Cohomology of Power Sets with Applications in Quantum Probability

J. M. Lindsay[*] and K. R. Parthasarathy

Indian Statistical Institute, (Delhi Centre), 7, S.J.S. Sansanwal Marg, New Delhi-110016, India

Abstract. Square integrable Wiener functionals may be represented as sums of multiple Itô integrals. This leads to an identification of such functionals with square integrable functions on the symmetric measure space of the Lebesgue space R_+. When the pointwise product of Wiener functionals is thus carried over, the product takes a pleasing form (cf. Wick's theorem) and various non-commutative perturbations of this "Wiener product" have been considered. Here we employ cohomological arguments to analyse deformations of an abstract Wiener product. This leads to the construction of Lévy fields which are neither bosonic nor fermionic, and also gives rise to homotopies between quasi-free boson and fermion fields. Finally we unify existence and uniqueness results for quantum stochastic differential equations by treating mixed noise differential equations.

Introduction

Any square integrable Wiener functional F has an expansion in terms of multiple Itô integrals:

$$F = f_0 + \sum_{n=1}^{\infty} \int \cdots \int_{\nabla^n} f_n(t_1, \ldots, t_n) \, dB_{t_1} \cdots dB_{t_n},$$

where ∇^n is the increasing quadrant $\{ \mathbf{t} \in \mathbb{R}_+^n : t_1 < \cdots < t_n \}$. The sequence $\{ f_n \}$ may be viewed as a single function f on the collection $\Gamma(\mathbb{R}_+)$ of all subsets of $\mathbb{R}_+$ having finite cardinality:

$$f(\varnothing) = f_0; \quad f(\sigma) = f_n(\mathbf{s}) \quad \text{for} \quad n = \#\sigma \geq 1,$$

where $s_1, s_2, \ldots, s_n$ is an enumeration of the set σ in increasing order. There is a natural measure λ on $\Gamma(\mathbb{R}_+)$, derived from Lebesgue measure on $\mathbb{R}_+$, for which the correspondence $F \to f$ is an isometric isomorphism from $\mathcal{W}$, the space of square integrable Wiener functionals, to $\mathcal{F} = L^2(\Gamma(\mathbb{R}_+), \lambda)$. Under pointwise

---

[*] Address from September 1988; Department of Mathematics, King's College, London WC2R2LS, UK

multiplication the Wiener functionals form an associative algebra and numerous subspaces of $\mathcal{W}$ are closed under this product, for example the Malliavin domain. A natural question arises: how is this algebraic structure manifested in $\mathscr{F}$? This is answered by the elegant formula

$$f * g : \sigma \to \sum_{\alpha \subset \sigma} \int_{\Gamma(\mathbb{R}_+)} f(\alpha \cup \omega) g(\omega \cup \bar\alpha) d\lambda(\omega), \qquad (0.1)$$

where $\bar\alpha$ denotes the complement of α in σ. Dense subspaces of $\mathscr{F}$ which are stable under $*$ may be identified, and if f and g correspond to Wiener functionals F and G respectively, then $f * g$ corresponds to the pointwise product FG.

The form of the Wiener product (0.1) immediately suggests the following generalisation: replace the Lebesgue measure space $\mathbb{R}_+$ by an abstract one (S, m), $(\Gamma(\mathbb{R}_+), \lambda)$ by the symmetric measure space $(\Gamma(S), \mu)$ of (S, m), and $*$ by the operation

$$f *_p g : \sigma \to \sum_{\alpha \subset \sigma} \int_{\Gamma(S)} p(\omega, \alpha, \bar\alpha) f(\alpha \cup \omega) g(\omega \cup \bar\alpha) d\mu(\omega). \qquad (0.2)$$

Incorporating an involution $\sim$ on $\Gamma(S)$, induced from a pointwise involution i on S, we may further generalise:

$$f \tilde{*}_p g : \sigma \to \sum_{\alpha \subset \sigma} \int_{\Gamma(S)} p(\omega, \alpha, \bar\alpha) f(\alpha \cup \tilde\omega) g(\omega \cup \bar\alpha) d\mu(\omega). \qquad (0.3)$$

Such an approach unifies the various products considered by Maassen, Lindsay and Meyer. The Wiener product is implicit in [Maa] and is highlighted in [Me 1] where algebraic variations, including the Clifford product, are explored. The Bose product, extensively studied in [L M 1] is obtained by taking the measure space to be the sum of $(\mathbb{R}_+, m_1)$ and $(\mathbb{R}_+, m_2)$, where m_1, m_2 are non-zero multiples of Lebesgue measure, and the involution to be $(\sigma_1, \sigma_2)^{\sim} = (\sigma_2, \sigma_1)$. The Fermi product [L M 2] is obtained by a similar modification to the Bose product as is required to form the Clifford product from the Wiener product—namely the introduction of a $\{\pm 1\}$-valued function p, which is dependent only on the relative position of the points of ω, α and $\bar\alpha$ on the line, in (0.3) and (0.2) respectively.

The analysis of functions p for which (0.3) is associative rests on an analysis of the solutions of the functional equation

$$q(\alpha, \beta) q(\alpha \cup \beta, \gamma) = q(\alpha, \beta \cup \gamma) q(\beta, \gamma) \quad \alpha, \beta, \gamma \in \Gamma(S) \text{ disjoint} \qquad (0.4)$$

for non-vanishing complex-valued functions q. If Γ and $\cup$ are replaced by a group G and its binary operation, then (0.4) becomes the condition for q to be a second order cocycle on G with values in $\mathbb{C}^\times$, the group of non-zero complex numbers. Such objects arise in the theory of projective representations of groups, in particular they play a vital rôle in quantum theory [Var]. Our analysis has been inspired by the work of E. P. Wigner, V. Bargmann and G. W. Mackey [BaW, Bar, Mac].

The second order cocycles on $\Gamma(S)$ with values in $\mathbb{C}^\times$ are classified in Sect. 2 and the associative products $*_p$ are described and classified in Sects. 3 and 4. The pointwise product of Poisson functionals also gives rise to a product on Fock space [Me 1], [L M 2]. An analysis of deformations of this Poisson product, similar to (0.2) and (0.3) for the Wiener product, will appear elsewhere ([LP]).

Several authors, particularly P-A. Meyer, have raised the subject of non-commutative Poisson and Wiener type products [Me 2].

The last two sections give some applications of these ideas. In Sect. 6 operator fields, which are deformations of the quasi-free boson and fermion fields, are constructed in a natural way from cocycles and multipliers on Γ. These fields satisfy local commutation relations and provide bridges between boson and fermion fields—two such bridges are discussed. A continuum of inequivalent Levy fields [A P] with common covariance is also obtained, showing that commutation relations do not follow from martingale (fair game) assumptions alone [P]. In Sect. 7 we unify existence and uniqueness results for quantum stochastic differential equations by establishing the explicit form (cf. [Maa]) of the unique solution of linear stochastic differential equations driven by mixed noise.

1. Set Notations

For a set S, $\Gamma(S)$ or Γ_S will denote the finite power set of S: $\{\alpha \subset S : \#\alpha < \infty\}$ which has the partition $\bigcup\limits_{n=0}^{\infty} \Gamma_n(S)$, where $\Gamma_n(S) = \{\sigma \subset S : \#\sigma = n\}$. When S is fixed and there is no danger of confusion we shall frequently drop mention of it in the notation, writing Γ, Γ_n etc. The cartesian product S^n will be understood to be the single point set with element 0 when $n = 0$ and the collection of coordinates $\{s_1, \ldots, s_n\}$ of a typical element $\mathbf{s} = (s_1, \ldots, s_n) \in S^n$ will, by convention, by empty when $n = 0$. To each element σ of Γ_n is associated $n!$ points of S^n—those points which have σ as the set of their coordinates. $S^{(n)}$ $(n = 0, 1, \ldots)$ will denote the subset of S^n consisting of those points $\mathbf{s}$ with distinct coordinates: $s_i \neq s_j$ for $i \neq j$. To any function f on $\Gamma(S)$ is naturally associated a sequence of symmetric functions f_n on $S^{(n)}$: $f_n(\mathbf{s}) = f(\{s_1, \ldots, s_n\})$, and conversely such a sequence determines a function on $\Gamma(S)$. In particular, by considering indicator functions, there is a natural correspondence between subsets E of $\Gamma(S)$ and sequences of symmetric subsets E_n of $S^{(n)}$.

