Lower Bounds for Volumes of Convex Bodies

By

William J. Firey[1])

1. Suppose $V(K)$ signifies the volume of a convex body K in Euclidean n-space. There are available in the literature, cf. [1], a number of inequalities of the type

$$V(K) \leqq f(p(K), q(K), \ldots),$$

where $p, q, \ldots$ are rigid motion invariants of K. However inequalities of the type

$$V(K) \geqq g(p(K), q(K), \ldots)$$

seem to be much less common: trivially we can estimate $V(K)$ from below by the volume of the insphere of K, and, for $n = 2$, Pál has shown in [4] that the equilateral triangle of height Δ has the least area among convex bodies of least width Δ.

Here we develop several inequalities of this second sort. For each direction or unit vector u, let $k(u)$ be the orthogonal projection of K onto the $(n-1)$-dimensional linear subspace $L(u)$ orthogonal to u and let $k'(u)$ be the orthogonal projection of K onto the orthogonal complement of $L(u)$. The volumes (of appropriate dimensionality) of $k(u)$ and $k'(u)$ will be denoted by $s(u)$ and $w(u)$; these are the brightness and width of K in the direction u. Set

$$\Delta = \min w(u), \quad D = \max w(u), \quad \sigma = \min s(u),$$

where these are attained extrema over the unit sphere Ω. Also let S denote the surface area of K. We shall prove that:

$$D\sigma/n \leqq V, \tag{1}$$

$$\Delta S/n(n+1) \leqq V. \tag{2}$$

Some consequences of (1) will also be taken up.

2. We first prove (1) for a special case. Let u be a direction in which the width of K is the maximum D, and suppose that K is a polyhedral convex body, none of whose $(n-1)$-dimensional faces contains a vector parallel to u. We remark, cf. [1], p. 51, that there is a chord of K in the direction u which has length D; let P and P' be its end points and choose Q to be an interior point of the chord.

[1]) This work was supported in part by a grant from the U.S. National Science Foundation, NSF-G19838.

Take the boundary of $k(u)$ as the directrix for a cylindrical surface Z whose generators have the direction u. Note that $Z \cap K$ is made up of $(n-2)$-dimensional polyhedra since no $(n-1)$-dimensional face of K contains a vector parallel to u. Using P and P' as vertices, we form two conical surfaces each having $Z \cap K$ as a directrix. These surfaces bound two closed cones C and C' each of which contains Q. We set $K' = C \cap C'$.

From the convexity of K it follows that $K' \subseteqq K$. Hence, to prove (1) in this case, it suffices to show that

$$V(K') \geqq D\sigma/n\,. \tag{3}$$

Let us decompose the $(n-2)$-dimensional polyhedra which make up $Z \cap K$ into simplexes S_i, $i = 1, 2, \ldots, r$. We form

$$T_i = \overline{S_i \cup Q \cup P}, \quad T_i' = \overline{S_i \cup Q \cup P'}$$

where the bar denotes convex closure. Then

$$V(K') = \sum_{i=1}^{r} [V(T_i) + V(T_i')]\,.$$

Denote by v_i the $(n-1)$-dimensional volume of $\overline{S_i \cup Q}$ and by $\bar{v}_i$ the volume of the projection of $\overline{S_i \cup Q}$ onto $L(u)$. If u_i is a unit vector orthogonal to $\overline{S_i \cup Q}$, we have

$$(u_i, u)\, v_i = \bar{v}_i$$

where (u_i, u) denotes the inner product. Write δ for the length of QP and δ' for that of QP'. Then

$$\begin{aligned} V(T_i) + V(T_i') &= (u, u_i)(\delta + \delta')\, v_i/n \\ &= D\,\bar{v}_i/n \end{aligned}$$

since $\delta + \delta' = D$. We sum over i and note that

$$\sum_{i=1}^{r} \bar{v}_i = s(u)$$

and so

$$V(K') = D\,s(u)/n \geqq D\sigma/n$$

from the minimal character of σ.

Standard approximation techniques and the continuity of $s(u)$, D and V as set functions allow us to conclude (1) generally.

