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Affine Translation Surfaces with Constant Sectional Curvature

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In this paper we characterize affine translation surfaces with constant Gaussian curvature. We show that such surfaces must be flat and that one of the defining curves must be planar.

1 Introduction

In 1993 Nomizu and Vrancken ([4]) introduced a new transversal plane for affine surfaces in affine 4-space. Such surfaces come equipped with a metric, g, which is invariant under the group of special affine motions. One class of surfaces, for which the induced metric is Lorentzian, is that of translation surfaces. By definition a *translation surface* is one which can be written, locally, as a sum of two curves. This class coincides with those surfaces which are both Lorentzian and harmonic.

Affine surfaces in 4-space have been investigated in the past ([1], [2]) using transversal planes which are, in general, distinct from those employed here. We should note that in the case $\nabla g = 0$ and $\triangle_g f = 0$, for the immersion f, that the transversal plane defined by Klingenberg [2] and our own coincide. This obtains in the case of translation surfaces and so the first result on affine translation surfaces with our equiaffine normal plane is

Theorem 1.0 ([2]) *Let M^2 be an affine translation surface in $\mathbf{R}^4$. M^2 is maximal ($\nabla_g f = 0$) iff M^2 is equivalent to an open subset of*

$$f(u,v) = (u, u^2, P_1(u), P_2(u)) + (0, 0, v, v^2/2),$$

for P_1, P_2 arbitrary functions of u.

In this paper we will classify those translation surfaces with constant Gaussian curvature, i.e., the sectional curvature of the Levi-Civita connection associated to g is constant. We will prove

Theorem 1.1 *The Gaussian curvature of a translation surface is* 0 *iff one of the defining curves is planar.*

Theorem 1.2 *If the Gaussian curvature of a translation surface is constant, then it is* 0.

2 Basic Equations for a Surface in $\mathbf{R}^4$

In what follows $f : M^2 \to \mathbf{R}^4$ will be a surface immersed in $\mathbf{R}^4$. We first give the fundamental equations for a surface in $\mathbf{R}^4$ equipped with an arbitrary transversal plane bundle σ, i.e., $(f_*)(TM) \oplus \sigma = T\mathbf{R}^4$. Eventually we will choose σ to have certain properties.

Given any transversal σ, we have the two fundamental equations.

$$D_X Y = \nabla_X Y + h(X,Y) \tag{1}$$
$$D_X \xi = -S_\xi X + \nabla_X^\perp \xi, \tag{2}$$

where $\nabla_X Y$ and $S_\xi X$ are in TM while $h(X,Y)$ and $\nabla_X^\perp \xi$ are in σ. Note that, in these equations, we have suppressed the mention of f_*.

Because the codimension is two, we can choose a local basis $\{\eta_1, \eta_2\}$ of σ and rewrite $h(X,Y)$ and $\nabla_X^\perp \eta_j$ as follows.

$$h(X,Y) = h^1(X,Y)\eta_1 + h^2(X,Y)\eta_2 \tag{3}$$
$$\nabla_X^\perp \eta_j = \tau_j^1(X)\eta_1 + \tau_j^2(X)\eta_2. \tag{4}$$

Beginning with $R^D(X,Y)Z = 0 = R^D(X,Y)\eta$, where R^D is the curvature tensor of the standard connection in $\mathbf{R}^4$, using the equations 1, 2, 3, 4 and calculating the tangential and σ components, we obtain the structure equations of the immersion. These equations are called the Gauss, Codazzi and Ricci equations of the immersion.