For each n, $\Gamma^{(n)}(S)$ will denote the subset

$$\{\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \Gamma(S)^n : \sigma_i \cap \sigma_j = \phi \text{ if } i \neq j\} \tag{1.1}$$

of $\Gamma(S)^n$, and $\Delta^n(S)$ will denote the set diagonal:

$$\Gamma(S)^n \setminus \Gamma^{(n)}(S). \tag{1.2}$$

For $\boldsymbol{\alpha} \in \Gamma^{(n)}$, $\alpha_1 \cup \cdots \cup \alpha_n$ will be abbreviated to $|\boldsymbol{\alpha}|$. In any summation over subsets α of an element σ of Γ, $\bar{\alpha}$ will always denote the complement of α in σ.

$\mathscr{F}(A; B)$ will denote the set of B-valued functions on A, and will be abbreviated to $\mathscr{F}(A)$ when $B = \mathbb{C}$. For a group valued function f we call $\{a : f(a) \neq \text{identity}\}$ the support of f. $\delta_\varnothing$ will always denote the function equal to one at $\varnothing$ and zero elsewhere in Γ, and an empty product will always be understood to be equal to one.

2. Second Order Cohomology on $\Gamma(S)$

In this section we describe the second order cohomology of $\Gamma(S)$ for an arbitrary set S. Since S will be fixed we shall frequently drop references to it in the notation. A second order cocycle on $\Gamma(S)$, with values ˙n the multiplicative group of non-zero

complex numbers $\mathbb{C}^\times$, is a map $q:\Gamma^{(2)}(S)\to\mathbb{C}^\times$ satisfying the cocycle identity

$$q(\alpha,\beta)q(\alpha\cup\beta,\gamma) = q(\alpha,\beta\cup\gamma)q(\beta,\gamma), \quad (\alpha,\beta,\gamma)\in\Gamma^{(3)}(S). \qquad (2.1)$$

The collection of such 2-cocycles forms an abelian group under the pointwise product which is also closed under complex conjugation and the *involution*

$$\tilde{q}(\alpha,\beta) = q(\beta,\alpha). \qquad (2.2)$$

If f is a map from Γ into $\mathbb{C}^\times$ then

$$\tau_f:(\alpha,\beta)\mapsto\frac{f(\alpha\cup\beta)}{f(\alpha)f(\beta)}, \quad (\alpha,\beta)\in\Gamma^{(2)} \qquad (2.3)$$

is called a *trivial cocycle*. For $f,g\in\mathscr{F}(\Gamma,\mathbb{C}^\times)$ to determine the same trivial cocycle, $h=f/g$ must be multiplicative (homomorphic): $h(\alpha\cup\beta)=h(\alpha)h(\beta)$ for $(\alpha,\beta)\in\Gamma^{(2)}$. The trivial cocycles form a subgroup of the group of 2-cocycles. We denote the quotient group by $\mathscr{H}^2(\Gamma)$ and say that 2-cocycles q_1,q_2 are *equivalent* written $q_1\approx q_2$, if their ratio is trivial.

If k is a map from $S^{(2)}$ to $\mathbb{C}^\times$ then

$$\pi_k:(\alpha,\beta)\mapsto\prod_{\substack{a\in\alpha\\b\in\beta}} k(a,b), \quad (\alpha,\beta)\in\Gamma^{(2)} \qquad (2.4)$$

is called a *product cocycle*. Thus a 2-cocycle is a product cocycle if it is multiplicative in each argument.

Proposition 2.1. *Let k be a map from $S^{(2)}$ into $\mathbb{C}^\times$. If k is symmetric then π_k is trivial, and conversely.*

Proof. If k is symmetric, then the function $f:\Gamma\to\mathbb{C}^\times$,

$$\alpha\mapsto\prod_{\{s,t\}\subset\alpha} k(s,t)$$

is well defined and

$$\tau_f(\alpha,\beta) = \frac{f(\alpha\cup\beta)}{f(\alpha)f(\beta)} = \prod_{a\in\alpha,b\in\beta} k(s,t) = \pi_k(\alpha,\beta).$$

In other words π_k is trivial. The converse is obvious. □

Putting $\beta=\varnothing$ in (2.1) we see that $q(\varnothing,\cdot)$ and $q(\cdot,\varnothing)$ are equal constant functions on Γ; dividing by $q(\varnothing,\varnothing)$ therefore gives an equivalent 2-cocycle which is *normalised*: $q(\varnothing,\varnothing)=1$. (The equivalence is effected by the function equal to $q(\varnothing,\varnothing)$ at $\varnothing$ and 1 elsewhere). Although $\Gamma(S)$ is a semigroup under the union operation, and even a group under symmetric difference, all the 2-cocycles on these structures turn out to be trivial. Our less restrictive definition, only involving disjoint elements, is therefore needed for obtaining non-trivial cocycles.

When $\sigma=\{s_1,\ldots,s_n\}$, $\tau=\{t_1,\ldots,t_m\}$, where $(\sigma,\tau)\in\Gamma^{(2)}$, $q(\sigma,\tau)$ will sometimes be written $q(s_1,\ldots,s_n;t_1,\ldots,t_m)$. The next result shows in particular that (normalized) 2-cocycles on Γ are determined by their restrictions to $\Gamma_1\times\Gamma$, i.e. $\{(\{s\},\tau)\in\Gamma^{(2)}\}$.

Proposition 2.2. *Let q be a 2-cocycle on $\Gamma(S)$*

(i) *For $(\sigma, \tau) \in \Gamma^{(2)}$, if $\sigma = \{s_1, \ldots, s_n\}$ $n \geq 1$, then*

$$q(\sigma, \tau) = q(s_n; \tau) \prod_{i=1}^{n-1} \frac{q(s_i; \tau \cup \{s_{i+1}, \ldots, s_n\})}{q(s_i; s_{i+1}, \ldots, s_n)}. \tag{2.5}$$

(ii) *$q/\tilde{q}$ is a product cocycle, where $\tilde{q}$ is defined by (2.2).*

Proof.

(i) Applying the cocycle identity to the triple $(\{s_1, \ldots, s_{n-1}\}, \{s_n\}, \tau)$ we obtain

$$q(\sigma, \tau) = q(s_n; \tau) \frac{q(s_1, \ldots, s_{n-1}; \tau \cup \{s_n\})}{q(s_1, \ldots, s_{n-1}; s_n)}.$$

Iterating this identity for the numerator and denominator there is a cancellation of terms of the form $q(s_1, \ldots, s_j; s_{j+1})$ for $j = 1, \ldots, n-2$, leaving the expression (2.5).

(ii) Let $(\alpha, \beta, \gamma) \in \Gamma^{(3)}$, then

$$q(\alpha \cup \beta, \gamma) = \frac{q(\beta, \gamma)}{q(\alpha, \beta)} q(\alpha, \gamma \cup \beta) = \frac{q(\beta, \gamma) q(\alpha, \gamma)}{q(\alpha, \beta) q(\gamma, \beta)} q(\gamma \cup \alpha, \beta)$$

$$= \frac{q(\beta, \gamma) q(\alpha, \gamma) q(\alpha, \beta)}{q(\alpha, \beta) q(\gamma, \beta) q(\gamma, \alpha)} q(\gamma, \alpha \cup \beta),$$

so that

$$q/\tilde{q}(\alpha \cup \beta, \gamma) = q/\tilde{q}(\alpha, \gamma) q/\tilde{q}(\beta, \gamma).$$

By symmetry $q/\tilde{q}$ is also multiplicative in the second argument. □

A 2-cocycle q is *symmetric* if $\tilde{q} = q$ and *skew-symmetric* if $\tilde{q} = q^{-1}$. Clearly all trivial cocycles are symmetric, and a 2-cocycle is both symmetric and skew-symmetric if and only if it is (skew-)symmetric and $\{\pm 1\}$-valued.

