Math. Z. 150, 239–260 (1976)

Mathematische Zeitschrift

Cobordism Theories of Unitary Manifolds with Singularities and Formal Group Laws

Urs Würgler

Mathematisches Institut der Universität Bern,
Sidlerstraße 5, CH-3000 Bern, Switzerland

1. Introduction

In [1] Baas has shown that to any collection $Q=\{Q_1, Q_2, \ldots, Q_n\}$ of stably almost complex manifolds Q_i there corresponds a cobordism theory $MUQ^*(-)$ based on manifolds with cone-like singularities of type $Q_1, \ldots, Q_n$. These cohomology theories (resp. the corresponding homology theories $MUQ_*(-)$) have proved to be a useful tool for example in the work of Johnson and Wilson on homological dimension of finitely generated BP_*-modules (see [2]) and it seems they will be important for a better understanding of $MU_*(X)$, as yet unpublished work of Morava suggests (see [3]).

In general, the theories $MUQ^*(-)$ are not multiplicative, but by their definition they are canonically modules over the multiplicative theory $MU^*(-)$. Moreover, if the sequence $\{q_1, \ldots, q_n\}$, where $q_i=[Q_i]$, is regular in the ring $MU^*(S^0)$, they are also modules over the ring $MUQ^*(S^0)=MU^*/(Q)$. A general cohomology theory $h^*(-)$ with analogous properties will be called an MU-module theory. As in the multiplicative case, to any MU-module theory $h^*(-)$ we can associate a formal group law $F_h(X, Y)$ over the coefficient ring $h^*(S^0)$. The purpose of this paper is to study the relationship between MU-module theories and formal group laws, and one of our main results (2.14) roughly says that a rather large class of MU-module theories are characterized by their coefficient rings and their formal group laws. This is proved by studying first the theories $MUQ^*(-)$ for Q an invariant regular sequence and for this it is useful to remark that for such sequences Q, $MUQ^*(X)$ has a natural profinite topology. We calculate $h^*(MUQ)$ as a comodule over $h^*(MU)$ (Q invariant and regular) and use part of this information to give a simple proof of a version of Landweber's filtration theorem which is used in the proof of some of our results.

The plan of the paper is as follows. In 2. we review properties of the theories $MUQ^*(-)$, introduce some notions needed later and state our main results. In 3. we introduce the category $\mathbf{Mod}_R^{\mathrm{prof}}$ of profinite graded modules over a locally finite graded ring R and study cohomology theories with values in such a category. In 4. our (technical) main result (4.17) is proved by an analysis of comodules in the category $\mathbf{Mod}_R^{\mathrm{prof}}$. Applying the results of 4. we consider in 5. operations for theories $MUQ^*(-)$ with Q invariant and regular and in 6. we study the connections between the theories $MUQ^*(-)$ and formal group laws and discuss some applications.

It is a pleasure for us to thank Professor P.S. Landweber for his interest in the present work. He discovered a lot of mistakes in earlier versions and his many suggestions were very helpful. In particular, he showed me the importance of invariant sequences for the problems studied here.

2. Preliminaries and Statement of the Main Results

Let $\mathbf{W}$ (resp. $\mathbf{W}^f$) be the category of pointed spaces of the homotopy type of a CW-complex (resp. finite CW-complex). All cohomology theories considered will be reduced and representable theories defined on $\mathbf{W}$. If $h^*(-)$ is a cohomology theory, we denote by h its representing spectrum and by $h^* = h^*(S^0)$ its coefficient object. Clearly, $h^*(-)$ may be extended on Boardman's stable category of CW-spectra by setting $h^*(X) = \{X, h\}^*$ for any spectrum X. By the homology theory associated to $h^*(-)$ we mean the theory $h_*(-) = \pi_*(h \wedge -)$. We shall write h_* for $h_*(S^0)$.

By $MU^*(-)$ we denote the complex cobordism theory. Let $Q = \{q_0, q_1, \ldots\}$ be a sequence of elements $q_i \in MU^*$ of degree $|q_i|$, $Q_n = \{q_0, \ldots, q_n\}$. According to Baas [1] there exist CW-spectra MUQ_n with the following properties:

(1) MUQ_n is (for all n) a module spectrum over MU.

(2) For all n there is a map of MU-module spectra $\mu_n: MU \to MUQ_n$ of degree 0.

(3) For all n there exist morphisms η_n: $MUQ_{n-1} \to MUQ_n$ of degree 0 and $\partial_n: MUQ_n \to MUQ_{n-1}$ of degree $-(|q_n|+1)$ such that the diagram

(2.1)
$$\begin{array}{ccc} MUQ_{n-1}^*(X) & \xrightarrow{\eta_n} & MUQ_n^*(X) \\ & {}_{\theta_n}\nwarrow \quad \swarrow_{\partial_n} & \\ & MUQ_{n-1}^*(X) & \end{array}$$

is exact for all finite CW-complexes X. θ_n, η_n and ∂_n are maps of MU-module spectra and θ_n is induced by multiplication with q_n.

(4) $\mu_n = \eta_n \circ \eta_{n-1} \circ \cdots \circ \eta_1$.

If the sequence Q happens to be infinite, we define the spectrum MUQ by $MUQ = \varinjlim (MUQ_n, \eta_{n+1})$ and we set $\mu_Q = \varinjlim \mu_n$.

Recall that a sequence $x_1, x_2, \ldots$ of elements of a ring A is called *A-regular*, if for all positive integers n

(i) $(x_1, \ldots, x_n) \neq A$

(ii) for all $i = 1, \ldots, n$ the element x_i is not a zero-divisor of $A/(x_1, \ldots, x_{i-1})$.

(2.2) **Lemma.** *Let Q be an MU^*-regular sequence (finite or infinite). Then $MUQ^* = MU^*/(Q)$ and μ_Q: $MU^* \to MUQ^*$ is the canonical projection. Moreover,* *for each finite CW-complex, $MUQ^*(X)$ is a module over MUQ^* and induced maps are homomorphisms of MUQ^*-modules.*

Proof. This follows, for example, from [3], appendix.

(2.3) *Remark.* Suppose Q is an MU^*-regular sequence and $q_0 \neq 0$. Then MUQ^i is a finite group in each dimension, so the $MUQ^i(X)$ are finite abelian groups for all finite CW-complexes X. For arbitrary CW-complexes Y we have

$$MUQ^*(Y) \cong \varprojlim MUQ^* \text{ (finite subcomplexes of } Y),$$

so in this case the MUQ^*-module structure may be extended on $\mathbf{W}$.

Following Landweber [8] we call a sequence $Q=\{q_0, q_1, \ldots\}$ of elements of MU^* *invariant*, if $s^E(q_0)=0$ for $E \neq 0$ and $s^E(q_i) \in (q_0, \ldots, q_{i-1})$ for all $E \neq 0$ and $i>0$. Here the E's denote exponent sequences $E=(e_1, e_2, \ldots)$ of non-negative integers and s^E is the Landweber-Novikov operation of index E and degree $2\|E\| \doteq 2\Sigma i e_i$. We will be interested in *invariant regular* sequences Q, such a sequence always satisfies $0 \neq q_0 \in MU^0$. Note that if Q is an invariant regular sequence, the same is true for $Q_n (n \geqq 0)$. There are many invariant regular sequences (cf. [6, 8]), some interesting ones will be considered later.

A cohomology theory $h^*(-)$ is called a *cohomology module*, if $h^*=h^*(S^0)$ is a commutative graded ring with unit element and if there exists a natural transformation $\alpha_h: h^* \otimes h^*(-) \to h^*(-)$ of degree 0 over $\mathbf{W}$ which turns $h^*(X)$ into a h^*-module for all X such that $\alpha(a \otimes \Sigma x) = \Sigma \alpha(a \otimes x)$ (Σ denotes the suspension isomorphism) and $\alpha: h^* \otimes h^* \to h^*$ agrees with the given ring multiplication on h^*. If $c: X \to S^0$ is the trivial map we set $1_X = c^*(1) \in h^0(X)$. A *transformation of cohomology modules* $\Theta: h^*(-) \to k^*(-)$ is a transformation of cohomology theories such that $\alpha_k \circ (\Theta_{S^0} \otimes \Theta) = \Theta \circ \alpha_h$. An *MU-module theory* is a cohomology module $h^*(-)$ which is also a module over the multiplicative theory $MU^*(-)$ in such a way that the natural transformation $\mu_h: MU^*(-) \to h^*(-)$ defined by $\mu_h(u) = \upsilon_h(u \otimes 1_X)$ is a transformation of cohomology modules. $\upsilon_h: MU^*(-) \otimes h^*(-) \to h^*(-)$ denotes the module map. μ_h will be called the Conner-Floyd map of the MU-module theory $h^*(-)$. A *transformation of MU-module theories* $\Theta: h^*(-) \to k^*(-)$ is a transformation of cohomology modules such that $\Theta \upsilon_h(x \otimes y) = \upsilon_k(x \otimes \Theta(y))$ for all $x \in MU^*(X)$, $y \in h^*(X)$.

