal normal subgroup K.

(3) $\phi(G)=1$. This implies that the Fitting subgroup $F(G)$ coincides with the unique minimal normal subgroup N.

If p is the largest prime divisor of $|G|$, then we are done because G/N possesses a Sylow tower. Of course, p is the largest prime divisor of $|K|$ (unless we have the trivial case $K=1$). The Sylow subgroup K_s of K to the largest prime divisor s of $|K|$ is normal in K, i.e. $K_s \leqq F(K)=F(G)=N$ and $s=p$. Therefore we may assume that $|G/K|=q \neq p$ is the largest prime divisor of $|G|$. By induction, the Sylow-q-subgroup $G_q N/N$ of G/N is normal in G/N. Hence $G_q N$ is normal in G, and from $q \nmid |K|$ it follows that $G=G_q N$ and $|G|=qp^a$. Since N is a minimal normal subgroup of G, we conclude that G_q is a maximal and, by hypothesis, modular subgroup of G. If G_q is not normal in G, then $G \simeq G/(G_q)_G$ is a non-abelian group of order pq, which is impossible in view of $p<q$. Therefore G_q is a normal subgroup of G, and the proposition is proved.

Every nilpotent group has submodular Sylow subgroups. Therefore we cannot expect restrictions on the structure of the Sylow subgroups in case they are all submodular. However, we have the following

Lemma 9. *If G is a group with submodular Sylow subgroups, then all Sylow subgroups in $G/F(G)$ are elementary abelian.*

Proof. Let p be a prime divisor of $|G|$ and $G_1, \ldots, G_r$ be the distinct Sylow-p-subgroups of G. By Proposition 2, G_i/S_i is elementary abelian where S_i denotes the largest subnormal subgroup of G contained in G_i. The subgroup $S=\langle S_1, \ldots, S_r\rangle$ generated by the S_i's is normal in G. Therefore S is contained in each G_i and coincides with the Sylow-p-subgroup $F_p(G)$ of $F(G)$. Now the assertion follows since $G_i/F_p(G)$ is isomorphic to a Sylow-p-subgroup of $G/F(G)$.

The above lemma implies that in a supersolvable group G with submodular Sylow subgroups, $G/F(G)$ is abelian of squarefree exponent. We show that the converse of this statement is also true.

Proposition 10. *The Sylow subgroups of the supersolvable group G are submodular in G if and only if $G/F(G)$ is abelian of squarefree exponent.*

Proof. We have to establish the sufficency of the condition. Let G be a finite supersolvable group such that $G/F(G)$ is abelian of squarefree exponent and assume the proposition is proved for all groups of smaller order having the same properties as G. Take two different maximal normal subgroups M_1 and M_2 of G with $F(G) \leqq M_1 \cap M_2$. Since $F(M_1)=F(M_2)=F(G)$, the Sylow subgroups of M_1 and M_2 are submodular and the proposition is proved if $|G/M_1| \neq |G/M_2|$.

Therefore we have to deal with the case that $G/F(G)$ is an elementary abelian p-group. All Sylow-q-subgroups for $q \neq p$ are now submodular and it remains to show the submodularity of the Sylow-p-subgroups. Choose a minimal normal subgroup N of G. We may assume that $|N|=q \neq p$. If G_p is a Sylow-p-subgroup of G, we have $G_p \cap F(G) \leqq C_{G_p}(N)$, which implies that $G_p/C_{G_p}(N)$ is elementary abelian and therefore cyclic of order p (or 1). Hence G_p is submodular in $G_p N$. Now $G_p N$ is submodular in G by induction. We conclude that G_p is submodular in G.

We now characterize arbitrary finite groups having all the Sylow subgroups submodular. For abbreviation, denote by $G_{p_1 \ldots p_s}$ the product of Sylow subgroups $G_{p_1}, \ldots, G_{p_s}$.