Proposition 2.3. *Every 2-cocycle on $\Gamma(S)$ is equivalent to a skew-symmetric product cocycle.*

Proof. Let q be a 2-cocycle on Γ which we suppose without loss to be normalised, and let π be the product cocycle determined by the function $(s, t) \to q(s; t)$. Define a sequence of function f_n on $S^{(n)}, n = 0, 1, 2, \ldots$ recursively by letting $f_0(0) = 1$ and, for $n \geq 1$,

$$f_n : (s_1, \ldots, s_n) \mapsto f_{n-1}(s_1, \ldots, s_{n-1})(q/\pi)(s_n; s_1, \ldots, s_{n-1}). \tag{2.6}$$

$f_1(s) = f_0(0)(q/\pi)(s; \phi) = 1$ and, for $n \geq 2$, the cocycle identity applied to the triple $(\{s_n\}, \{s_{n-1}\}, \{s_1, \ldots, s_{n-2}\})$ yields

$$f_n(s_1, \ldots, s_n) = f_{n-2}(s_1, \ldots, s_{n-2})(q/\pi)(s_n; s_1, \ldots, s_{n-1})(q/\pi)(s_{n-1}; s_1, \ldots, s_{n-2})$$

$$= f_{n-2}(s_1, \ldots, s_{n-2})(q/\pi)(s_n; s_{n-1})(q/\pi)(s_{n-1}, s_n; s_1, \ldots, s_{n-2})$$

$$= f_{n-2}(s_1, \ldots, s_{n-2})(q/\pi)(s_{n-1}, s_n; s_1, \ldots, s_{n-2}). \tag{2.7}$$

Thus, if f_j is symmetric for $j < n$, by (2.6) f_n is symmetric in its first $n-1$ arguments, and by (2.7) it is also symmetric in its last two arguments; in other words f_n is symmetric. Since f_0 and f_1 are symmetric, it follows by induction that each of the

functions f_n is symmetric. The corresponding function f on Γ satisfies

$$(q/\pi)(s;\tau) = \frac{f(\tau \cup \{s\})}{f(\tau)f(\{s\})},$$

so that $q = \tau_f \pi$. Now let l be a logarithm of the function $(s,t) \to q(s;t)$ on $S^{(2)}$, then $\pi = \pi_k \pi_{k'}$ where $k = \exp(l - \tilde{l})/2$ and $k' = \exp(l + \tilde{l})/2$. Since k' is symmetric $\pi_{k'}$ is trivial, τ_g say; on the other hand k is skew-symmetric and $q = \tau_{fg}\pi_k$, so that the proof is complete. $\square$

Corollary 2.4.

(i) *The trivial cocycles are the symmetric cocycles.*
(ii) *Two skew-symmetric product cocycles π_{k_1} and π_{k_2} are equivalent if and only if k_1/k_2 is skew-symmetric and $\{\pm 1\}$-valued.*

Since $\mathbb{C}^\times$ and $\mathbb{C}^\times/\{\pm 1\}$ are isomorphic groups the above degeneracy is not revealed in the second cohomology group of Γ_S.

Theorem 2.5.

$$\mathscr{H}^2(\Gamma_S) \cong \mathscr{A}(S^{(2)}; \mathbb{C}^\times),$$

the multiplicative group of skew-symmetric $\mathbb{C}^\times$-valued functions on $S^{(2)}$.

In terms of any given partition $S_1 \cup \cdots \cup S_n$ of S we have the following alternative representation for 2-cocycles on Γ_S.

Proposition 2.6. *Let $S_1 \cup \cdots \cup S_n$ be a partition of S. Any 2-cocycle on Γ_S is equivalent to a 2-cocycle of the form $\prod_{i \geq j} q_{ij}$, where each q_{ij} is a product cocycle, $(s,t) \to q_{ij}(s;t)$ has support in $S_i \times S_j \cup S_j \times S_i$ and the q_{ii} are skew-symmetric.*

Proof. By Proposition 2.3 any 2-cocycle is equivalent to a skew-symmetric product cocycle π which may be written $\prod_{i,j} \pi_{ij}$, where $\pi_{ij}(\alpha, \beta) = \pi(\alpha \cap S_i, \beta \cap S_j)$. Define a function f and product cocycles q_{ij} $(i \geq j)$ as follows:

$$f(\alpha) = \prod_{i<j} \pi_{ij}(\alpha, \alpha); q_{ij} = \begin{cases} \pi_{ij}/\tilde{\pi}_{ji} & \text{if } i > j \\ \pi_{ii} & \text{if } i = j \end{cases}.$$

Then, by the multiplicativity of π_{ij} (in each argument),

$$\left(\tau_f \prod_{i \geq j} q_{ij}\right)(\alpha, \beta) = \prod_{i<j} \frac{\pi_{ij}(\alpha \cup \beta, \alpha \cup \beta)}{\pi_{ij}(\alpha, \alpha)\pi_{ij}(\beta, \beta)} \frac{\prod_{i>j} \pi_{ij}(\alpha, \beta)}{\prod_{i<j} \pi_{ij}(\beta, \alpha)} \prod_i \pi_{ii}(\alpha, \beta)$$

$$= \prod_{i,j} \pi_{ij}(\alpha, \beta),$$

and the result follows. $\square$

Each 2-cocycle q determines an algebraic structure on $\mathscr{F}(\Gamma_S)$ as follows:

$$f \circ_q g : \sigma \to \sum_{\alpha \subset \sigma} q(\alpha, \bar{\alpha}) f(\alpha) g(\bar{\alpha}), \tag{2.8}$$

where $\bar{\alpha}$ denotes the complement of α in σ:$\sigma\backslash\alpha$. The cocycle identity expresses the associativity of this product. The resulting complex, associative algebra $\mathscr{F}(\Gamma_S, q)$ has unit $q(\varnothing, \varnothing)^{-1}\delta_\varnothing$ and is non-commutative unless q is trivial.

M_f will always denoted the pointwise multiplication operator $g \mapsto fg$ on complex valued functions defined on the same space as f.

Proposition 2.7. *Let q_1 and q_2 be 2-cocycles on Γ_S and $f \in \mathscr{F}(\Gamma_S)$. Then the following are equivalent:*

(i) $q_1/q_2 = \tau_f$,
(ii) $M_f:\mathscr{F}(\Gamma, q_1) \to \mathscr{F}(\Gamma, q_2)$ *is an isomorphism.*

Proof. Immediate.

Remark. Algebras of functions of several variables under products of the form

$$f \circ_q g(\sigma) = \sum q(\alpha, \beta) f(\alpha) g(\beta), \quad \sigma \in \Gamma(S_1) \times \cdots \times \Gamma(S_n)$$

(where the sum is over pairs (α, β) for which each $\alpha_i \cup \beta_i$ is a partition of σ_i) may be reduced to the one variable case through the obvious identification of $\Gamma(S_1) \times \cdots \times \Gamma(S_n)$ with $\Gamma(S_1 \amalg \cdots \amalg S_n)$.

When $q = (-1)^n$ where n is defined, in terms of an ordering of S, by $n(\alpha, \beta) = \#\{(a, b) \in \alpha \times \beta : a > b\}$, $\mathscr{F}(\Gamma, q)$ is isomorphic to the algebra of anti-symmetric functions on $\bigcup_{n=0}^{\infty} S^{(n)}$ under the antisymmetric product. When q is the unit cocycle $\mathscr{F}$ is (naturally) isomorphic to the symmetric function algebra. Taking a cue from P-A. Meyer [Me 2] we call $\mathscr{F}(\Gamma_S, q)$ *Wick-Grassmann algebras* on S. By heeding the previous remark one can see that algebras of functions of several variables which are symmetric in some of the variables and anti-symmetric in the remaining variables are also covered by (2.8).

3. Wiener–Clifford Algebras

In this section we construct associative algebras of measurable functions on $\Gamma(S, m)$, the symmetric measure space of a measure space (S, m) [Gui], by means of three argument multipliers. Study of the isomorphism classes of these algebras shows that the generic product is obtained by a welding of two paradigm products: the *Wiener product* [Maa] and the *Clifford product* [Me 1].

Let $X = (S, m)$ be a σ-finite, non-atomic measure space. The correspondence between Γ_S and $\bigcup_{n=0}^{\infty} S^{(n)}$ induces a measurable structure on Γ_S:$E \subset \Gamma_S$ is measurable if each E_n is of the form $E'_n \cap S^{(n)}$ for some E'_n which is measurable in the product space X^n. A measure μ_X is then defined by

$$\mu_X(E) = \sum_{n=0}^{\infty} (n!)^{-1} m^n(E'_n), \tag{3.1}$$

where m^0 is point measure on S^0 and m^n (for $n \geq 1$) is the product measure on S^n. We denote the resulting measure space, the *symmetric measure space* of X, by $\Gamma(X)$ or Γ_X and usually abbreviate $d\mu_X(\sigma)$ to $d\sigma$.

Remark. Non-atomic σ-finite measures take all intermediate values between 0 and the (possibly infinite) total measure. It follows that each set diagonal $\Delta^n(S)$ is (contained in) a null set of the corresponding product measure. In particular μ_X is unambiguously defined.