A transformation $\upsilon_h: MU^*(-) \otimes h^*(-) \to h^*(-)$ with the properties mentioned above will also be called a *complex orientation* ($\mathbb{C}$-orientation) of the cohomology module $h^*(-)$. Note that in general there may exist a lot of different $\mathbb{C}$-orientations on a given cohomology module.

(2.4) *Examples.* (1) Every multiplicative cohomology theory $h^*(-)$ with the property that the canonical complex line bundle η over $P_\infty \mathbb{C}$ is $h^*(-)$-orientable is an MU-module theory. The possible $\mathbb{C}$-orientations of $h^*(-)$ are—via the Conner-Floyd map—in one-one correspondence with the possible $h^*(-)$-orientations of η.

(2) For any MU^*-regular sequence $Q=\{q_0, q_1, \ldots\}$ with $0 \neq q_0 \in MU^0$, $MUQ^*(-)$ is an MU-module theory. This follows from Lemma (2.2), the Remark (2.3) and the general properties of the spectra MUQ mentioned at the beginning

of this section. In general, the theories $MUQ^*(-)$ are not multiplicative, cf. the Remark (2.5) (*d*) in [3].

(2.5) *Remark.* Because we always assume $h^*(-)$ to be representable, the $\mathbb{C}$-orientation of an MU-module theory is given by a map $\upsilon_h\colon MU \wedge h \to h$ of spectra (of degree 0) which satisfies the usual conditions.

(2.6) *Remark.* A homology theory $h_*(-)=\pi_*(h \wedge -)$ is called a *homology module*, if the dual cohomology theory $h^*(-)=\{-, h\}^*$ is a cohomology module.

(2.7) *Remark.* Note that if $\Theta\colon h^*(-) \to k^*(-)$ is a transformation of MU-module theories, the diagram of natural transformations

$$\begin{array}{ccc} h^*(-) & \xrightarrow{\Theta} & k^*(-) \\ {\scriptstyle \mu_k}\nwarrow & & \nearrow{\scriptstyle \mu_k} \\ & MU^*(-) & \end{array}$$

is automatically commutative.

If $h^*(-)$ is a cohomology module, the differentials of the spectral sequence $H^*(-, h^*) \Rightarrow h^*(-)$ are h^*-linear. If $h^*(-)$ is an MU-module theory, the same argument as for $\mathbb{C}$-oriented multiplicative theories shows that the spectral sequence $H^*(MU, h^*) \Rightarrow h^*(MU)$ is trivial. So we get an isomorphism of h^*-modules

(2.8) $h^*(MU) \cong h^*[[\mu_h(s^E)]]$.

Recall that to any $\mathbb{C}$-oriented multiplicative cohomology theory $h^*(-)$ we can associate a 1-dimensional commutative formal group law

(2.9) $F_h(X, Y) = X + Y + \sum_{i,j \geq 1} a_{ij} X^i Y^j \in h^*[[X, Y]]$

(X and Y of degree 2) which describes the Euler class of the tensor product of two complex line bundles as a function of the Euler classes of the factors. A well known theorem of Quillen asserts that $F_{MU}(X, Y)$ is universal for 1-dimensional commutative formal group laws over commutative graded rings with unit. Using the Conner-Floyd map μ_h, we define the formal group law $F_h(X, Y)$ of an MU-module theory $h^*(-)$ by

(2.10) $F_h(X, Y) = (\mu_h)_* F_{MU}(X, Y)$.

For any formal group law $F(X, Y)$ and any natural number n the power series $[n]_F(X)$ is defined recursively by

(2.11) $[1]_F(X) = X, \quad [n]_F(X) = F([n-1]_F(X), X)$.

Note that

(2.12) $[p]_F(X) = pX + \sum_{k \geq 1} c_k X^k, \quad |c_k| = -2(k-1)$.

(2.13) **Definition.** Suppose $F(X, Y)$ is a formal group law over the (graded) ring A of characteristic $p > 0$. $F(X, Y)$ is called *n-flat* ($n \geq 1$), if the coefficients of (2.12) satisfy

(i) $c_{p-1} = c_{p^2-1} = \cdots = c_{p^{n-1}-1} = 0$,

(ii) multiplication by c_{p^n-1} on A is monic,

(iii) multiplication by c_{p^m-1} on $A/(c_{p^n-1}, \ldots, c_{p^{m-1}-1})$

is monic for all $m > n$.

Examples of n-flat formal group laws exist for all n and p (see 6.). We are now in position to state

(2.14) **Theorem.** *Let A be a commutative graded ring of characteristic $p > 0$ and suppose A^q is a finite abelian group for all $q \in \mathbb{Z}$. Let n be a positive integer and suppose $F(X, Y)$ is an n-flat formal group law over A. There exists an MU-module theory $A_F^*(-)$ over the category W with $A_F^*(S^0) = A$ and formal group law $F(X, Y)$. Moreover, any two MU-module theories with this property are isomorphic as MU-module theories.*

Theorem (2.14) is an existence and uniqueness statement for MU-module theories with prescribed formal group (A, F). See (6.7) for a more precise statement.

The crucial step in proving (2.14) (or (6.7)) is Theorem (4.17) which asserts that if Q is an invariant regular sequence, $h^*(-)$ an MU-module theory with locally finite coefficients such that $\mu_h(Q) = 0$, there is an isomorphism

(2.15) $\mathrm{Hom}^*_{MU}(MUQ, h) \cong E_{h^*}[[\beta_0, \beta_1, \ldots]]$.

Here $\mathrm{Hom}^*_{MU}(MUQ, h)$ stands for the abelian group of maps of MU-module spectra $MUQ \to h$ and the right-hand side of (2.15) means the completed exterior algebra over h^* with generators β_i of degree $1 - |q_i|$, considered as an h^*-module. (2.15) contains in particular a complete enumeration of all transformations of MU-module theories $MUQ^*(-) \to h^*(-)$.

(2.15) leads to some insight into the structure of $MUQ^*(MUQ)$. In 5. we construct operations

$$s_Q^E \colon MUQ^*(-) \to MUQ^{*+2\|E\|}(-)$$

(one for each exponent sequence E) with the property that

$$s_Q^E \circ \mu_Q = \mu_Q \circ s^E$$

for all E and prove the following

(2.16) **Theorem.** *Let $Q = \{q_0, q_1, \ldots\}$ be an invariant regular sequence. There exists an isomorphism of MUQ^*-modules*

$$\Phi_Q \colon MUQ^*(MUQ) \cong MUQ^* \hat{\otimes} \mathbb{Z}/(q_0)\,[s_Q^E] \hat{\otimes} E[\beta_0, \beta_1, \ldots]$$

where $E[\beta_0, \beta_1, \ldots]$ denotes the exterior algebra over $\mathbb{Z}/(q_0)$ in the variables β_i of degree $-(|q_i| - 1)$. For any sequence $C = (\varepsilon_0, \varepsilon_1, \ldots)$ with $\varepsilon_i = 0$ or 1, set $\beta^C = \beta_0^{\varepsilon_0} \beta_1^{\varepsilon_1} \ldots$. Then, for all $u \in MU^*(X)$, $x \in MUQ^*(X)$ and all E, C the operations s_Q^E and β^C satisfy

(2.17) $$s_Q^E(ux) = \sum_{F+G=E} s^F(u) \cdot s_Q^G(x)$$

$$\beta^C(ux) = u\beta^C(x).$$

Note that (2.16) contains more information than (2.12) of [3] where $P(n)^*(P(n))$ is calculated as a module over $P(n)^*$ ($P(n)^*(-)$ is the BP-analog of $MUQ^*(-)$ for some special sequence Q).

3. A Category of Profinite Graded R-modules

Let R be a graded commutative ring with unit element and denote by $\mathbf{Mod}_R$ the category of (graded) R-modules. Because R is commutative, we will not distinguish between left and right actions of R. An object M of $\mathbf{Mod}_R$ is called *locally finite*, if M^q is a finite abelian group for all q. Let us assume for the rest of this section that R itself is locally finite.

(3.1) **Definition.** A *profinite graded R-module* is an inverse limit (over a directed set) of locally finite R-modules.

It follows from the definition that a graded profinite R-module carries a natural topology: If $M=\varprojlim_\alpha M_\alpha$, M_α locally finite, we consider the M_α as discrete R-modules and give M the limit topology. So, in each dimension q, M^q is a profinite abelian group, i.e. a compact, Hausdorff and totally disconnected abelian group. Let $\mathbf{Mod}_R^{\mathrm{prof}}$ be the subcategory of $\mathbf{Mod}_R$ whose objects are profinite R-modules and whose morphisms are homomorphisms of R-modules which are continuous in each dimension. Let M be an object of $\mathbf{Mod}_R^{\mathrm{prof}}$, $M=\varprojlim_\alpha M_\alpha$, and let $p_\alpha\colon M\to M_\alpha$ be the canonical projections. Because the M_α are discrete and from properties of the limit topology one easily sees that the system

(3.2) $\{\bar{M}_\alpha=\ker(M\xrightarrow{p_\alpha}M_\alpha)\}$

is a basis of open neighborhoods of $0\in M$ (in each dimension). Because the M^q are compact they carry a uniquely determined uniform structure which induces the given topology and the M^q are complete with respect to this uniform structure. From this one sees that

(3.3) $M\cong\varprojlim_\alpha M/\bar{M}_\alpha$.