Choose a local frame $u = \{X_1, X_2\}$ on M. We define

$$G_u(Y,Z) = \frac{1}{2}([X_1, X_2, D_Y X_1, D_Z X_2] + [X_1, X_2, D_Z X_1, D_Y X_2]), \tag{5}$$

which is the same as

$$G_u(Y,Z) = \frac{1}{2}[X_1, X_2, \xi_1, \xi_2] \left(\begin{vmatrix} h^1(X_1,Y) & h^1(X_2,Z) \\ h^2(X_1,Y) & h^2(X_2,Z) \end{vmatrix} + \begin{vmatrix} h^1(X_1,Z) & h^1(X_2,Y) \\ h^2(X_1,Z) & h^2(X_2,Y) \end{vmatrix} \right) \tag{6}$$

We have used $[X,Y,Z,W]$ to denote the determinant of four vectors in $\mathbf{R}^4$. The second expression is more useful for calculations, while the first shows that $G_u(Y,Z)$ is independent of the choice of σ, and basis $\{\xi_1, \xi_2\}$. Note that the non-degeneracy of G_u is independent of the choice of frame. Thus we will say that M is non-degenerate if G_u is non-degenerate for some choice of frame. In this case if we set

$$det_u G_u = \begin{vmatrix} G_u(X_1,X_1) & G_u(X_1,X_2) \\ G_u(X_1,X_2) & G_u(X_2,X_2) \end{vmatrix},$$

which is non-zero, then we can define an invariant metric g by

$$g(Y,Z) = \frac{G_u(Y,Z)}{\sqrt[3]{det_u G_u}}. \tag{7}$$

We note that the expression on the right-hand side of the equation can be shown to be independent of u. We will only consider surfaces that are nondegenerate with respect to G, and note that, for non-degenerate translation surfaces, the metric is Lorentzian.

Call a local frame field $\{Y_1, Y_2\}$ a normalized null frame if $g(Y_j, Y_j) = 0$ and $g(Y_1, Y_2) = 1$. Following [4] we can find a basis $\{\eta_1, \eta_2\}$ of any transversal bundle σ with the following properties.

$$\begin{array}{ll} [Y_1, Y_2, \eta_1, \eta_2] = 2 & \\ h^1(Y_1,Y_1) = 1 & h^2(Y_1,Y_1) = 0 \\ h^1(Y_2,Y_2) = 0 & h^2(Y_2,Y_2) = 1 \\ h^1(Y_1,Y_2) = 0 & h^2(Y_1,Y_2) = 0. \end{array}$$

There is also a metric $g^\perp$ which can be defined on σ such that $g^\perp(\eta_j, \eta_j) = 0$ and $g^\perp(\eta_1, \eta_2) = -2$. Finally we can fix a transveral plane bundle σ by requiring that

$$(\nabla g)(Y_j, Y_j, Y_j) = 0 = (\nabla g)(Y_i, Y_j, Y_i),$$

where $i, j = 1, 2$. This condition implies that $\nabla \omega_g = 0$, i.e., (∇, ω_g) form an equiaffine structure.

3 Flat Translation Surfaces

Assume that we have a translation surface given, locally, by

$$f(s,t) = \alpha(s) + \beta(t). \tag{8}$$

Denoting $\frac{\partial}{\partial s}$ by ∂s and $\frac{\partial}{\partial t}$ by ∂t, we see, using 6, that

$$g(\partial s, \partial s) = 0 = g(\partial t, \partial t) \quad and \quad g(\partial s, \partial t) = (-d/2)^{\frac{1}{3}},$$

where $d = [\alpha', \beta^{\cdot}, \alpha'', \beta^{\cdot\cdot}]$. Here differentiation with respect to s is denoted by $'$ and with respect to t by $\cdot$, for functions of one variable. We use subscripts for partial differentiation with respect to s or t. For convenience we write $(-d/2)^{\frac{1}{3}} = \epsilon c^2$, with $\epsilon = \pm 1$ and assume that $c \neq 0$.

We can also determine, using $Y_1 = \frac{\partial s}{c}$ and $Y_2 = \epsilon\frac{\partial t}{c}$, that the canonical transversal plane σ is spanned by

$$\eta_1 = \frac{c\alpha'' - 2\alpha' c_s}{c^3} \qquad \eta_2 = \frac{c\beta^{\cdot\cdot} - 2\beta^{\cdot} c_t}{c^3}.$$

Before beginning the proof of Theorem 1.1, we have a crucial lemma.