The algebras considered here result from the convolution-like products:

$$f *_p g : \sigma \to \sum_{\alpha \subset \sigma} \int p(\omega, \alpha, \bar{\alpha}) f(\alpha \cup \omega) g(\omega \cup \bar{\alpha}) d\omega, \tag{3.2}$$

where the requirement of associativity imposes some restriction on the form of p.

From now on $\mathscr{F}(Y; Z)$ will denote the space of *measurable* maps between the measure spaces Y and Z. We introduce two further function spaces:

$$\mathscr{F}_r(\Gamma^n) = \{ f \in \mathscr{F}(\Gamma^n; \mathbb{C}^\times) : f \text{ has relatively compact range and } f = 1$$
$$\text{on the set diagonal } \Delta^n(S) \},$$

$$\mathscr{K}(X) = \{ f \in \mathscr{F}(\Gamma_X) : \int a^{\#\sigma} |f(\sigma)|^2 d\sigma < \infty \, \forall a > 0 \} = \bigcap_{a > 0} \mathscr{D}(a^N),$$

where a^N is the multiplication operator: $(a^N f)(\sigma) = a^{\#\sigma} f(\sigma)$, on $L^2(\Gamma_X)$. The algebraic structures will be on $\mathscr{K}(X)$ and the products will be determined by functions from $\mathscr{F}_r(\Gamma^3)$. Recalling out set notations (Sect. 1) the following elementary identities are crucial:

Proposition 3.1.

(i) *For* $f \in \mathscr{F}(\Gamma^{(2)}), (\sigma, \tau) \in \Gamma^{(2)}$,

$$\sum_{\gamma \subset \sigma \cup \tau} f(\gamma, \bar{\gamma}) = \sum_{\alpha \subset \sigma} \sum_{\beta \subset \tau} f(\alpha \cup \beta, \bar{\alpha} \cup \bar{\beta}). \tag{3.3}$$

(ii) *If* $g \in \mathscr{F}(\Gamma^n)$ *is integrable in the product measure* μ^n, *then*

$$\int \cdots \int g(\sigma_1, \ldots, \sigma_n) d\sigma_1 \cdots d\sigma_n = \int \sum_{|\alpha| = \sigma} g(\alpha) d\sigma. \tag{3.4}$$

Proof. First suppose that g is non-negative, and $\mathbf{j} = (j_1, \ldots, j_n) \in \mathbb{N}^n$ let $g_{\mathbf{j}}$ be the function on $S^{j_1} \times \cdots \times S^{j_n} = S^{j_1 + \cdots + j_n}$ corresponding to $g|_{\Gamma_{j_1} \times \cdots \times \Gamma_{j_n}}$ which is symmetric in its first j_1 arguments, and in its second j_2 arguments, and so on. Then for $(s_1, \ldots, s_N) \in S^{(N)}$,

$$\sum_{|\alpha| = \{s_1, \ldots, s_N\}} g(\alpha_1, \ldots, \alpha_n) = \sum_{j_1 + \cdots + j_n = N} (j_1! \cdots j_n!)^{-1} \sum_{\pi \in \mathscr{S}_N} g_{\mathbf{j}}(s_{\pi(1)}, \ldots, s_{\pi(N)}),$$

so that, by the invariance of the product measure m^N under (the natural action of) the permutation group $\mathscr{S}_N$, and regrouping courtesy of Fubini,

$$\int_{\Gamma_N} \sum_{|\alpha| = \sigma} g(\alpha) d\sigma = (N!)^{-1} \sum_{j_1 + \cdots + j_n = N} (j_1! \cdots j_n!)^{-1} \sum_{\pi \in \mathscr{S}_N} \int \cdots \int g_{\mathbf{j}}(\mathbf{u}_1, \ldots, \mathbf{u}_n) d\mathbf{u}_1 \cdots d\mathbf{u}_n$$

$$= \sum_{j_1 + \cdots + j_n = N} \int_{\Gamma_{j_1}} \cdots \int_{\Gamma_{j_n}} g(\sigma_1, \ldots, \sigma_n) d\sigma_1 \cdots d\sigma_n,$$

where we have abbreviated $dm^k(\mathbf{u})$ to $d\mathbf{u}$. By summing over N we obtain the identity (3.4). Since an integrable function is expressible as a linear combination of non-negative integrable functions (ii) follow by linearity. (i) is immediate. $\quad\square$

Remark. Proposition 3.1 (ii) also follows by repeated application of the case $n = 2$

which is proved in [L M 1]:

$$\int \sum_{|\alpha|=\sigma} f(\alpha) d\sigma = \int \sum_{\beta \subset \sigma} \sum_{|\alpha|=\beta} f(\alpha_1, \ldots, \alpha_{n-1}, \bar{\beta}) d\sigma$$

$$= \int \int \sum_{|\alpha|=\beta} f(\alpha_1, \ldots, \alpha_{n-1}, \gamma_n) d\beta \, d\gamma_n$$

$$\vdots$$

$$= \int \cdots \int f(\gamma_1, \ldots, \gamma_n) d\gamma_1, \ldots, d\gamma_n.$$

We have included a full proof here since much of the subsequent analysis depends on (3.4) which we shall refer to as the *integral-sum identity/lemma*.

Proposition 3.2. *For* $f, g \in \mathcal{K}(X)$ *and* $p \in \mathcal{F}(\Gamma^3)$ *essentially bounded,* $f *_p g$ *is well-defined and satisfies the* L^2-*norm inequality*

$$\| a^N(f *_p g) \| \leq \sup |p| \, \| (a\sqrt{3})^N f \| \, \| (a\sqrt{3})^N g \|, \quad a \geq 1, \tag{3.5}$$

while if $q \in \mathcal{F}(\Gamma^2)$ *is essentially bounded and* $f \circ_q g$ *is defined by (2.8),*

$$\| a^N(f \circ_q g) \| \leq \sup |q| \, \| (a\sqrt{2})^N f \| \, \| (a\sqrt{2})^N g \|, \quad a > 0. \tag{3.6}$$

In particular, $f *_p g, f \circ_q g \in \mathcal{K}(X)$.

Proof. (cf. [L M 2]) An application of the integral sum identity (3.4) gives for $f \in \mathcal{K}(X)$,

$$\int 3^{\#\sigma} |f(\sigma)|^2 d\sigma = \int \sum_{\alpha \subset \sigma} 2^{\#\alpha} |f(\sigma)|^2 d\sigma = \int \int 2^{\#\alpha} |f(\alpha \cup \beta)|^2 d\alpha \, d\beta;$$

in other words, $f^\alpha : \beta \mapsto f(\alpha \cup \beta)$ is square integrable for almost all α and

$$\| \sqrt{3}^N f \|^2 = \int 2^{\#\alpha} \| f^\alpha \|^2 d\alpha. \tag{3.7}$$

By Cauchy–Schwartz and more applications of (3.4),

$$(\sup |p|)^{-2} \int |(a^N(f*g))(\sigma)|^2 d\sigma \leq \int \left\{ \sum_{\alpha \subset \sigma} \int |(a^N f)(\alpha \cup \omega)(a^N g)(\omega \cup \bar\alpha)| \, d\omega \right\}^2 d\sigma$$

$$\leq \int 2^{\#\sigma} \sum_{\alpha \subset \sigma} \| (a^N f)^\alpha \|^2 \, \| (a^N g)^{\bar\alpha} \|^2 d\sigma$$

$$= \int \sum_{\alpha \subset \sigma} 2^{\#\alpha} \| (a^N f)^\alpha \|^2 \, 2^{\#\bar\alpha} \| (a^N g)^{\bar\alpha} \|^2 d\sigma$$

$$= \int 2^{\#\alpha} \| (a^N f)^\alpha \|^2 d\alpha \int 2^{\#\beta} \| (a^N g)^\beta \|^2 d\beta$$

$$= \| (a\sqrt{3})^N f \|^2 \, \| (a\sqrt{3})^N g \|^2,$$

in particular $f *_p g$ is defined almost everywhere and (3.5) holds. Again by the integral-sum lemma,

$$\| a^N(f \circ_q g) \|^2 = \int |a^{\#\sigma} \sum_{\alpha \subset \sigma} q(\alpha, \bar\alpha) f(\alpha) g(\bar\alpha)|^2 d\sigma$$

$$\leq \int (2a^2)^{\#\sigma} \sum_{\alpha \subset \sigma} |q(\alpha, \bar\alpha) f(\alpha) g(\bar\alpha)|^2 d\sigma$$