Clearly, the quotients $M/\bar{M}_\alpha$ are again locally finite.

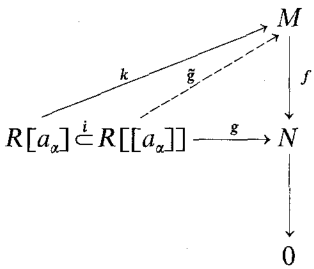
(3.4) **Lemma.** *The category* $\mathbf{Mod}_R^{\mathrm{prof}}$ *is abelian.*

Proof. It is obvious that $\mathbf{Mod}_R^{\mathrm{prof}}$ is an additive category. Because the topologies in view are compact and Hausdorff, any continuous bijection in $\mathbf{Mod}_R^{\mathrm{prof}}$ is an isomorphism, so we have only to show that any morphism $f\colon M\to N$ in $\mathbf{Mod}_R^{\mathrm{prof}}$ has a kernel and a cokernel. But this follows from (3.3) and (again) the fact that M and N are compact Hausdorff spaces in each dimension.

Let $\{a_\alpha\}_{\alpha\in A}$ be a set of indeterminates of degree $|a_\alpha|$ and consider the R-module $R[[a_\alpha]]$ whose elements of degree q are the infinite sums $\sum_\alpha\lambda_\alpha a_\alpha$, $\lambda_\alpha\in R$ and $|\lambda_\alpha|+|a_\alpha|=q$. Clearly, $R[[a_\alpha]]=\varprojlim_U R[a_\alpha;\alpha\in U]$ where $\{U\}$ is the directed set of all finite subsets of A. So, $R[[a_\alpha]]$ is an object of $\mathbf{Mod}_R^{\mathrm{prof}}$ and it is easily seen that – in $\mathbf{Mod}_R^{\mathrm{prof}}$ – $R[[a_\alpha]]$ is isomorphic to the product $\prod_{\alpha\in A}R\cdot a_\alpha$.

(3.5) **Lemma.** *$R[[a_\alpha]]$ is projective in* $\mathbf{Mod}_R^{\mathrm{prof}}$.

Proof. Consider the inclusion $R[a_\alpha] \subset R[[a_\alpha]]$. We give $R[a_\alpha]$ the subspace topology. Because $R[[a_\alpha]]$ is a product it follows that $R[a_\alpha]$ is a discrete space and dense in $R[[a_\alpha]]$. Consider the diagram



where f is an epimorphism in $\mathbf{Mod}_R^{\mathrm{prof}}$. Because $R[a_\alpha]$ is projective in $\mathbf{Mod}_R$ (the category of R-modules), there exists a homomorphism of R-modules k so that $fk = gi$. Because $R[a_\alpha]$ is a discrete space, k is automatically (uniformly) continuous and because $R[a_\alpha]$ is dense in $R[[a_\alpha]]$, it follows by standard arguments that k may be extended on $R[[a_\alpha]]$ by a morphism $\tilde{g}$ in $\mathbf{Mod}_R^{\mathrm{prof}}$. Again because $R[a_\alpha]$ is dense in $R[[a_\alpha]]$ one sees that $\tilde{g}$ is a lift of g.

For later use we need a sort of tensor product in the category $\mathbf{Mod}_R^{\mathrm{prof}}$. Let M, N be profinite R-modules, $M = \varprojlim M_\alpha$, $N = \varprojlim N_\beta$. We define their tensor product $M \underset{R}{\boxtimes} N$ in $\mathbf{Mod}_R^{\mathrm{prof}}$ by

(3.6) $$M \underset{R}{\boxtimes} N = \varprojlim_{\alpha,\beta} \left(M \underset{R}{\otimes} N / \left[\mathrm{im}\left(\bar{M}_\alpha \underset{R}{\otimes} N\right) + \mathrm{im}\left(M \underset{R}{\otimes} \bar{N}_\beta\right)\right]\right)$$

where "im" refers to the canonical maps $\bar{M}_\alpha \underset{R}{\otimes} N \to M \underset{R}{\otimes} N$ resp. $M \underset{R}{\otimes} \bar{N}_\beta \to M \underset{R}{\otimes} N$. It is easily established that the product $M \underset{R}{\boxtimes} N$ is an object of $\mathbf{Mod}_R^{\mathrm{prof}}$, is associative and commutative and satisfies $M \underset{R}{\boxtimes} R \cong M$.

It is well known (see [14] for example) that the inverse limit functor is exact on the category of profinite abelian groups. From this it follows that $\varprojlim$ is exact on the category $\mathbf{Mod}_R^{\mathrm{prof}}$. Because for any profinite R-module M,

$$R[[a_\alpha]] \underset{R}{\boxtimes} M \cong M[[a_\alpha]] \cong \varprojlim_U M[a_\alpha; \alpha \in U]$$

(U a finite subset of A), we get

(3.7) **Lemma.** *The functor $R[[a_\alpha]] \underset{R}{\boxtimes} -$ is exact on* $\mathbf{Mod}_R^{\mathrm{prof}}$.

Let $h^*(-)$ be a cohomology module defined on $\mathbf{W}$ and suppose that the coefficient ring h^* of $h^*(-)$ is locally finite. For any CW-complex X denote by $\{X_\alpha\}$ the directed system of all finite subcomplexes of X. It is well known (see for example [14], p. 309), that under the present assumption on h^* we have

(3.8) $$h^*(X) \cong \varprojlim_\alpha h^*(X_\alpha).$$

This shows that $h^*(X)$ is naturally a profinite h^*-module. Since every continuous map $f: X \to Y$ of CW-complexes induces a morphism $\{X_\alpha\} \to \{Y_\beta\}$ of directed sets we get

(3.9) **Lemma.** *Any cohomology module $h^*(-)$ with locally finite coefficient ring h^* takes values in* $\mathbf{Mod}_{h^*}^{\mathrm{prof}}$.

(3.10) *Remark.* We note that in the above lemma the category $\mathbf{W}$ may be replaced by the category $\mathbf{S}(-1)$ of (-1)-connected CW-spectra.

(3.11) *Remark.* For an infinite CW-complex X, $h^*(X)$ is usually considered as a graded topological group with topology induced by the skeleton filtration

$$F_p h^*(X) = \ker\{h^*(X) \to h^*(X^{p-1})\}.$$

The (algebraic) isomorphism $h^*(X) \cong \varprojlim_\alpha h^*(X_\alpha)$ is continuous by well known properties of the limit topology (where $h^*(X)$ is given the filtration topology), but in general it is not a homeomorphism.

Clearly, over $\mathbf{W}^f$, any cohomology module with locally finite coefficients takes values in $\mathbf{Mod}_h^{\mathrm{prof}}$. An easy argument shows (see for example [10], p. 36) that for any cohomology module $h^*(-)$ defined over $\mathbf{W}^f$ and with locally finite coefficients, there exists a unique extension $Eh^*(-)$ over $\mathbf{W}$ such that

$$(3.12)\quad \begin{array}{ccc} \mathbf{W}^f & \xrightarrow{h^*(-)} & \mathbf{Mod}_h^{\mathrm{prof}} \\ \cap & & \| \\ \mathbf{W} & \xrightarrow{Eh^*(-)} & \mathbf{Mod}_h^{\mathrm{prof}} \end{array}$$

commutes. This extension is given by

(3.13) $Eh^*(X) = \varprojlim h^*$ (finite subcomplexes of X).

(3.14) **Lemma.** *Let $h^*(-)$ be an MU-module theory with locally finite coefficients. The module map v_h: $MU \wedge h \to h$ induces a natural isomorphism in* $\mathbf{Mod}_h^{\mathrm{prof}}$

$$t: h^*(MU) \underset{h^*}{\boxtimes} h^*(X) \xrightarrow{\cong} h^*(MU \wedge X)$$

over the categories $\mathbf{W}$ *or* $\mathbf{S}(-1)$.

For the proof of (3.14) we will need the following remark. Suppose M is a graded R-module generated by a set $\{m_\alpha\}$ of elements. Let $\{U\}$ be the set of all R-submodules of M such that

a) M/U is locally finite,

b) almost all m_α lie in U.

Then we call $\hat{M} = \varprojlim_U M/U$ the *profinite completion* of M. As in the case of profinite groups one shows that the profinite completion is functorial and that $\hat{M} \cong M$ if $M \in \mathbf{Mod}_R^{\mathrm{prof}}$.