Lemma 3.1 *If M^2 is a translation surface in $\mathbf{R}^4$ then the induced connection ∇ equals the Levi-Civita connection of the affine metric g.*

Proof of Lemma 3.1: As above, suppose that the surface is given by $f(s,t) = \alpha(s) + \beta(t)$. We note first that

$$D_{\partial s} f_s = \alpha'' = \frac{c^3}{c}\left[\frac{2c_s}{c^3}\alpha' + \eta_1\right],$$

so that $\nabla_{\partial s}\partial s = \frac{2c_s}{c}\partial s$ and, similarly, $\nabla_{\partial t}\partial t = \frac{2c_t}{c}\partial t$. It is clear, of course, that $\nabla_{\partial s}\partial t = \nabla_{\partial t}\partial s = 0$.

Using the fact that $\{\partial s, \partial t\}$ is a null basis with respect to g and $g(\partial s, \partial t) = \epsilon c^2$, one can see that $\hat{\nabla}_X Y = \nabla_X Y$. □

Proof of Theorem 1.1: The Gaussian curvature of g is zero iff

$$\hat{R}(\partial t, \partial s)\partial s = 0 = \hat{\nabla}_{\partial t}\hat{\nabla}_{\partial s}\partial s - \hat{\nabla}_{\partial s}\hat{\nabla}_{\partial t}\partial s = 2\left(\frac{\partial}{\partial t}\left(\frac{c_s}{c}\right)\right)\partial s.$$

This holds iff

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} ln(c^2) = 0.$$

We note finally that the sectional curvature of g equals zero iff the determinant d is a product of a function of s and a function of t.

Now assume that one of the curves, say $\alpha(s)$, is planar. Then $-d = [\alpha', \alpha'', \beta^{\cdot}, \beta^{\cdot\cdot}]$ has the block form

$$\begin{bmatrix} A(s) & B \\ 0 & C(t) \end{bmatrix}$$

and so is a product and the Gaussian curvature is zero.

Conversely, let us assume that the Gaussian curvature is 0. Then d is a product of a function of s and a function of t. In fact, by reparametrizing α and β we may assume that d is a constant.

Because $\alpha', \alpha'', \beta^{\cdot}$, and $\beta^{\cdot\cdot}$ span $\mathbf{R}^4$ we can write

$$\begin{aligned} \alpha''' &= b_1\alpha' + b_2\beta^{\cdot} + b_3\alpha'' + b_4\beta^{\cdot\cdot} \\ \beta^{\cdot\cdot\cdot} &= c_1\alpha' + c_2\beta^{\cdot} + c_3\alpha'' + c_4\beta^{\cdot\cdot}. \end{aligned}$$

By differentiating d with respect to s and t we find

$$[\alpha', \beta^{\cdot}, \alpha''', \beta^{\cdot\cdot}] = 0 = [\alpha', \beta^{\cdot}, \alpha'', \beta^{\cdot\cdot\cdot}]$$

or $b_3d = 0 = c_4d$, so that $b_3 = c_4 = 0$. Furthermore we can calculate

$$\alpha_t''' = 0 = (b_{1t} + c_1b_4)\alpha' + (b_{2t} + b_4c_2)\beta^{\cdot} + b_4c_3\alpha'' + (b_{4t} + b_2)\beta^{\cdot\cdot},$$

which implies that $b_4c_3 = 0$ and $b_2 + b_{4t} = 0$. Similarly, from $\beta_s^{\cdot\cdot\cdot} = 0$ we get $c_1 + c_{3s} = 0$. If, at some p, $b_4(p) \neq 0$, then c_3 is zero in a neighborhood of p, as is c_1. This yields $\beta^{\cdot\cdot\cdot} = c_2\beta^{\cdot}$ and β is planar. If at some point $c_3(p) \neq 0$, we see that α is planar. □