$$\leq (\sup|q|)^2 \int\int (2a^2)^{\#\alpha+\#\beta}|f(\alpha)g(\beta)|^2\,d\alpha\,d\beta$$
$$= \{\sup|q|\,\|(a\sqrt{2})^N f\|\,\|(a\sqrt{2})^N g\|\}^2,$$

giving (3.6). □

Proposition 3.3. *Let* $p\in\mathscr{F}_r(\Gamma^3)$ *be essentially bounded and satisfy the identity*

$$p(\omega_2\cup\omega_3,\alpha_1,\alpha_2\cup\alpha_3)p(\omega_1,\alpha_2\cup\omega_3,\alpha_3\cup\omega_2)$$
$$= p(\omega_1\cup\omega_2,\alpha_1\cup\alpha_2,\alpha_3)p(\omega_3,\alpha_1\cup\omega_2,\alpha_2\cup\omega_1) \tag{3.8}$$

for almost all $(\omega,\alpha)\in\Gamma^6$, *then* $*_p$ *is associative.*

Proof. By the non-atomicity of the measure m, $\{\sigma\in\Gamma:\sigma\cap\tau\neq\varnothing\}$ is a μ-null set for each $\tau\in\Gamma$. Applying (3.3) with $(\sigma,\tau)=(\omega,\bar\alpha_1)$ and then (3.4) with $n=2$ and $(\alpha_1,\sigma)=(\omega_2,\omega)$ gives

$$f*(g*h)(\sigma) = \sum_{\alpha\subset\sigma}\int p(\omega,\alpha_1,\bar\alpha_1)f(\omega\cup\alpha_1)g*h(\omega\cup\bar\alpha_1)d\omega$$

$$= \sum_{\alpha_1\subset\sigma}\int\int\sum_{\alpha_2\subset\bar\alpha_1}\sum_{\omega_2\subset\omega} p(\omega,\alpha_1,\bar\alpha_1)p(\omega_1,\bar\omega_2\cup\alpha_2,\omega_2\cup\bar\alpha_2)f(\alpha_1\cup\omega)$$
$$\cdot g(\omega_1\cup\bar\omega_2\cup\alpha_2)h(\omega_1\cup\omega_2\cup\bar\alpha_2)d\omega\,d\omega_1$$

$$= \sum_{|\alpha|=\sigma}\int\int\int p(\omega_2\cup\omega_3,\alpha_1,\alpha_2\cup\alpha_3)p(\omega_1,\omega_3\cup\alpha_2,\omega_2\cup\alpha_3)f(\alpha_1\cup\omega_2\cup\omega_3)$$
$$\cdot g(\omega_1\cup\alpha_2\cup\omega_3)h(\omega_1\cup\omega_2\cup\alpha_3)d\omega_1\,d\omega_2\,d\omega_3.$$

On the other hand, applying (3.3) with $(\sigma,\tau)=(\omega,\bar\alpha_3)$ and then (3.4) with $(\alpha_1,\sigma)=(\omega_2,\omega)$ gives for $(f*g)*h(\sigma)$,

$$\sum_{\alpha_3\subset\sigma}\int p(\omega,\bar\alpha_3,\alpha)f*g(\omega\cup\bar\alpha_3)h(\omega\cup\alpha_3)d\omega$$

$$= \sum_{\alpha_3\subset\sigma}\int\int\sum_{\bar\alpha_2\subset\alpha_3}\sum_{\omega_2\subset\omega} p(\omega,\bar\alpha_3,\alpha_3)p(\omega_3,\omega_2\cup\bar\alpha_2,\bar\omega_2\cup\bar\alpha_2)$$
$$\cdot f(\omega_3\cup\omega_2\cup\bar\alpha_2)g(\omega_3\cup\bar\omega_2\cup\alpha_2)h(\omega\cup\alpha_3)d\omega\,d\omega_3$$

$$= \sum_{|\alpha|=\sigma}\int\int\int p(\omega_1\cup\omega_2,\alpha_1\cup\alpha_2,\alpha_3)p(\omega_3,\omega_2\cup\alpha_1,\omega_1\cup\alpha_2)$$
$$\cdot f(\alpha_1\cup\omega_2\cup\omega_3)g(\omega_1\cup\alpha_2\cup\omega_3)h(\omega_1\cup\omega_2\cup\alpha_3)d\omega_1\,d\omega_2\,d\omega_3.$$

Thus if p satisfies (3.8) $*_p$ is associative. □

Thus the a.e. equality (3.8) is a sufficient condition on an essentially bounded p for the associativity of $*_p$. When X is a separable measure space the condition is also sufficient. Since the proof does not illuminate the rest of the paper, where separability plays no rôle, we defer it to [LP]. In view of (3.5) $*_p$ descends to the quotient of $\mathscr{K}(X)$ by the μ_X-null functions. By abuse of notation we also denote the quotient by $\mathscr{K}(X)$.

Elements p of $\mathscr{F}_r(\Gamma^3)$, satisfying (3.8) for *all* (ω,α) in $\Gamma^{(6)}$ will be called *multipliers*. We do not know the answer to the following natural question: suppose that p belongs to $\mathscr{F}_r(\Gamma^3)$ and $*_p$ is associative, then does p agree almost everywhere with a multiplier on X? A positive answer to this would permit a more liberal definition of multipliers.

The first interesting non-trivial multiplier is the *Clifford multiplier* given by

$$p(\omega, \alpha, \beta) = (-1)^{n(\omega \cup \beta, \omega \cup \alpha)}, \tag{3.10}$$

where $n(\sigma, \tau) = \#\{(s,t)\sigma \times \tau : s > t\}$ and $>$ is some measurable ordering on X. We shall call pairs $(\mathcal{K}(X), *_p)$ *Wiener–Clifford algebras* on X.

Proposition 3.4. *p is a multiplier if and only if there are $f \in \mathcal{F}_r(\Gamma)$ and $q \in \mathcal{F}_r(\Gamma^2)$ such that*

(i) *q satisfies the cocycle identity (2.1),*
(ii) *$\tau_f q \tilde{q} = \tau_f^2 q^4 \equiv 1$,*

(iii) *$p(\omega, \alpha, \beta) = f(\omega) \dfrac{q(\alpha, \beta)}{q(\alpha, \omega) q(\omega, \beta)} \quad \forall (\omega, \alpha, \beta) \in \Gamma^{(3)}.$* \tag{3.11}

Proof. Since p satisfies (3.8) if and only if non-zero multiples of p do, and since such multiples may be absorbed by f in (3.11), we may assume p to be normalised: $p(\phi, \phi, \phi) = 1$. Suppose p is a multiplier and define q and f by

$$q(\alpha, \beta) = p(\phi, \alpha, \beta); \quad f(\omega) = p(\omega, \phi, \phi). \tag{3.12}$$

Putting $\omega = \varnothing$ in (3.8) we see that (up to a null set) q is a 2-cocycle, and putting $\alpha = \varnothing$ we obtain the cyclic relation

$$f(\omega_2 \cup \omega_3) p(\omega_1, \omega_3, \omega_2) = f(\omega_1 \cup \omega_2) p(\omega_3, \omega_2, \omega_1). \tag{3.13}$$

Putting $\alpha_2 = \alpha_3 = \omega_2 = \omega_3 = \varnothing$ in (3.8) and applying (3.13) gives

$$f(\omega) = p(\omega, \alpha, \varnothing) q(\alpha, \omega) = \frac{f(\omega \cup \alpha)}{f(\alpha)} q(\omega, \alpha) q(\alpha, \omega),$$

so that

$$\tau_f q \tilde{q} = 1. \tag{3.14}$$

Putting $\omega_3 = \varnothing$ in (3.13) and using (3.14) gives

$$p(\omega_1, \varnothing, \omega_2) = \frac{f(\omega_1 \cup \omega_2)}{f(\omega_2)} q(\omega_2, \omega_1) = \frac{f(\omega_1)}{q(\omega_1, \omega_2)}. \tag{3.15}$$

Putting $\alpha_1 = \alpha_2 = \omega_2 = \varnothing$ in (3.8) and applying (3.15) now yields

$$p(\omega_1, \omega_3, \alpha_3) = \frac{p(\omega_1, \varnothing, \alpha_3) p(\omega_3, \varnothing, \omega_1)}{p(\omega_3, \varnothing, \alpha_3)} = \frac{f(\omega_1) f(\omega_3)}{f(\omega_3)} \frac{q(\omega_3, \alpha_3)}{q(\omega_1, \alpha_3) q(\omega_3, \omega_1)}.$$