Proof of (3.14) Let X be a CW-complex or a (-1)-connected spectrum. There is a natural transformation

$$r: h^*(X)[\mu(s^E)] \to h^*(MU \wedge X)$$

defined by

$$r(\sum_E x_E \cdot \mu(s^E)) = \sum_E v_h(s^E \wedge x_E).$$

From the remark before the proof we see that r induces a natural transformation

$$\begin{array}{ccc} \widehat{h^*(X)\,[\mu(s^E)]} & \xrightarrow{\hat{r}} & \widehat{h^*(MU \wedge X)} \\ \| & & \| \\ h^*(X)\,[[\mu(s^E)]] & & h^*(MU \wedge X). \end{array}$$

Now $h^*(X)\,[[\mu(s^E)]]$ and $h^*(MU \wedge X)$ are both additive cohomology theories and $\hat{r}$ is a transformation of cohomology theories. From (2.8) it follows that $\hat{r}$ is an isomorphism for $X = S^0$, so $\hat{r}$ is an equivalence. From (3.6) one knows that

$$h^*(MU) \underset{h^*}{\boxtimes} h^*(X) \cong h^*(X)\,[[\mu(s^E)]],$$

so the lemma follows.

Let E be an MU-module spectrum. Suppose $q \in MU^n$ is represented by the map $\varphi: S^n \to MU$ and denote by θ_q^E the composite

$$\theta_q^E: S^n \wedge E \xrightarrow{\varphi \wedge \mathrm{id}} MU \wedge E \xrightarrow{v_E} E.$$

If $q \in MU^*$, we denote by $\langle q \rangle$ the invariant ideal of MU^* generated by q, i.e. the intersection of all ideals of MU^* which are invariant under the action of all Landweber-Novikov operations s^E and which contain q.

(3.15) **Lemma.** *Let $h^*(-)$ be an MU-module theory with locally finite coefficients and suppose $\mu_h(\langle q \rangle) = 0$ for some $q \in MU^*$. Then*

$$0 = h^*(\Theta_q^E):\ h^*(E) \to h^*(S^{|q|} E)$$

for every (-1)-connected MU-module spectrum E.

Proof. Recall that $h^*(MU) = h^*[[\mu_h(s^E)]]$. This is a product in $\mathbf{Mod}_h^{\mathrm{prof}}$. We get

$$\begin{aligned} \varphi^*(x) &= \varphi^*(\prod_E \lambda_E \mu_h(s^E)) \\ &= \prod_E \lambda_E \mu_h \varphi^*(s^E) \\ &= \prod_E \lambda_E \mu_h(s^E(q)) = 0 \end{aligned}$$

because μ_h vanishes on $\langle q \rangle$. So $\varphi^* = 0$. Using Lemma (3.14) we get a commutative diagram

$$\begin{array}{ccc} h^*(S^n \wedge E) & \xrightarrow{\cong} & h^*(S^n) \underset{h^*}{\boxtimes} h^*(E) \\ \uparrow{\scriptstyle (\varphi \wedge \mathrm{id})^*} & & \uparrow{\scriptstyle \varphi^* \otimes \mathrm{id}} \\ h^*(MU \wedge E) & \xrightarrow{\cong} & h^*(MU) \underset{h^*}{\boxtimes} h^*(E). \end{array}$$

The result follows.

Let $Q = \{q_0, q_1, \ldots\}$ be an MU^*-regular sequence and suppose $q_0 \neq 0, q_0 \in MU^0$. Consider the exact triangle of spectra

$$(3.16)\quad \begin{array}{ccccc} MUQ_{n-1} & \xrightarrow{\Theta_n^{n-1}} & MUQ_{n-1} & \longrightarrow & C_\Theta \\ \uparrow & & & & \big| \\ \big| & & & & \big| \\ \llcorner & \text{---} & \text{---} & \text{---} & \lrcorner \end{array}$$

Here $\Theta_n^{n-1} = \Theta_{q_n}^E$ with $E = MUQ_{n-1}$ and C_Θ is the cofibre of Θ_n^{n-1}. Set $MUQ_0 = MU$. Because $MUQ^*(-)$ takes values in $\mathbf{Mod}_{MUQ}^{\mathrm{prof}}$, it follows that the diagram (2.1) is exact for all (-1)-connected spectra X. From this it follows easily that

$$(3.17)\quad C_\Theta \cong MUQ_n \quad (\text{in } \mathbf{S})$$

for all $n \geq 0$. So, for all n, the triangles

$$(3.18)\quad \begin{array}{ccccc} MUQ_{n-1} & \xrightarrow{\Theta_n^{n-1}} & MUQ_{n-1} & \xrightarrow{\eta_n} & MUQ_n \\ \uparrow & & & & \big| \\ \big| & & & & \big| \\ \llcorner & \text{---} & \partial^n & \text{---} & \lrcorner \end{array}$$

are exact.

Remark. In fact (3.15) may be proved by using Adams lemma ([15], p. 20) instead of (3.14). This shows that for (3.15) the assumption h^* locally finite may be droped.

4. Comodules over the Coalgebra $h^*(MU)$

Let $h^*(-)$ be an MU-module theory with locally finite coefficients. Set $\tilde{s}^E = \mu_h(s^E) \in h^*(MU)$. From Lemma (3.14) it follows that the multiplication map

$$m\colon MU \wedge MU \to MU$$

induces a morphism

$$(4.1)\quad \Psi = t^{-1} \circ m^*\colon h^*(MU) \to h^*(MU \wedge MU) \cong h^*(MU) \underset{h^*}{\boxtimes} h^*(MU)$$

in the category $\mathbf{Mod}_h^{\mathrm{prof}}$. The Cartan formula for the Landweber-Novikov operations s^E implies that Ψ is given on the generators $\tilde{s}^E$ by the formula

$$(4.2)\quad \Psi(\tilde{s}^E) = \sum_{G+F=E} \tilde{s}^G \boxtimes \tilde{s}^F.$$

From the properties of the product map m it follows that the pair $(h^*(MU), \Psi)$ is a coalgebra in $\mathbf{Mod}_h^{\mathrm{prof}}$.

By a $h^*(MU)$-comodule we mean an object M of $\mathbf{Mod}_h^{\mathrm{prof}}$ together with a morphism

$$\Psi_M\colon M \to h^*(MU) \underset{h^*}{\boxtimes} M$$

of $\mathbf{Mod}_h^{\mathrm{prof}}$ which satisfies the usual identities. The category of $h^*(MU)$-comodules and morphisms of $h^*(MU)$-comodules is denoted by $\mathbf{Com}_h$.

(4.3) **Lemma.** $\mathbf{Com}_h$ *is an abelian category.*

Proof. As is well known this follows if we show that the functor $h^*(MU) \underset{h^*}{\boxtimes} -$ is exact on the category $\mathbf{Mod}_h^{\mathrm{prof}}$. But this is a consequence of (3.7) and (2.8). Let

$$S\colon \mathbf{Com}_h \to \mathbf{Mod}_h^{\mathrm{prof}}$$

be the forgetful functor and

$$F\colon \mathbf{Mod}_h^{\mathrm{prof}} \to \mathbf{Com}_h$$

the functor which assigns to each object M of $\mathbf{Mod}_h^{\mathrm{prof}}$ the extended $h^*(MU)$-comodule

(4.4) $F(M) = h^*(MU) \underset{h^*}{\boxtimes} M$

with coaction map

(4.5) $\Psi_{F(M)}\colon h^*(MU) \underset{h^*}{\boxtimes} M \xrightarrow{\Psi \otimes \mathrm{id}} h^*(MU) \underset{h^*}{\boxtimes} (h^*(MU) \underset{h^*}{\boxtimes} M).$

Note that

$$e\colon \mathrm{Hom}_{\mathbf{Com}_h}(M, F(N)) \to \mathrm{Hom}_{\mathbf{Mod}_h^{\mathrm{prof}}}(S(M), N),$$

where e assigns to f the composite $M \xrightarrow{f} h^*(MU) \underset{h^*}{\boxtimes} N \xrightarrow{\varepsilon \boxtimes \mathrm{id}} h^* \underset{h^*}{\boxtimes} N \cong N$ is an isomorphism (ε denotes the augmentation), so the functors S and F are adjoint.

(4.6) **Lemma.** *Suppose L and M are objects of* $\mathbf{Com}_h$. *If $S(L)$ is projective in* $\mathbf{Mod}_h^{\mathrm{prof}}$ *and $M = F(N)$ for some N, then*

$$\mathrm{Ext}^{*,*}_{\mathbf{Com}_h}(L, M) = 0.$$

Proof. From what has been said above, this is a standard application of relative homological algebra. Every object M of $\mathbf{Com}_h$ of the form $M = F(N)$ is a relative injective with respect to the class of split monomorphisms in $\mathbf{Mod}_h^{\mathrm{prof}}$. But so long as L is projective in $\mathbf{Mod}_h^{\mathrm{prof}}$, we may compute $\mathrm{Ext}_{\mathbf{Com}_h}(L, M)$ by resolving M by relative injectives. The lemma follows.

Let E be a (-1)-connected MU-module spectrum with structure map υ_E: $MU \wedge E \to E$ and suppose $h^*(-)$ is an MU-module theory with locally finite coefficients. Using Lemma (3.14) we get a morphism in $\mathbf{Mod}_h^{\mathrm{prof}}$

(4.7) $\upsilon_E^*\colon h^*(E) \to h^*(MU \wedge E) \cong h^*(MU) \underset{h^*}{\boxtimes} h^*(E)$

which turns $h^*(E)$ into a $h^*(MU)$-comodule.