Theorem 4. *Let G be a group and $p_1 > p_2 > \ldots > p_t$ the distinct prime divisors of $|G|$. The Sylow subgroups of G are submodular in G if and only if the following conditions are satisfied:*

(i) *G possesses a Sylow tower*

$$1 < G_{p_1} < G_{p_1 p_2} < \ldots < G_{p_1 \ldots p_t} = G.$$

(ii) *If G_{p_j} is a Sylow-p_j-subgroup of G such that*

$$[G_{p_1 \ldots p_i}, G_{p_j}] \nleqq G_{p_1 \ldots p_{i-1}} \quad \text{for } j > i,$$

then $p_j | p_i - 1$.

(iii) *$G/F(G)$ has elementary abelian Sylow subgroups.*

Proof. Suppose G is a finite group with submodular Sylow subgroups. Conditions (i) and (iii) were already shown in Proposition 9 and Lemma 9. To establish (ii) we argue by induction on $|G|$ and assume that (ii) holds in G/G_{p_1}.

Let $j > 1$ and $[G_{p_1}, G_{p_j}] \neq 1$. If $G_{p_1 p_j}$ is a proper subgroup of G, we conclude $p_j | p_1 - 1$ by induction. Therefore we may assume $G = G_{p_1 p_j}$, and in addition $\phi(G_{p_1}) = 1$, because $[G_{p_1}, G_{p_j}] \neq 1$ if and only if $[G_{p_1}/\phi(G_{p_1}), G_{p_j}] \neq 1$. Now G_{p_1} is elementary abelian and, by Maschke's theorem, the direct product of minimal G_{p_j}-invariant subgroups. Without loss of generality one can assume that G_{p_1} is itself G_{p_j}-invariant. But then G_{p_j} is a maximal and hence modular subgroup of G. This implies the assertion: by Lemma 3, $G/(G_{p_j})_G$ is a non-abelian group of order $p_1 p_j$, i.e. $|G_{p_1}| = p_1$ and $p_j | p_1 - 1$.

Conversely, assume G to be a finite group satisfying (i), (ii) and (iii). We want to show, again by induction on $|G|$, that the Sylow subgroups of G are submodular.

Since the imposed conditions are also fulfilled in $G_{p_1 \ldots p_{t-1}}$, we only need to prove the submodularity of G_{p_t}. By induction, $G_{p_t p_1}/G_{p_1}$ is submodular in G/G_{p_1}, i.e. $G_{p_t p_1}$ is submodular in G so that we may set $G = G_{p_t p_1}$. Furthermore we can assume that G_{p_1} is a minimal normal subgroup of G as well as $(G_{p_t})_G = 1$.

Now (iii) implies that G_{p_t} is cyclic of order p_t and clearly $[G_{p_1}, G_{p_t}] \neq 1$. By (ii), $p_t | p_1 - 1$ which leads to $|G_{p_1}| = p_1$. Therefore G is of order $p_1 p_t$ and G_{p_t} is (sub)modular in G.

From the remark at the beginning of this section it is clear that the Fitting length $f(G)$ of a finite group G in which all Sylow subgroups are modular is at most 2. However, the situation is different when we merely assume that the Sylow subgroups are submodular:

Let $G_0 = C_{p_1} \lambda C_{p_0}$, $p_0 | p_1 - 1$, be a non-abelian group of order $p_0 p_1$. For $i > j$ define

$$G_i = N_i \lambda G_{i-1}$$

where

(i) N_i is an elementary abelian p_i-group such that $p_j | p_i - 1$ for all $j = 0, 1, \ldots, i-1$;

(ii) N_i is a faithful irreducible G_{i-1}-module.

By construction, the group G_i satisfies the conditions (i)–(iii) from Theorem 4 and has therefore submodular Sylow subgroups. The Fitting length of G_i equals $i+1$: $F(G_i) = N_i$ and $F(G_i/N_j) = N_{j-1}/N_j$ for all $j = 1, \ldots, i$, $N_1 = C_{p_1}$ and $N_0 = C_{p_0}$. Thus we have

Proposition 11. *For every natural number n there exists a finite group G_n such that*

(i) $f(G_n) = n$.

(ii) *G_n possesses submodular Sylow subgroups.*

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