Thus p is composed from a function f and a 2-cocycle q, which stand in the reciprocal relation (3.14) through the formula (3.11). It remains to show that a function p given by such a pair (q, f) through (3.11) is a multiplier if and only if $\tau_f^2 q^4 \equiv 1$. Let p be such a function, then the ratio

$$\frac{p(\omega_2 \cup \omega_3, \alpha_1, \alpha_2 \cup \alpha_3) p(\omega_1, \alpha_2 \cup \omega_3, \alpha_3 \cup \omega_2)}{p(\omega_1 \cup \omega_2, \alpha_1 \cup \alpha_2, \alpha_3) p(\omega_3, \alpha_1 \cup \omega_2, \alpha_2 \cup \omega_1)} \tag{3.16}$$

may be expressed as a product $P_1 P_2 P_3$, where

$$P_1 = \frac{f(\omega_2 \cup \omega_3) f(\omega_1)}{f(\omega_1 \cup \omega_2) f(\omega_3)},$$

$$P_2 = \frac{q(\alpha_1, \alpha_2 \cup \alpha_3)}{q(\alpha_1 \cup \alpha_2, \alpha_3)} \frac{q(\omega_1 \cup \omega_2, \alpha_3)}{q(\omega_1, \alpha_3 \cup \omega_2)} \frac{q(\alpha_1 \cup \omega_2, \omega_3)}{q(\alpha_1, \omega_2 \cup \omega_3)} \frac{q(\omega_3, \alpha_2 \cup \omega_1)}{q(\alpha_2 \cup \omega_3, \omega_1)}$$

$$= \frac{q(\alpha_1, \alpha_2)}{q(\alpha_2, \alpha_3)} \frac{q(\omega_2, \alpha_3)}{q(\omega_1, \omega_2)} \frac{q(\omega_2, \omega_3)}{q(\alpha_1, \omega_2)} \frac{q(\omega_3, \alpha_2)}{q(\alpha_2, \omega_1)},$$

$$P_3 = \frac{q(\alpha_2 \cup \omega_3, \alpha_3 \cup \omega_2)}{q(\alpha_1 \cup \omega_2, \alpha_2 \cup \omega_1)} \frac{q(\alpha_1 \cup \alpha_2, \omega_1 \cup \omega_2)}{q(\omega_2 \cup \omega_3, \alpha_2 \cup \alpha_3)}.$$

Moreover, by (3.14)

$$P_1 P_2 = \frac{q(\omega_2, \omega_1)}{q(\omega_3, \omega_2)} \frac{q(\alpha_1, \alpha_2)}{q(\alpha_2, \alpha_3)} \frac{q(\omega_2, \alpha_3)}{q(\alpha_1, \omega_2)} \frac{q(\omega_3, \alpha_2)}{q(\alpha_2, \omega_1)}.$$

Applying the cocycle identity successively to the triples $(\alpha_1 \cup \omega_2, \alpha_2, \omega_1)$, $(\alpha_1 \cup \alpha_2, \omega_2, \omega_1)$ and $(\omega_2, \alpha_1, \alpha_2)$, and using the fact that $q/\tilde{q}$ is a product cocycle,

$$\frac{q(\alpha_1 \cup \alpha_2, \omega_1 \cup \omega_2)}{q(\alpha_1 \cup \omega_2, \alpha_2 \cup \omega_1)} = \frac{q(\alpha_1 \cup \alpha_2, \omega_1 \cup \omega_2)}{q(\alpha_1 \cup \alpha_2 \cup \omega_2, \omega_1)} \frac{q(\alpha_2, \omega_1)}{q(\alpha_1 \cup \omega_2, \alpha_2)}$$

$$= \frac{q(\alpha_1 \cup \alpha_2, \omega_2)}{q(\omega_2, \omega_1)} \frac{q(\alpha_2, \omega_1)}{q(\omega_2, \alpha_1 \cup \alpha_2)} \frac{q(\omega_2, \alpha_1)}{q(\alpha_2, \alpha_1)}$$

$$= \frac{q(\alpha_2, \omega_1)}{q(\omega_2, \omega_1)} \frac{q(\omega_2, \alpha_1)}{q(\alpha_1, \alpha_2)} q/\tilde{q}(\alpha_1, \omega_2) \, q/\tilde{q}(\alpha_2, \omega_2)$$

$$= \frac{q(\alpha_2, \omega_1)}{q(\omega_2, \omega_1)} \frac{q(\alpha_1, \omega_2)}{q(\alpha_1, \alpha_2)} q/\tilde{q}(\alpha_2, \omega_2),$$

and similarly,

$$\frac{q(\alpha_2 \cup \omega_2, \alpha_3 \cup \omega_2)}{q(\omega_2 \cup \omega_3, \alpha_2 \cup \alpha_3)} = \frac{q(\alpha_2, \alpha_3)}{q(\omega_2, \alpha_3)} \frac{q(\omega_3, \omega_2)}{q(\omega_3, \alpha_2)} q/\tilde{q}(\alpha_2, \omega_3).$$

Therefore the ratio (3.16) is $(q/\tilde{q})^2(\alpha_2, \omega_2)$ or by (3.14),

$$\tau_f^2(\alpha_2, \omega_2) \, q^4(\alpha_2, \omega_2).$$

Thus the identity (3.8) holds for a function p given by (3.11), where q is a 2-cocycle and $\tau_f q \tilde{q} \equiv 1$, if and only if $\tau_f^2 q^4 \equiv 1$, and the proof is complete.   □

A function $f: \Gamma_S \to \mathbb{C}$ is called *multiplicative* if

$$f(\alpha \cup \beta) = f(\alpha) f(\beta) \quad \text{for } \alpha, \beta \text{ disjoint.}$$

Thus multiplicative functions are of the form

$$\sigma \to \prod_{s \in \sigma} \phi(s),$$

where ϕ is a function on S. Notation: ε_ϕ. If ϕ_1 and ϕ_2 are measurable functions on X which agree a.e. m, then $\varepsilon_{\phi_1} = \varepsilon_{\phi_2}$ a.e. μ. If $\phi \in L^1(X)$ then $\varepsilon_\phi \in L^1(\Gamma_X)$ and $\int \varepsilon_\phi d\mu = \exp\{\int \phi \, dm\}$. When $\phi \in L^2(X)$ ε_ϕ is called the *exponential vector* determined by ϕ.

Note that a product cocycle in $\mathscr{F}_r(\Gamma^2)$ must be of unit modulus (outside a null set in $\Gamma^{(2)}$).

Lemma 3.5. *Let* $q \in \mathcal{F}_r(\Gamma^2)$ *be a 2-cocycle and* $f \in \mathcal{F}_r(\Gamma^1)$, *then the following are equivalent*:

(i) $\tau_f q \tilde{q} = \tau_f^2 q^4 \equiv 1$,
(ii) $q = \tau_h \pi_k$, *where* $k\tilde{k} = 1$, $k(S^2) \subset \{1, i, -i\}$ *and* $f h^2$ *is multiplicative*.

Proof. Suppose that q is of the form (ii), then $\tau_f q \tilde{q} = \tau_{fh^2} \pi_k \tilde{\pi}_k = \tau_{fh^2} \equiv 1$ and $\tau_f^2 q^4 = (\tau_{fh^2})^2 (\pi_k)^4 \equiv 1$, so (ii) implies (i). Suppose now that q, f satisfy (i), let g_0 be a measurable square root of f and let $\eta = \tau_{g_0} q$. Then

$$\eta \tilde{\eta} = \tau_f q \tilde{q} \equiv 1; \quad \eta^4 = \tau_{f^2} q^4 \equiv 1,$$

so that η is skew-symmetric with range in $\{1, i, -1, -1\}$. Let $k_1(s, t) = \eta(s, t)$, then $\eta = \tau_{g_1} \pi_{k_1}$ for some $g_1 \in \mathcal{F}_r(\Gamma^1)$. Define k_2 by

$$k_2(s, t) = \begin{cases} -1, & \text{if } k_1(s, t) = -1; \\ 1, & \text{otherwise.} \end{cases}$$

Then, since k_2 is symmetric, π_{k_2} is trivial: τ_{g_2}, say. Letting $h = g_0^{-1} g_1 g_2$ and $k = k_1 k_2$ we have $q = \tau_h \pi_k$ and

$$1 \equiv \tau_f q \tilde{q} = \tau_{fh^2} \pi_{k\tilde{k}} = \tau_{fh^2},$$

so that $f h^2$ is multiplicative. Since k has range in $\{1, i, -i\}$ the proof is complete. $\square$