If E is an MU-module spectrum and X an arbitrary spectrum, we will always consider $E \wedge X$ as an MU-module spectrum with structure map $\upsilon_{E \wedge X}$ given by

(4.8) $\upsilon_{E \wedge X}\colon MU \wedge (E \wedge X) \xrightarrow{\upsilon_E \wedge \mathrm{id}} E \wedge X.$

(4.9) **Lemma.** *Suppose $h^*(-)$ is an MU-module theory with locally finite coefficients, X a (-1)-connected spectrum. Then $(h^*(MU \wedge X), \upsilon_{MU \wedge X}^*)$ is an extended*

$h^*(MU)$-comodule, i.e. we have an isomorphism

$$h^*(MU\wedge X)\cong F(h^*(X))=h^*(MU)\underset{h^*}{\boxtimes} h^*(X)$$

in $\mathbf{Com}_h$.

Proof. Applying Lemma (3.14) twice, we get a natural isomorphism

$$h^*(MU\wedge MU\wedge X)\cong h^*(MU)\underset{h^*}{\boxtimes} h^*(MU)\underset{h^*}{\boxtimes} h^*(X).$$

The lemma follows from the commutativity of the diagram

$$\begin{array}{ccc} h^*(MU\wedge X) & \xrightarrow{(\upsilon_{E\wedge X})^*} & h^*(MU\wedge MU\wedge X) \\ \uparrow\cong & & \uparrow\cong \\ h^*(MU)\underset{h^*}{\boxtimes} h^*(X) & \xrightarrow{\upsilon^*_{MU}\boxtimes \mathrm{id}} & h^*(MU)\underset{h^*}{\boxtimes} h^*(MU)\underset{h^*}{\boxtimes} h^*(X). \end{array}$$

Let $E[\tau_0,\dots,\tau_n]$ be the exterior algebra over h^* in the variables τ_i. As an h^*-module, $E[\tau_0,\dots,\tau_n]$ is generated by the elements τ^C, where $C=(\varepsilon_0,\dots,\varepsilon_n)$, $\varepsilon_i=0$ or 1 and $\tau^C=\tau^{\varepsilon_0}\cdot\dots\cdot\tau_n^{\varepsilon_n}$. If M is a $h^*(MU)$-comodule, we always consider $M\underset{h^*}{\boxtimes} E[\tau_0,\dots,\tau_n]$ as a $h^*(MU)$-comodule with coaction map Ψ given by $\Psi(m\boxtimes\tau^C)=\Psi(m)\boxtimes\tau^C$. Note that $M\underset{h^*}{\boxtimes} E[\tau_0,\dots,\tau_n]$ is isomorphic to the direct sum (in $\mathbf{Com}_h$) of as many copies (up to possible dimension shifts) of M as there are h^*-module generators in $E[\tau_0,\dots,\tau_n]$.

Let $h^*(-)$ be an MU-module theory with locally finite coefficients and suppose there is given an exact triangle of (-1)-connected MU-module spectra and morphisms of MU-module spectra

(4.10) $$\begin{array}{ccccc} M & \xrightarrow{\theta} & M & \xrightarrow{\eta} & L \\ \uparrow & & & & \downarrow \\ & & \xleftarrow{\quad\partial\quad} & & \end{array}$$

(4.11) **Lemma.** *Suppose $h^*(\theta)=0$, $h^*(M)$ is projective in $\mathbf{Mod}_h^{\mathrm{prof}}$ and $h^*(M)$ is an extended $h^*(MU)$-comodule. Then there is an isomorphism of $h^*(MU)$-comodules*

$$\Phi\colon h^*(L)\xrightarrow{\cong} h^*(M)\underset{h^*}{\boxtimes} E[\beta]$$

where β has degree $-(\text{degree }\theta-1)$.

Proof. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & h^*(M)\underset{h^*}{\boxtimes} h^* & \xrightarrow{\mathrm{id}\boxtimes\lambda} & h^*(M)\underset{h^*}{\boxtimes} E[\beta] & \xrightarrow{\mathrm{id}\boxtimes\pi} & h^*(M)\underset{h^*}{\boxtimes} h^* & \longrightarrow & 0 \\ & & \downarrow\cong & & & & \downarrow\cong & & \\ 0 & \longrightarrow & h^*(M) & \xrightarrow{\partial^*} & h^*(L) & \xrightarrow{\eta^*} & h^*(M) & \longrightarrow & 0. \end{array}$$

The vertical isomorphisms are the canonical ones and we consider them as identifications. λ is given by $\lambda(x)=x\beta$ and π is the unique homomorphism of h^*-modules such that $\pi(1)=1$, $\pi(\beta)=0$. Because all maps in the diagram are morphisms of $h^*(MU)$-comodules and the rows are exact, they both are elements of $\mathrm{Ext}^{*,*}_{\mathbf{Com}_h}(h^*(M), h^*(M))$. Now $h^*(M)$ is projective in $\mathbf{Mod}_h^{\mathrm{prof}}$ and an extended $h^*(MU)$-comodule, so the lemma follows from (4.6).

(4.12) **Proposition.** *Let $h^*(-)$ be an MU-module theory with locally finite coefficients, Q an invariant MU^*-regular sequence and suppose $\mu_h(Q)=0$. For all $n\geq 0$ there are isomorphisms of $h^*(MU)$-comodules*

$$h^*(MUQ_n)\cong h^*(MU)\underset{h^*}{\boxtimes} E[\beta_0, \ldots, \beta_{n-1}]$$

where degree $(\beta_i)=-(|q_i|-1)$, and the morphisms

$$\eta_n^*: h^*(MUQ_n)\to h^*(MUQ_{n-1})$$

are split epic in $\mathbf{Com}_h$.

Proof. From (3.15) and (3.18) it follows that for all n we have an exact sequence of $h^*(MU)$-comodules

$$0\longrightarrow h^*(MUQ_{n-1})\xrightarrow{\partial_n^*} h^*(MUQ_n)\xrightarrow{\eta_n^*} h^*(MUQ_{n-1})\longrightarrow 0.$$

Using (4.6) and (3.5) the result follows by induction on n.

Now the next point is to show how (4.12) can be used to construct transformations of MU-module theories $MUQ^*(-)\to h^*(-)$. Let M be a $h^*(MU)$-comodule with structure map Ψ_M. An element $a\in M$ is called *primitive* (with respect to Ψ_M), if $\Psi_M(a)=1\boxtimes a$.

(4.13) **Lemma.** *Let $h^*(-)$ be an MU-module theory with locally finite coefficients and suppose $h^q(S^0)=0$ for $q>0$. Let E be a (-1)-connected MU-module spectrum and $g: E\to h$ a map of spectra. g is a morphism of MU-module spectra if and only if it is a primitive element of the $h^*(MU)$-comodule $h^*(E)$.*

Proof. By definition, g is a map of MU-module spectra iff the diagram

$$(*)\quad \begin{array}{ccc} MU\wedge E & \xrightarrow{\upsilon_E} & E \\ \downarrow{\scriptstyle \mathrm{id}\wedge g} & & \downarrow{\scriptstyle g} \\ MU\wedge h & \xrightarrow{\upsilon_h} & h \end{array}$$

commutes in $\mathbf{S}$. Because h is (-1)-connected, $(*)$ induces a diagram (compare (3.14))

$$\begin{array}{ccc} h^*(h) & \xrightarrow{g^*} & h^*(E) \\ \downarrow{\scriptstyle \upsilon_h^*} & & \downarrow{\scriptstyle \upsilon_E^*} \\ h^*(MU)\underset{h^*}{\boxtimes} h^*(h) & \xrightarrow{\mathrm{id}\boxtimes g^*} & h^*(MU)\underset{h^*}{\boxtimes} h^*(E). \end{array}$$

But $g = g^*(\mathrm{id}_h) \in h^*(E)$ and $(*)$ commutes iff

$$\begin{aligned} \upsilon_E^*(g) &= \upsilon_E^*(g^*(\mathrm{id}_h)) \\ &= (\mathrm{id} \boxtimes g^*)\, \upsilon_h^*(\mathrm{id}_h) \\ &= (\mathrm{id} \boxtimes g^*)\,(1 \boxtimes \mathrm{id}_h) \\ &= 1 \quad \boxtimes g. \end{aligned}$$

This proves the lemma.

The next lemma shows, that the restriction $h^q(S^0) = 0$ for $q > 0$ in (4.13) is not essential.