The following example shows that the factor $1/n$ is the largest possible. Let $x_1, x_2, \ldots, x_n$ be Cartesian coordinates of a point x and let P, P' be the points $(\pm a, 0, \ldots, 0)$. Let k be the set of points for which

$$x_1 = 0, \quad |x_2| + \cdots + |x_n| \leqq 1\,.$$

Choose K to be $\overline{k \cup P \cup P'}$. For $a \geqq 1$ we have $D = 2a$. The minimum of $s(u)$ will be in a direction u lying in the 2-dimensional linear subspace spanned by a vector from the origin to P and a vector from the origin to one of the vertices of k. Let

$u(\theta) = (\cos\theta, 0, \ldots, \sin\theta)$, and $\theta_0 = \operatorname{Arctan} a$. For $s(u(\theta)) = f(\theta)$ we have

$$f(\theta) = 2^{n-1}\cos\theta/(n-1)! \quad \text{for} \quad 0 \leqq \theta \leqq (\pi/2) - \theta_0,$$
$$f(\theta) = 2^{n-1} a \sin\theta/(n-1)! \quad \text{for} \quad (\pi/2) - \theta_0 \leqq \theta \leqq \pi/2.$$

From this representation we conclude that

$$\sigma = 2^{n-1} a/(n-1)!\,\sqrt{1+a^2},$$

and so

$$D\sigma/n = 2^n a^2/n!\,\sqrt{1+a^2} < V = 2^n a/n!.$$

Hence $nV/D\sigma$ can be made arbitrarily close to one by choosing a sufficiently large.

The example suggests the cases of equality, namely if and only if K is degenerate. In this case $\sigma = V = 0$. We omit the proof.

3. In this section, in addition to (2), we prove that

$$V \leqq DS/2n. \tag{4}$$

We let H be the support function of K and $S(\omega)$ the value of the surface area function of K corresponding to the Borel set ω on the unit sphere Ω, cf. [2]. Then

$$\int_\Omega H(u)\, S(d\Omega)/n \int_\Omega S(d\Omega) = V/S.$$

Further we use V_1 as an abbreviation for the mixed volume

$$\int_\Omega H(-u)\, S(d\Omega)/n$$

of K and its image under a reflection in the origin. Since

$$H(u) + H(-u) = w(u),$$

we have

$$\int_\Omega w(u)\, S(d\Omega)/\int_\Omega S(d\Omega) = n(V + V_1)/S.$$

The left side of this last equation is a weighted arithmetic mean of $w(u)$ and so we conclude that

$$\Delta \leqq n(V + V_1)/S \leqq D. \tag{5}$$

We now make use of the following bounds on V_1, cf. [1], pp. 52—53 and p. 105:

$$V \leqq V_1 \leqq nV \tag{6}$$

which shows that

$$\Delta S/n(n+1) \leqq V \leqq DS/2n.$$

We take up the cases of equality in (2) and (4).

Consider the right side of (5) and the left side of (6). In the latter, there is equality if and only if K has a centre of symmetry. We suppose H is defined relative to this centre and so

$$H(u) = H(-u).$$

Let Ω_0 be the union of those Borel sets ω for which $S(\omega) = 0$. Then

$$\int_{\Omega} w(u)\, S(d\Omega) = \int_{\Omega - \Omega_0} w(u)\, S(d\Omega)$$

and so, from the continuity of w, there is equality in (5) if and only if

$$w(u) = D \quad \text{over} \quad \Omega - \Omega_0 .$$

Hence

$$H(u) = D/2 \quad \text{over} \quad \Omega - \Omega_0 .$$

Therefore the boundary of K contains pieces of the spherical surface of radius $D/2$ centred at the origin. The remainder of the boundary is of zero area. From these observations and the continuity of H over Ω, it follows that K is a sphere. Hence there is equality in (4) if and only if K is a sphere.

In the right side of (6) there is equality if and only if K is a simplex or K is degenerate.

This is not specifically mentioned in [1]; however the discussion there, on pp. 52—53, with a little amplification gives the cases of equality. Inequality (6) rests on

$$(6') \qquad n\, H(u) \geqq H(-u)$$

where the origin is taken to be the centroid of K. There is equality if and only if K is a pyramid (possibly degenerate) having a base with outer normal u. From this it easily follows that, if there is equality in (6′) in all directions u for which $S(u) \neq 0$, then K is a simplex or is degenerate. Hence the cases of equality in (6).

From (5) we conclude that the heights of such a simplex are all equal. Thus, there is equality in (2) if and only if K has equal altitudes or is degenerate.