If $g \in \mathcal{F}(X; \mathbb{C}^\times)$ and ε is the multiplicative function determined by a measurable square root of g, then $\varepsilon_g(\omega) = \varepsilon(\alpha \cup \beta)^{-1} \varepsilon(\alpha \cup \omega) \varepsilon(\omega \cup \beta)$. By letting g be the function $s \mapsto (fh^2)(\{s\})$, Lemma 3.5 combines with Proposition 3.4 to give:

Proposition 3.6. *Let p be a multiplier on* Γ_X, *then*

$$p(\omega, \alpha, \beta) = \frac{f(\alpha \cup \omega) f(\omega \cup \beta)}{f(\alpha \cup \beta)} \frac{\pi(\alpha, \beta)}{\pi(\alpha, \omega) \pi(\omega, \beta)}, \tag{3.17}$$

where $f \in \mathcal{F}_r(\Gamma^1)$ *and* $\pi = \pi_k$, *where* $k \in \mathcal{F}(S^2)$ *is skew-symmetric with range in* $\{1, \pm i\}$.

Note that the functions k that arise in (3.17) are all the functions of the form

$$k_F : (s, t) \to \begin{cases} i & \text{on} \quad F; \\ -i & \text{on} \quad \tilde{F}; \\ 1 & \text{elsewhere,} \end{cases} \tag{3.18}$$

where F is a (measurable) subset of $S^{(2)}$ satisfying $F \cap \tilde{F} = \varnothing$.

Writing $p = (f, k)$ for a multiplier with representation (3.17) we define *equivalence* for multipliers by $(f, k) \approx (f', k')$ if π_k and $\pi_{k'}$ are equivalent cocycles, modulo a null set—in other words $\pi_{k'} = \tau_g \pi_k$ a.e. for some $g \in \mathcal{F}_r(\Gamma^1)$. We denote the group of equivalence classes of multipliers by $\mathcal{M}(\Gamma_X)$. The justification for this definition of equivalence will come later.

Proposition 3.7. *Let* $p = (f, k)$, $p' = (f', k')$ *be multipliers on* Γ_X. *Then the following are equivalent*:

(i) $p \approx p'$,
(ii) $\text{supp } k \, \Delta \, \text{supp } k'$ *is null.*

Proof. Let E be the symmetric set $\operatorname{supp} k \Delta \operatorname{supp} k'$ and let k_1 be the symmetric function agreeing with $k'k^{-1}$ outside E and equal to 1 on E. Then π_{k_1} agrees with $\pi_{k'}\pi_k^{-1}$ outside the set $\hat{E} = \{(\{s\} \cup \alpha, \{t\} \cup \beta): (s,t) \in E, \alpha, \beta \in \Gamma\}$. Since $\hat{E}$ is μ^2-null if and only if E is m^2-null the result follows □

Theorem 3.8.

$$\mathcal{M}(\Gamma_X) \cong (\mu(\Gamma_2), \Delta),$$

the measure algebra of Γ_2 considered as a group under symmetric difference.

Proof. By Proposition 3.7, $\Phi: [(f,k)] \to [\operatorname{supp} k]$ is a well-defined injective map from $\mathcal{M}(\Gamma_X)$ to the equivalence classes of symmetric measurable subsets of $S^{(2)}$. Let E be such a subset of $S^{(2)}$ then, if $F \cup \tilde{F}$ is a measurable partition of E. $\Phi([(1, k_F)]) = [E]$, so Φ is bijective. Now suppose that $E = E_1 \Delta E_2$, where E_1, E_2 are symmetric subsets of $S^{(2)}$, and let $F \cup \tilde{F}$ be a measurable partition of E and $F_3 \cup \tilde{F}_3$ a measurable partition of $E_1 \cap E_2$. Define $k_1 = k_{F_3 \cup F_1}$ and $k_2 = k_{\tilde{F}_2 \cup F_1}$, where $F_i = F \cap E_i$ $(i = 1,2)$, then $\operatorname{supp} k_i = E_i$ and $\operatorname{supp} k_1 k_2 = E$. Thus $\Phi^{-1}([E_1 \Delta E_2]) = \Phi^{-1}([E_1])\Phi^{-1}([E_2])$, and the theorem now follows since the measure algebra of Γ_2 is isomorphic to the symmetric measure algebra of $X^{(2)}$. □

Suppose that $p = (f,k)$ and $p' = (f',k')$ are equivalent multipliers so that $\pi_{k'} = \tau_h \pi_k$ almost everywhere and, for a.a. σ, $h(\sigma) = \prod_{\{s,t\} \subset \sigma} (k/k')(s,t)$ and is $\{\pm 1\}$-valued. Then $M_{hf/f'}$ is an isomorphism from $(\mathcal{K}(X), *_p)$ to $(\mathcal{K}(X), *_{p'})$, justifying our notion of equivalence for multipliers. Again Wiener–Clifford algebras of functions of several variables may be reduced to one variable function algebras via the natural identification of $\Gamma(X_1) \times \cdots \times \Gamma(X_n)$ with $\Gamma(X)$, where X is the sum of the measure spaces $X_1, \ldots, X_n$. In particular super-symmetric Wiener–Clifford algebras arise by choosing multipliers p of the form $(\omega, \alpha, \beta) \to \prod_i p(\omega \cap S_i,$ $\alpha \cap S_i, \beta \cap S_i)$, where $P|_{\Gamma^{(3)}(S_i)}$ is a Clifford multiplier for $i = 1, \ldots, k$, say, and the Wiener multiplier $(\equiv 1)$ for $i = k+1, \ldots, n$.

4. Twisted Wiener–Clifford Algebras

In this section we construct and classify a further class of algebras by means of an involution on S, that is a map $i: S \to S$ whose square is the identity map. The Bose and Fermi products, which are discussed extensively in [L M 1], are included and also some interesting new products. The results of this section subsume those of the previous one since the involution may be the identity map.

Let $Z = (S, m, i, S_+)$, where $X = (S, m)$ is a σ-finite, non-atomic measure space, i is a measurable involution on S and S_+ is a measurable subset of S such that if $S_0 = \{s \in S: i(s) = s\}$ and $S_- = i(S_+)$, then $S_0 \cup S_+ \cup S_-$ is a partition of S. Let $\sim$ be the involution on $\Gamma^n(S)$ induced by i: $\tilde{\sigma} = (\tilde{\sigma}_n, \ldots, \tilde{\sigma}_1)$, where $\{s_1, \ldots, s_n\}^\sim = \{i(s_1), \ldots, i(s_n)\}$, and for $f \in \mathcal{F}(\Gamma^n)$ let $\tilde{f}$ be the induced map: $\tilde{f}(\sigma) = f(\tilde{\sigma})$. Note that this does not conflict with the previous use of $\sim$. For Z as above, $\Gamma(Z)$ or Γ_Z will denote the measure space $\Gamma(S, m)$ together with this extra structure. $\sim$ (and i) will be referred to as the *twist*.

We are interested in those elements p of $\mathscr{F}_r(\Gamma^3)$ for which the twisted convolution

$$f \tilde{*}_p g : \sigma \mapsto \sum_{\alpha \subset \sigma} \int p(\omega, \alpha, \tilde{\alpha}) f(\alpha \cup \tilde{\omega}) g(\omega \cup \tilde{\alpha}) d\omega \qquad (4.1)$$

defines an associative product on $\mathscr{K}(Z) := \mathscr{K}(S, m + \tilde{m})$. A similar inequality to (3.5) holds so that $\mathscr{K}(Z)$ is closed under $\tilde{*}_p$ which again descends to the quotient modulo μ_X-null functions.

Proposition 4.1. *For $p \in \mathscr{F}_r(\Gamma^3)$, let $D_p \in \mathscr{F}(\Gamma^6)$ be given by*

$$(\boldsymbol{\alpha}, \boldsymbol{\omega}) \mapsto p(\omega_2 \cup \omega_3, \alpha_1, \alpha_2 \cup \alpha_3) p(\omega_1, \omega_3 \cup \alpha_2, \omega_2 \cup \alpha_3)$$
$$- p(\omega_1 \cup \omega_2, \alpha_1 \cup \alpha_2, \alpha_3) p(\omega_3, \alpha_1 \cup \tilde{\omega}_2, \alpha_2 \cup \tilde{\omega}_1), \qquad (4.2)$$

then the vanishing of D_p (a.e.) is sufficient for the associativity of $\tilde{*}_p$ on $\mathscr{K}(Z)$.