(4.14) **Lemma.** *Let $h^*(-)$ be a cohomology theory, $k^*(-)$ its (-1)-connected cover. If $h^*(-)$ is an MU-module theory, $k^*(-)$ admits the structure of an MU-module theory such that the canonical transformation π: $k^*(-) \to h^*(-)$ becomes a transformation of MU-module theories.*

Proof. Because $k^q(S^0) = h^q(S^0)$ for $q \leqq 0$, k^* is a subring of h^*. Using the description

$$k^q(X) = \mathrm{im}\, \{h^q(X/X^{q-1}) \to h^q(X/X^{q-2})\}$$

for $k^*(-)$ (X a CW-complex), it is easily seen that the natural transformation

$$k^* \otimes h^*(X) \to h^* \otimes h^*(X) \xrightarrow{\alpha_h} h^*(X)$$

factors through $k^*(-)$. So $k^*(-)$ becomes a cohomology module. The canonical map π: $k^*(-) \to h^*(-)$ is a transformation of cohomology modules by construction of the k^*-module structure on $k^*(-)$. Now recall that the (-1)-connected cover k of the spectrum h is characterized by the property that if E is any (-1)-connected spectrum and g: $E \to h$ a map of spectra there exists a unique lift $\tilde{g}$: $E \to k$, such that the diagram

(4.15)
$$\begin{array}{ccc} & & k \\ & \overset{\tilde{g}}{\nearrow} & \downarrow \pi \\ E & \xrightarrow{g} & h \end{array}$$

commutes. As $MU \wedge k$ is (-1)-connected, the unique lift of the map

$$MU \wedge k \xrightarrow{\mathrm{id} \wedge \pi} MU \wedge h \xrightarrow{\upsilon_h} h$$

turns k into an MU-module spectrum. π is clearly a map of MU-module spectra. μ_k: $MU \to k$ is a transformation of cohomology modules because it is the unique lift of μ_h: $MU \to h$ and by the construction of the k^*-module structure on $k^*(-)$. This proves the lemma.

Let $\{E_{n-1} \xrightarrow{\eta_n} E_n\}_{n \geqq 1}$ be a system of (-1)-connected MU-module spectra and morphisms of MU-module spectra, $h^*(-)$ an MU-module theory with locally finite coefficients. Clearly, $E = \varinjlim E_n$ is again an MU-module spectrum and the canonical maps μ_n: $E_n \to E$ are morphisms of MU-module spectra. The easy proof of the following lemma is left to the reader.

(4.16) **Lemma.** *If* $\{a_n\}_{n>0}$ *is a sequence of primitive elements* $a_n \in h^*(E_n)$ *such that* $\eta_n^*(a_{n-1}) = a_n$ *for all n,* $a = \varprojlim a_n \in h^*(E)$ *is again primitive.*

For any two MU-module spectra E, F denote by $\mathrm{Hom}_{MU}^*(E, F)$ the graded abelian group of maps of MU-module spectra. If $E^*(-)$, $F^*(-)$ are in addition MU-module theories, let $\mathrm{Nat}_{MU}(E, F)$ be the set of transformations of MU-module theories. Finally, let ε: $E_A[\beta_0, \beta_1, \ldots] \to A$ be the homomorphism of A-modules defined by $\varepsilon(\beta^C) = 0$ for $C \neq 0$, $\beta(1) = 1$, and denote by $\Pr\{M\}$ the set of primitive elements of the $h^*(MU)$-comodule M.

(4.17) **Theorem.** *Let* $h^*(-)$ *be an* MU*-module theory with locally finite coefficients,* Q *an invariant regular sequence and suppose* $\mu_h(Q) = 0$. *There is a degree-preserving isomorphism*

$$\mathrm{Hom}_{MU}^*(MUQ, h) \cong E_{h^*}[[\beta_0, \beta_1, \beta_2, \ldots]]$$

and an element θ *of* $\mathrm{Hom}_{MU}^*(MUQ, h)$ *lies in* $\mathrm{Nat}_{MU}(MUQ, h)$ *iff* $\varepsilon(\theta) = 1$.

Proof. Note first that by (4.14) and the universal property of the (-1)-connected cover construction (4.15) we may assume that the spectrum h is (-1)-connected. From (4.12), (4.13) and (4.16) we know that

$$\begin{aligned}\mathrm{Hom}_{MU}^*(MUQ, h) &\cong \Pr\{h^*(MUQ)\} \\ &\cong \Pr\{h^*(MU) \underset{h^*}{\boxtimes} E[[\beta_0, \beta_1, \ldots]]\} \\ &\cong \Pr\{h^*(MU)\} \underset{h^*}{\boxtimes} E[[\beta_0, \beta_1, \ldots]].\end{aligned}$$

Next one sees that there is an isomorphism f: $\Pr\{h^*(MU)\} \cong h^*$ such that $f(\mu_h) = 1$: if $\theta \in \mathrm{Hom}_{MU}^*(MU, h)$, θ satisfies $\theta(xy) = x \cdot \theta(y)$. Taking $y = 1$ we get $\theta(x) = x \cdot \theta(1)$, so θ is determined by $\theta(1)$. Define f by $f(\theta) = \theta(1)$. Because any element of h^* clearly determines a primitive element of $h^*(MU)$, f is an isomorphism. Note that $f(\mu_h) = \mu_h(1) = 1$. So we get

$$\mathrm{Hom}_{MU}^*(MUQ, h) \cong E[[\beta_0, \beta_1, \ldots]].$$

Because μ_Q: $MU^* \to MUQ^*$ is epic, an element $\theta \in \Pr\{h^0(MUQ)\}$ corresponds to a transformation of MU-module theories iff $\mu_Q^*(\theta) = \mu_h \in h^0(MU)$, i.e. iff $\varepsilon(\theta) = 1$.

From (4.17) it follows that, under the conditions of the theorem, the set $\mathrm{Nat}_{MU}(MUQ, h)$ is not void. How many elements there are in $\mathrm{Nat}_{MU}(MUQ, h)$ depends of Q and h^*: let ε_i be 0 or 1 and $n > 0$ and set

(4.18) $$J(h, Q)_n = \left\{\lambda \in h^* \mid \lambda \neq 0, |\lambda| = -\sum_{i=0}^{n-1} \varepsilon_i(1 - |q_i|), \sum_i \varepsilon_i \neq 0\right\},$$

$$J(h, Q) = \bigcup_{n \geqq 0} J(h, Q)_n.$$

It is easily seen that $\mathrm{Nat}_{MU}(MUQ, h)$ contains exactly one element iff $J(h, Q) = \varnothing$, so in this case there is a *canonical* transformation $MUQ^*(-) \to h^*(-)$. This remark applies for example if $Q = (p)$ and $h^*(-)$ is such that $p \cdot h^* = 0$, $h^{-1} = 0$. This implies that $MU\mathbb{Z}_p^*(-)$ is universal for MU-module theories $h^*(-)$ such that $p \cdot h^* = 0$, $h^{-1} = 0$.

5. Landweber-Novikov Operations in $MUQ^*(-)$

As a first application of Theorem (4.17) we construct in this section a family of operations s_Q^E in the theory $MUQ^*(-)$ (Q an invariant regular sequence) with properties analogous to those of the Landweber-Novikow operations in $MU^*(-)$. Although we are not able to determine the behaviour of the operations s_Q^E under composition, they may be used to give a very simple proof of a version of Landweber's filtration theorem.

(5.1) **Theorem.** *Let Q be an invariant regular sequence. There exists a family $\{s_Q^E\}$ of natural and stable operations (one for each exponent sequence $E=(e_1, e_2, \ldots)$)*

$$s_Q^E: MUQ^i(-) \to MUQ^{i+2\|E\|}(-)$$

such that

(i) $$s_Q^E(ux) = \sum_{F+G=E} s^F(u) \cdot s_Q^G(x)$$

where $u \in MU^*(X)$ and $x \in MUQ^*(X)$

(ii) *the diagram*

$$\begin{array}{ccc} MU^*(-) & \xrightarrow{s^E} & MU^*(-) \\ \downarrow & & \downarrow \\ MUQ^*(-) & \xrightarrow{s_Q^E} & MUQ^*(-) \end{array}$$

commutes.

Before proving Theorem (5.1) we briefly recall the construction of the operations s^E as presented e.g. in [9]. Let $\{t_1, t_2, \ldots\}$ be a sequence of indeterminates of degree $(t_i) = -2i$. Set $t_0 = 1$. As usual we write t^E for the expression $t_1^{e_1} t_2^{e_2} \ldots$, E an exponent sequence. If $h^*(-)$ is a cohomology module, we denote by $h^*(-)[\mathbf{t}]$ the cohomology module $h^*(-) \bigotimes_{h^*} h^*[t_1, t_2, \ldots]$ over $\mathbf{W}^f$. $MU^*(-)[\mathbf{t}]$ is a multiplicative cohomology theory in a canonical way. If $C \in MU^2(P_\infty \mathbb{C})$ denotes the Euler class of η, $C \otimes 1 \in MU^2(P_\infty \mathbb{C})[\mathbf{t}]$ defines a $\mathbb{C}$-orientation for $MU^*(-)[\mathbf{t}]$. From the well known universal property of $MU^*(-)$ it follows that there exists a unique transformation of multiplicative cohomology theories

(5.2) $s_{\mathbf{t}}: MU^*(-) \to MU^*(-)[\mathbf{t}]$

such that

(5.3) $$s_{\mathbf{t}}(C) = \sum_{i \geq 0} C^{i+1} \otimes t_i.$$

The Landweber-Novikov operations s^E may be defined by the formula

(5.4) $$s_{\mathbf{t}}(x) = \sum_E s^E(x) \otimes t^E.$$

Proof of (5.1). $MUQ^*(-)[\mathbf{t}]$ is a cohomology module in an obvious way. Using the transformation $s_{\mathbf{t}}$ of (5.2) we define a natural homomorphism

$$v_{Q,\mathbf{t}}: MU^*(X) \otimes MUQ^*(X)[\mathbf{t}] \to MUQ^*(X)[\mathbf{t}]$$

by the formula

$$\upsilon_{Q,\mathbf{t}}(u\otimes x\otimes t^F)=\sum_E \upsilon_Q(s^E(u)\otimes x)\otimes t^{E+F}.$$