4. We remark that in (1) and (2) the constants are not only the best possible, but we cannot replace the minimum width in (2) by the maximum width D or even the arithmetic mean of the widths. For, if K is a degenerate convex body lying in some linear subspace L of dimension $n - 1$ and if, as a convex body in L, K is not degenerate, then $w(u) = 0$ if and only if u is orthogonal to L. Hence the arithmetic mean of the widths over Ω is positive. So also is S which is twice the $(n - 1)$-dimensional volume of K. However $V = 0$ and so $S\,\bar{B}/n(n + 1)$, where $\bar{B}$ is the mean width, is not necessarily bounded above by V. *A fortiori*, we cannot replace Δ by D. From this we conclude that, in (1), we cannot replace σ by the arithmetic mean of $s(u)$ over Ω since this last is proportional to S in virtue of Cauchy's surface area formula.

5. A convex body is of constant width or of constant brightness when $w(u)$ or $s(u)$ is constant over Ω. Spheres are the only figures which are both of constant width and of constant brightness. Further, from Urysohn's inequality

$$V \leqq \varkappa_n (\bar{B}/2)^n , \qquad \text{equality if and only if } K \text{ is a sphere,}$$

where $\varkappa_n$ is the volume of the unit sphere in n-space, we know that a sphere of constant width $\bar{B}$ is the unique figure of greatest volume among convex bodies of the same constant width. Also, from the isepiphanic inequality

(7) $$V \leqq \varkappa_n (S/n\varkappa_n)^{n/(n-1)}, \qquad \text{equality if and only if } K \text{ is a sphere,}$$

and Cauchy's surface area formula

(8) $$S = \int_\Omega s(u)\, d\Omega/\varkappa_{n-1},$$

we conclude that a sphere of constant brightness σ is the unique figure of greatest volume among convex bodies of the same constant brightness. All these matters are discussed in [1].

From (1) we can obtain lower bounds for volumes of bodies of constant brightness σ. We have, cf. [1] p. 110,

$$\bar{B} \geqq 2(S/n\varkappa_n)^{1/(n-1)}, \qquad \text{equality if and only if } K \text{ is a sphere.}$$

Whence, by (1) and

$$\bar{B} \leqq D,$$

$$\sigma = \varkappa_{n-1} S/n\varkappa_n,$$

this last from (8), we obtain

$$V > 2\varkappa_{n-1}(S/n\varkappa_n)^{n/(n-1)}/n.$$

The inequality is strict because a body of constant brightness cannot be degenerate. We may put the matter this way: the volume of a body of constant brightness is greater than $2\varkappa_{n-1}/n\varkappa_n$ times that of a sphere of the same brightness.

Inequality (1) also permits us to compare the volume of a body of constant width with that of a sphere of the same width. We proceed inductively, beginning with the plane case. From the work of LEBESGUE [3], we have for the area A of a convex body of constant width:

$$A \geqq (\pi - \sqrt{3})\, \bar{B}^2/2$$

with equality if and only if K is a Reuleaux triangle.

The inequality to be proved is

(10) $$V \geqq (\pi - \sqrt{3})\, \bar{B}^n/n!$$

which is true for $n = 2$ and so we suppose it established for $n = r$. We remark that $k(u)$ is of constant width $\bar{B}$ if K is. Hence

$$\sigma \geqq (\pi - \sqrt{3})\, \bar{B}^r/r!$$

if K is a convex body of constant width in $(r + 1)$-dimensional space. We apply (1) with $D = \bar{B}$ and complete the inductive argument. If $n > 2$, inequality (10) is a strict inequality and, in fact, not the best possible. That (10) is strict in these cases follows from the conditions for equality in (1) since a body of constant width cannot be degenerate. On the other hand, with the use of BLASCHKE's selection theorem, one concludes that there are convex bodies of least volume for a given constant width. Hence (10) is not the best possible inequality. Similarly (9) is not the best possible inequality.

We conclude by noting that an argument similar to the preceding one establishes for a general convex body K:

$$V \geqq \Delta^n / n! \sqrt{3}.$$

Here we commence with PÁL's result for $n = 2$. Again, for $n > 2$, the inequality is not the best possible.

References

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Eingegangen am 13. 1. 1964

Anschrift des Autors:
William J. Firey
Department of Mathematics
Oregon State University
Corvallis (Oregon), USA