4 Constant Curvature Translation Surfaces

To prove Theorem 1.2 we switch to null vector fields $\{Y_1, Y_2\}$ and a basis $\{\eta_1, \eta_2\}$ of the affine normal plane as in Section 1. Because the connection is metric, it is easy to see that there are functions a_1, a_8 so that

$$D_{Y_1}Y_1 = a_1Y_1 + \eta_1 \qquad D_{Y_2}Y_1 = -a_8Y_1 \tag{9}$$

$$D_{Y_1}Y_2 = -a_1Y_2 \qquad D_{Y_2}Y_2 = a_8Y_2 + \eta_2. \tag{10}$$

Furthermore, from the Codazzi equations we get $\tau_1^1(Y_1) = 2a_1$, $\tau_1^1(Y_2) = -2a_8$, $\tau_2^1(Y_1) = 0 = \tau_1^2(Y_2)$, while the Gauss equation gives $S_1Y_2 = -kY_1$, $S_2Y_1 = -kY_2$, where k is the sectional curvature of g. Thus

$$D_{Y_1}\eta_1 = b_1Y_1 + b_2Y_2 + 2a_1\eta_1 + b_3\eta_2 \tag{11}$$

$$D_{Y_2}\eta_1 = kY_1 - 2a_8\eta_1 \tag{12}$$

$$D_{Y_1}\eta_2 = kY_2 - 2a_1\eta_2 \tag{13}$$

$$D_{Y_2}\eta_2 = b_4Y_1 + b_5Y_2 + b_6\eta_1 + 2a_8\eta_2, \tag{14}$$

where $b_1, \ldots, b_6$ are additional functions on the surface.

We can simplify the equations 9-14 using the next lemma.

Lemma 4.1 *By rescaling the null frame $\{Y_1, Y_2\}$ we can set $a_1 = \frac{1}{2}$, and $a_8 = -k$.*

Proof of Lemma 4.1: We choose a new normalized null frame by setting $U_1 = \phi Y_1$, $U_2 = \frac{1}{\phi}Y_2$. We want $a_1 = \frac{1}{2}$, and $a_8 = -k$, i.e.,

$$\nabla_{U_1}U_1 = (1/2)U_1 \qquad \textit{and} \qquad \nabla_{U_2}U_2 = -kU_2. \tag{15}$$

This is equivalent to solving the system

$$\begin{aligned} Y_1\phi &= (1/2) - \phi a_1 \\ Y_2\phi &= \phi a_8 + k\phi^2 \end{aligned}$$

We can find such a ϕ iff ϕ is integrable, i.e.

$$Y_1(Y_2\phi) - Y_2(Y_1\phi) = [Y_1, Y_2]\phi,$$

which we calculate is equivalent to

$$\phi(Y_1a_8 + 2a_1a_8 + Y_2a_1 + k) = 0.$$

This does, in fact, hold, because the factor inside the parentheses is equivalent to the Gauss equation. □

At this point we have the following values from 9 and 10:

$$h^1(Y_1, Y_1) = 1 \qquad h^2(Y_1, Y_1) = 0 \tag{16}$$
$$h^1(Y_1, Y_2) = 0 \qquad h^2(Y_1, Y_2) = 0 \tag{17}$$
$$h^1(Y_2, Y_2) = 0 \qquad h^2(Y_2, Y_2) = 1, \tag{18}$$

and, from 11 -14 we have

$$S_1Y_1 = -b_1Y_1 - b_2Y_2 \tag{19}$$
$$S_1Y_2 = -kY_1 \tag{20}$$
$$S_2Y_1 = -kY_2 \tag{21}$$
$$S_2Y_2 = -b_4Y_1 - b_5Y_2 \tag{22}$$

and

$$\tau_1^1(Y_1) = 0 \qquad \tau_1^1(Y_2) = 2k \tag{23}$$
$$\tau_1^2(Y_1) = b_3 \qquad \tau_1^2(Y_2) = 0 \tag{24}$$
$$\tau_2^1(Y_1) = 0 \qquad \tau_2^1(Y_2) = b_6. \tag{25}$$
$$\tag{26}$$

The Codazzi and Ricci equations thus yield

$$Y_2b_1 = 2kb_1 - b_3b_4 \qquad Y_2b_2 = 4kb_2 - b_3b_5 \tag{27}$$
$$Y_1b_4 = -2b_4 - b_1b_6 \qquad Y_1b_5 = -b_2b_6 - b_5 \tag{28}$$

and

$$3k = b_3b_6 \tag{29}$$
$$Y_2b_3 = 5kb_3 - b_2 \tag{30}$$
$$Y_1b_6 = -b_4 - (5/2)b_6 \tag{31}$$