It is easy to see that $\upsilon_{Q,\mathbf{t}}$ turns the cohomology module $MUQ^*(-)[\mathbf{t}]$ into an MU-module theory with Conner-Floyd map $(\mu_Q\otimes \mathrm{id})\circ s_{\mathbf{t}}$. Now from Theorem (4.17) we know there exist a transformation of MU-module theories $s_{\mathbf{t}}^Q$ which makes the diagram

$$\begin{array}{ccc} MU^*(-) & \xrightarrow{s_{\mathbf{t}}} & MU^*(-)[\mathbf{t}] \\ \downarrow & & \downarrow{\scriptstyle \mu_Q\otimes \mathrm{id}} \\ MUQ^*(-) & \xrightarrow{s_{\mathbf{t}}^Q} & MUQ^*(-)[\mathbf{t}] \end{array}$$

commutative. If we define the operations s_Q^E by

(5.5) $\quad s_{\mathbf{t}}^Q(x)=\sum_E s_Q^E(x)\otimes t^E,$

the properties stated in (5.1) are easily checked. Using (3.12) we can extend everything on the category **W**.

Let SQ^* be the abelian group of MUQ^0-linear combinations of the operations s_Q^E, i.e.

(5.6) $\quad SQ^*=MUQ^0[s_Q^E],$

and let S^* be the Landweber-Novikov algebra. Obviously, the s_Q^E give rise to homology operations $s_E^Q\colon MUQ_*(-)\to MUQ_*(-)$ of degree $-2\|E\|$. By SQ_* we denote the (graded) abelian group of MUQ_0-linear combinations of the s_E^Q. An ideal $A\subset MUQ^*$ is called *SQ^*-invariant*, if $s_Q^E(A)\subset A$ for all exponent sequences E. SQ_*-invariant ideals of MUQ_* are defined similarly.

Recall that $MU^*=MU_{-*}=\mathbb{Z}[x_1,x_2,\dots]$, and that for each prime p the generators $x_i\in MU^{2i}$ may be chosen such that all Chern numbers of x_{p^n-1} are divisible by p. If we fix a set of polynomial generators x_i of MU^* with this property the sequences $Q(p,n)=\{p,x_{p-1},\dots,x_{p^{n-1}-1}\}$ and $Q(p,\infty)=\bigcup_{n>0} Q(p,n)$ are obviously regular and they are invariant according to a theorem of Landweber ([6], Theorem 2.7). In fact Landweber shows that the ideals $I(p,n)=(Q(p,n))$ are the only non-trivial finitely generated invariant prime ideals of MU^*.

(5.7) **Lemma.** *Let Q be an invariant MU^*-regular sequence. A finitely generated ideal P of MUQ^* is an SQ^*-invariant prime ideal iff it is of the form $P=I(p,n)/(Q)$ for some Landweber ideal $I(p,n)$.*

Proof. Because the diagram

$$\begin{array}{ccc} MU^*(S^0) & \xrightarrow{s^E} & MU^*(S^0) \\ {\scriptstyle \mu_Q}\downarrow & & \downarrow{\scriptstyle \mu_Q} \\ MUQ^*(S^0) & \xrightarrow{s_Q^E} & MUQ^*(S^0) \end{array}$$

commutes for all exponent sequences E by 5.1, μ_Q is surjective and the inverse image $\mu_Q^{-1}(P)$ of a finitely generated prime ideal is again a finitely generated prime ideal this is immediate from Landweber's theorem mentioned above.

Call an MUQ_*-module M *connective*, if $M_i=0$ for $i<N$ (some N) and let **MUQ** be the category whose objects are connective and finitely presentable MUQ_*-modules together with operations $s_E^Q=s_E^Q(M)$: $M\to M$ of degree $-2\|E\|$ (one for each exponent sequence E) which are additive and satisfy the relation

(5.8) $s_E^Q(u\cdot x)=\sum_{F+G=E} s_F^Q(u)\cdot s_G^Q(x)$

for $u\in MUQ_*$, $x\in M$. We further suppose that for all $a\in MU_*$,

(5.9) $s_E^Q(\mu_Q(a))=\mu_Q(s_E(a))$

where μ_Q: $MU_*\to MUQ_*$ is the canonical projection and the s_E are Landweber-Novikov homology operations. The morphisms of **MUQ** are homomorphisms of MUQ_*-modules which commute with the operations s_E^Q. Note that it follows from (5.1) that for all invariant regular sequences Q, the homology theory $MUQ_*(-)$ takes values in **MUQ** (over the category of finite complexes).

(5.10) **Lemma.** *Suppose Q is a finite invariant MU^*-regular sequence. Each object $M\neq 0$ of MUQ admits a finite filtration*

$$0=M_0\subset M_1\subset\cdots\subset M_n=M$$

in the category MUQ such that for $0\leqq i<n$, $M_{i+1}/M_i=MUQ_*/P_i$ (up to possible dimension shift) where the $P_i\subset MUQ_*$ are ideals of the form $I(p,m)/(Q)$.

Remark. This is a variant of Landweber's filtration theorem ([7], Theorem 3.3'). A similar statement (in the BP-case) appears in [12] and [13] and our argument is modelled on the one given in [13]. Note however, that because we have a better knowledge on $MUQ^*(MUQ)$, our category **MUQ** seems to be more transparent than the category considered in [13].

Proof. Because Q is finite, in the exact sequence

$$0\to(Q)\to MU_*\to MUQ_*=MU_*/(Q)\to 0$$

the first and second terms are coherent MU_*-modules, so MUQ_* is also a coherent MU_*-module. Any non-trivial object M of **MUQ** contains a non-zero element a of lowest degree. From (5.8), (5.9) it follows that $s_E^Q(MUQ_*\cdot a)\subset MUQ_*\cdot a$, so $MUQ_*\cdot a$ is a subobject of M in **MUQ**. Again from (5.8) and (5.9) it follows that the annihilator ideal $\mathrm{Ann}_{MU_*}(a)\subset MU_*$ is S_*-invariant. Because $\mathrm{Ann}_{MU_*}(a)\cong MUQ_*\cdot a$, $\mathrm{Ann}_{MU_*}(a)$ is finitely generated and contains (Q). Using induction and the preceding remarks one easily shows that any object M of **MUQ** admits a finite filtration

(*) $0=M_0\subset M_1\subset\cdots\subset M_r=M$

in the category **MUQ** such that M_{s+1}/M_s is stably isomorphic to MU_*/J_s where J_s is some finitely generated S_*-invariant ideal of MU_* containing (Q). The result follows now from [7], Theorem 3.3

(5.10) *will be essential in the next section.*

We close this section by indicating a

Proof of Theorem (2.16). 2.16 is an easy consequence of (4.12), (2.8), (5.1), (4.13) and a limit argument.

6. The Relationship between $MUQ^*(-)$ and Formal Groups

Let $F_{MU}(X, Y)$ be the universal formal group law of complex cobordism theory. For any prime p we have the series

(6.1) $[p]_{F_{MU}}(X) = pX + \sum_{k \geq 1} c_k X^k.$

As is well known (see for example [6]) the coefficients c_{p^k-1} of (6.1) may be taken as generators of the Landweber ideals $I(p, n)$ and we will do so from now on. We shall write $MU(p, n)$ for $MUQ(p, n)$ and $\mu_{(p,n)}$ for $\mu_{Q(p,n)}$. Let $F_{(p,n)}(X, Y)$ be the formal group law $(\mu_{(p,n)})_* F_{MU}(X, Y)$. Recall from (2.13) the notion of a n-flat formal group law.

(6.2) **Lemma.** *The formal group law $F_{(p,n)}(X, Y)$ over the ring $MU(p, n)^*$ is universal for n-flat formal group laws over rings of characteristic $p > 0$.*

Proof. Let $F(X, Y)$ be a n-flat formal group law over the ring A of characteristic p and denote by $\varphi\colon MU^* \to A$ the unique homomorphism with $\varphi_*(F_{MU}) = F$. Because $\varphi_*[p]_{F_{MU}}(X) = [p]_F(X)$ and $F_{(p,n)}$ is clearly n-flat, the result is obvious.

Note that (6.2) implies in particular, that any n-flat formal group law over a ring A of characteristic p induces a canonical $MU(p, n)^*$-module structure on A.

(6.3) **Lemma.** *Let A be a graded ring of characteristic p. If $F(X, Y)$ is a n-flat formal group law over A, the functor* $- \underset{MU(p,n)_*}{\otimes} A$ *is exact on the category* **MU(p, n)**.

Proof. We have to show that for all objects M of **MU(p, n)**, $\mathrm{Tor}_1^{MU(p,n)_*}(M, A) = 0$. In view of Lemma (5.10), this is equivalent to the statement

$$\mathrm{Tor}_1^{MU(p,n)_*}(MU_*/I(p, m), A) = 0$$

for all $m > n$. But this follows easily from the definition of a n-flat formal group law.

(6.4) *Remark.* Note that (6.3) may obviously been generalized by allowing A to be an $MU(p, n)_*$-module with appropriate properties. With a little more care the converse of (6.3) may also be proved.

(6.5) *Remark.* Note that the formal group laws $F_{(p,n)}(X, Y)$ are n-flat for all n and p.

From (6.3) it follows immediately that, over the category $\mathbf{W}^f$, any n-flat formal group law $F(X, Y)$ over a ring of characteristic p can be realised by an MU-module theory, namely by

(6.6) $A_F^*(-) = MU(p, n)^*(-) \underset{MU(p,n)}{\otimes} A.$

If A happens to be locally finite, $A_F^*(-)$ can uniquely been extended on the category $\mathbf{W}$ by (3.12). The next theorem shows that the realisation (6.6) is unique up to isomorphism.

(6.7) **Theorem.** *Let $h^*(-)$ be an MU-module theory with locally finite coefficient ring h^* of characteristic p. If the formal group law $F_h(X, Y)$ is n-flat, there exists an equivalence*

$$(6.8)\quad \Phi: E\big(MU(p,n)^*(-) \bigotimes_{MU(p,n)^*} h^*\big) \xrightarrow{\cong} h^*(-)$$

of MU-module theories over **W** *such that $\Phi_{\mathrm{pt}} = \mathrm{id}$. Moreover, if $J(h, I(p,n)) = \varnothing$, Φ is unique.*

Proof. We have just seen that the left side of (6.8) is an MU-module theory with coefficient object h^* and formal group F_h. From (4.17) we know there exists a transformation of MU-module theories $\varphi: MU(p,n)^*(-) \to h^*(-)$, the comparison theorem for additive cohomology theories implies that the transformation $\Phi = \varphi \otimes \mathrm{id}_{h^*}$ has the desired properties. The uniqueness statement is a consequence of (4.18).

(6.9) *Remark.* The notion of a n-flat formal group law does not contain formal group laws over rings of characteristic p which satisfy $[p]_F(X) = 0$. This case has been treated in [11].

We illustrate Theorem (6.7) by some examples.

(6.10) *Example.* Let $K^*(-, \mathbb{Z}_p)$ be complex K-theory with coefficients $\mathbb{Z}_p$. We have $K^*(S^0, \mathbb{Z}_p) = \mathbb{Z}_p[t, t^{-1}]$, $|t| = -2$, and the formal group law of $K^*(-, \mathbb{Z}_p)$ satisfies $[p]_F(X) = t^{p-1} X^p$. Because t^{p-1} is a unit of $\mathbb{Z}_p[t, t^{-1}]$, (6.7) applies and we get a *unique* isomorphism of MU-module theories (for all primes p!)

$$(6.11)\quad E\big(MU^*(-, \mathbb{Z}_p) \bigotimes_{MU^*|(p)} \mathbb{Z}_p[t, t^{-1}]\big) \cong K^*(-, \mathbb{Z}_p).$$

This is the Conner-Floyd theorem mod p.

(6.12) *Example.* For any prime p consider the polynomial ring $Z_p[v_n]$, v_n an indeterminate of degree $-2(p^n - 1)$, $n > 0$. From Lemma (4.3) of [6] we know that for all p, n there is a formal group law $G_{((p,n)}(X, Y)$ over $\mathbb{Z}_p[v_n]$ so that

$$(6.13)\quad [p]_{G(p,n)}(X) = v_n X^{p^n}.$$

Over $\mathbb{Z}_p[v_n, v_n^{-1}]$, $G_{(p,n)}(X, Y)$ becomes n-flat, so by (6.6),

$$(6.14)\quad K^*(p,n)(-) = E\big(MU(p,n)^*(-) \bigotimes_{MU(p,n)^*} \mathbb{Z}_p[v_n, v_n^{-1}]\big)$$

is an MU-module theory with coefficient object $\mathbb{Z}_p[v_n, v_n^{-1}]$ and formal group $G_{(p,n)}(X, Y)$. A simple dimension consideration shows that $J(K(p,n), Q(p,n)) = \varnothing$, so, by (6.7), the theories $K(p,n)^*(-)$ are – up to a canonical isomorphism – uniquely determined by their coefficients and their formal group law. The exotic K-theories $K(p,n)^*(-)$ have been studied by Morava, see also [3]. Note that this characterisation of $K(p,n)^*(-)$ implies a rather strong improvement of Theorem (3.1), (*b*) of [3].

In [15] Adams has shown that for each prime p, $K^*(-\mathbb{Z}_{(p)})$ contains a multiplicative theory $G^*(-)$ as a direct summand. If we put coefficients $\mathbb{Z}_p$ into $G^*(-)$, $G^*(-, \mathbb{Z}_p)$ becomes a multiplicative theory for p odd and an MU-module theory for $p = 2$. Moreover, $G^*(S^0, \mathbb{Z}_p) = \mathbb{Z}_p[u, u^{-1}]$ where degree $u = -2(p-1)$ and the formal group of $G^*(-, \mathbb{Z}_p)$ satisfies $[p]_F(X) = u \cdot X^p$.

From what has been said above we immediatly see that there is a canonical isomorphism of MU-module theories

(6.15) $K(p,1)^*(-)\cong G_p^*(-,\mathbb{Z}_p)$.

(6.16) *Example.* Considerations completely analogous to those of examples (6.10) and (6.12) show that the theories $P(n)^*(-)$ considered in [3] are canonically isomorphic to

$$E\big(MU(p,n)^*(-)\underset{MU(p,n)^*}{\otimes} BP^*/I_n\big).$$

(6.17) *Remark.* Note that (6.6) and (6.7) immediatly imply (2.14).

We conclude the paper by some remarks.

(6.18) *Remark.* As is easy to see, the obvious BP-analog of Theorem (4.17) holds. This is significant because it allows one to determine completely the structure of a very large subalgebra of $P(n)^*(P(n))$. These calculations will appear elsewhere, they depend on the special structure of the ring $P(n)^*=BP^*/I_n$.

(6.19) *Remark.* In [3], Proposition 4.14, Johnson and Wilson showed that there is a natural homomorphism (we use their notation)

$$\hat{\Lambda}\colon P(n)_*(-)\to K(n)_*(-)\otimes\mathbb{Z}_p[v_{n+1}, v_{n+2}, \ldots]$$

which, for $n<2(p-1)$, induces a natural isomorphism

$$\Lambda\colon B(n)_*(-)\cong K(n)_*(-)\otimes\mathbb{Z}_p[v_{n+1}, v_{n+2}, \ldots].$$

An inspection of their proof shows that the BP-version of Theorem (2.16) implies that the condition $n<2(p-1)$ can be droped. In particular, it follows that there is a natural isomorphism

$$K(n)^*(X)\cong \mathrm{Hom}_{K(n)_*}(K(n)_*(X), K(n)^*)$$

for all n.

References

1. Baas, N.A.: On bordism theory of manifolds with singularities. Math. Scand. **33**, 279–302 (1973)
2. Johnson, D.C., Wilson, S.: Projective dimension and Brown-Peterson homology. Topology **12**, 327–353 (1973)
3. Johnson, D.C., Wilson, S.: BP-operations and Morava's extraordinary K-theories. Math. Z. **144**, 55–75 (1975)
4. Fröhlich, A.: Formal groups. Lecture notes in Mathematics **74**, Berlin-Heidelberg-New York: Springer 1968
5. Landweber, P.S.: Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$. Preprint
6. Landweber, P.S.: Annihilator ideals and primitive elements in complex bordism. Illinois J. Math. **17**, 273–284 (1973)
7. Landweber, P.S.: Associated prime ideals and Hopf algebras. J. Pure and Applied Algebra **3**, 43–58 (1973)
8. Landweber, P.S.: Invariant regular ideals in Brown-Peterson Homology. Preprint
9. Quillen, D.: Elementary proofs of some results of cobordism theory using Steenrod operations. Advances Math. **7**, 28–56 (1971)

10. Sullivan, D.: Genetics of homotopy theory and the Adams conjecture. Ann. of Math. II. Ser. **100**, 1–79 (1974)
11. Würgler, U.: Über Kohomologietheorien mit formaler Gruppe der Charakteristik p. Commentarii math. Helvet. **48**, 531–536 (1973)
12. Yagita, N.: The exact functor theorem for BP_*/In-theory. Preprint
13. Yosimura, Z.: Projective dimension of Brown-Peterson homology with $(p, v_1, \ldots, v_{n-1})$ coefficients. Preprint
14. Yosimura, Z.: On cohomology theories of infinite CW-complexes. Publ. Res. Inst. math. Sci. Kyoto Univ. **8**, 295–310 (1972/73)
15. Adams, J.: Lectures on generalized cohomology. Lecture Notes in Mathematics **99**, Berlin-Heidelberg-New York: Springer 1969

Received February 26, 1976