If we assume that $k \neq 0$, we can set $b_6 = 3k/b_3$, and get from 31

$$Y_1b_3 = \frac{b_4b_3^2}{3k} + \frac{5}{2}b_3. \tag{32}$$

Proof of Theorem 1.2: We break the proof of Theorem 2 into three parts. We are assuming we have a constant curvature tranlation surface with $k \neq 0$ and will derive a contradiction. We first assume $b_2 b_4 \neq 0$ and set

$$Y_1 b_2 = 5kb_3 + 2b_2 + g, \tag{33}$$

with g to be determined.

From the integrability condition for b_3 we find

$$Y_2 b_4 = -4kb_4 - \frac{3kg}{b_3^2} + \frac{2b_2 b_4}{b_3}, \tag{34}$$

while from b_2 we get

$$Y_2 g = 12kb_2 + 5kg - \frac{b_3^2 b_4 b_5}{3k}. \tag{35}$$

Using these values we calculate the integrability condition for b_4 and find

$$Y_1 g = \frac{b_3^2}{3k}\left(\frac{4b_4 g}{b_3} - \frac{2b_2 b_4^2}{3k} + 3b_4 k - \frac{3b_1 b_2 k}{b_3^2} + \frac{15kg}{2b_3^2}\right). \tag{36}$$

At this point we can calculate that the integrability condition of g yields

$$\frac{4}{b_3} g^2 + g\left(20k - \frac{8b_2 b_4}{3k}\right) + \left(\frac{4b_2^2 b_3 b_4^2}{9k^2} - 16b_2 b_3 b_4\right) = 0. \tag{37}$$

Solving for g we find

$$g = \frac{2b_2 b_3 b_4 k - 15b_3 k^3 \pm \sqrt{21}\sqrt{4b_2 b_3^2 b_4 k^4 - 15b_3^2 k^6}}{6k^2}. \tag{38}$$

If we now differentiate 37 with respect to Y_1 and Y_2 and use 38 we get

$$b_1 = \frac{b_3 b_4\left(-2b_2 b_3 b_4 k + 15b_3 k^3 \pm \sqrt{21}\sqrt{b_3^2 k^4 (4b_2 b_4 - 15k^2}\right)}{6b_2 k^3}. \tag{39}$$

$$b_5 = -\frac{3b_2\left(2b_2 b_3 b_4 k - 15b_3 k^3 \pm \sqrt{21}\sqrt{b_3^2 k^4 (4b_2 b_4 - 15k^2}\right)}{2b_3^3 b_4 k}. \tag{40}$$

Substituting these values into 27 gives

$$\frac{5b_3}{b_2}(36k^2 - 7b_2 b_4) = 0 \tag{41}$$

or $b_4 = 36k^2/7b_2$. From 28 we then get, (with the choice of the +) that

$$\frac{36\left(-13b_3 k^3 + 7\sqrt{13}\sqrt{b_3^2 k^2}\right)}{49b_2^2 k} = 0,$$

or $b_3k = 0$, which is a contradiction.

To end the proof we must take care of the cases when $b_2 = 0$ and $b_4 = 0$.

If $b_2 = 0$ then we can see from 27 that $b_5 = 0$. If we now set $Y_2b_4 = g$, a function to be determined, we get, from the integrability of b_3 that

$$g = k\left(-4b_4 + \frac{15k}{b_3}\right). \tag{42}$$

The integrability of b_4 shows that $b_4 = b_1 = 0$. Finally, the integrability of b_3 gives $b_3k = 0$. which is a contradiction.

Finally, if we assume that $b_4 = 0$ we set $Y_1b_2 = g$ and find that $g = 2b_2 + 5kb_3$. The integrability of b_2 shows that $b_2 = 0$ which again forces $kb_3 = 0$. □

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