Proof. The same applications of Lemma 3.1 as were used in the proof of Proposition 3.2 lead to the expression

$$\sum_{|\alpha| = \sigma} \int\int\int D_p(\boldsymbol{\alpha}, \boldsymbol{\omega}) f(\alpha_1 \cup \tilde{\omega}_2 \cup \tilde{\omega}_3) g(\tilde{\omega}_1 \cup \alpha_2 \cup \omega_3) h(\omega_1 \cup \omega_2 \cup \alpha_3) d\omega_1 d\omega_2 d\omega_3$$

for $\{f \tilde{*} (g \tilde{*} h) - (f \tilde{*} g) \tilde{*} h\}(\sigma)$ and the result follows. $\quad\square$

Elements p of $\mathscr{F}_r(\Gamma_Z^3)$ for which D_p vanishes identically will be called multipliers on Γ_Z. In our analysis of multipliers on Γ_Z we continue to denote by τ_f and π_k the trivial and product cocycles determined by $f \in \mathscr{F}_r(\Gamma^1)$ and $k \in \mathscr{F}_r(S^2)$ respectively.

Proposition 4.2. *p is a multiplier on Γ_Z if and only if there are $f \in \mathscr{F}_r(\Gamma^1)$ and $q \in \mathscr{F}_r(\Gamma^2)$ such that*

(i) *q satisfies the cocycle identity (2.1),*

(ii) *$\tau_f q \tilde{q} = 1$,* $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (4.3)

(iii) *$\dfrac{q(\alpha, \beta)\, q(\alpha, \tilde{\beta})}{q(\beta, \alpha)\, q(\tilde{\beta}, \alpha)} = 1$,* $\qquad\qquad\qquad\qquad\qquad\qquad\quad$ (4.4)

(iv) *$p(\omega, \alpha, \beta) = f(\omega) \dfrac{q(\alpha, \beta)}{q(\alpha, \tilde{\omega})\, q(\omega, \beta)} \quad \forall (\omega, \alpha, \beta) \in \Gamma^{(3)}$.* $\qquad$ (4.5)

Proof. Applying identical steps to (4.2) as were applied to (3.7) in the proof of Proposition 3.4 leads to the identities (4.3) and (4.5), q being a 2-cocycle on $\Gamma(S)$. Further identical steps lead to the expression

$$\frac{q(\alpha_2, \tilde{\omega}_2)\, q(\alpha_2, \omega_2)}{q(\tilde{\omega}_2, \alpha_2)\, q(\omega_2, \alpha_2)}$$

for the ratio (3.16)˜. Thus p is a multiplier on Γ_Z if and only if p is of the form (4.5), where (q, f) satisfies (4.3) and q is a 2-cocycle satisfying (4.4). $\quad\square$

Lemma 4.3. *Let $q \in \mathscr{F}_r(\Gamma^2)$ be a 2-cocycle on $\Gamma(S, m)$ and $f \in \mathscr{F}_r(\Gamma^1)$, then the following are equivalent:*

(i) $\tau_f q\tilde{q} = 1$,

(ii) $q = \tau_h \pi_k$ for some $h \in \mathscr{F}_r(\Gamma^1)$, where $k\tilde{k} = 1$ and $f h\tilde{h}$ is multiplicative.

Proof. Suppose that (q, f) satisfies (i). Then, by Proposition 2.3, $q = \tau_{h_1} \pi_{k_1}$ for some $h_1 \in \mathscr{F}_r(\Gamma^1)$ and product cocycle π_{k_1}, moreover $k_2 = (k_1\tilde{k}_1)^{-1}$ is both $\sim$-invariant and symmetric. Let k_3 be a measurable square root of k_2 which is also $\sim$-invariant and symmetric. Then $\pi_{k_3} = \tau_g$ for some $g \in \mathscr{F}_r(\Gamma^1)$, and if $k = k_1 k_3$, $h = h_1 g^{-1}$, then the pair (h, k) satisfies (ii). The converse is immediate.

Proposition 4.4. *Let p be a multiplier on Γ_Z, then*

$$p(\omega, \alpha, \beta) = \varepsilon(\omega) \frac{g(\alpha \cup \tilde{\omega}) g(\omega \cup \beta)}{g(\alpha \cup \beta)} \frac{\pi(\alpha, \beta)}{\pi(\alpha, \tilde{\omega})\pi(\omega, \beta)}, \tag{4.6}$$

where $\varepsilon \in \mathscr{F}_r(\Gamma^1)$ is multiplicative, $g \in \mathscr{F}_r(\Gamma^1)$ and $\pi = \pi_k$, where k is determined by a skew-symmetric $\{\pm i, 1\}$-valued function ξ on $S_0 \times S_0$, a skew-symmetric function γ on $S_+ \times S_+$ and a $\{\pm 1\}$-valued function η on $S_0 \times S_+$ as follows:

$$k(s, t) = \xi(s, t) \text{ on } S_0 \times S_0, = \begin{cases} \gamma(s, t) & \text{on } S_+ \times S_+ \\ \gamma(\tilde{s}, \tilde{t}) & S_- \times S_- \\ \gamma(\tilde{t}, s) & S_+ \times S_- \\ \gamma(t, \tilde{s}) & S_- \times S_+ \end{cases} = \begin{cases} \eta(s, t) & \text{on } S_0 \times S_+ \\ \eta(t, \tilde{s}) & S_- \times S_0 \\ 1 & S_+ \times S_0 \\ 1 & S_0 \times S_- \end{cases} \tag{4.7}$$

Proof. Combining Proposition 4.2 and Lemma 4.3 we obtain the form (4.6) for p in which $\varepsilon \in \mathscr{F}_r(\Gamma_1)$ is multiplicative and $\pi = \pi_{k_1}$, where k_1 satisfies

$$k_1\tilde{k}_1 = 1, \tag{4.8a}$$

$$\frac{k_1(s, t)\, k_1(s, \tilde{t})}{k_1(t, s)\, k_1(\tilde{t}, s)} = 1. \tag{4.8b}$$

By the analysis of Sect. 3 we may assume that k_1 has the required form on $S_0 \times S_0$, and by taking a measurable logarithm of k_1 on $S_+ \times S_+$ (and arguing as in the proof of Proposition 2.3) we may assume that $\gamma := k_1|_{S_+ \times S_+}$ has the required form also. Let $\zeta = k_1|_{S_+ \times S_-}$, $\delta = k_1|_{S_+ \times S_0}$ and $\alpha = k_1|_{S_0 \times S_+}$ then by (4.8b) $\eta: (s, t) \mapsto \delta(t, s)\, \alpha(s, t)^{-1}$ is $\{\pm 1\}$-valued on $S_0 \times S_+$, thus k_1 has the following form on the rest of $S^{(2)}$:

$$k_1(s, t) = \begin{cases} \gamma(\tilde{s}, \tilde{t}) & \text{on } S_- \times S_- \\ \zeta(t, s)\, \gamma(t, \tilde{s})\, \gamma(\tilde{s}, t)^{-1} & S_- \times S_+ \\ \delta(\tilde{t}, s)^{-1} & S_0 \times S_- \\ \delta(t, s)\, \eta(s, t) & S_0 \times S_+ \\ \delta(\tilde{s}, t)^{-1}\eta(t, \tilde{s}) & S_- \times S_0 \end{cases}.$$

Define k_2 on $S^{(2)}$ by

$$k_2(s, t) = \begin{cases} \gamma(\tilde{t}, s)\, \zeta(s, t)^{-1} & \text{on } S_+ \times S_- \\ \gamma(\tilde{s}, t)\, \zeta(t, s)^{-1} & S_- \times S_+ \end{cases} = \begin{cases} \delta(s, t)^{-1} & \text{on } S_+ \times S_0 \\ \delta(t, s)^{-1} & S_0 \times S_+ \\ \delta(\tilde{t}, s) & S_0 \times S_- \\ \delta(\tilde{s}, t) & S_- \times S_0 \end{cases}$$

and 1 elsewhere, then k_2 satisfies (4.8) and is symmetric, so π_{k_2} is trivial. The product $k_1 k_2$ is of the required form and the proof is complete. $\quad\square$

We write (f, g, k) for the multiplier (4.6) where $\varepsilon = \varepsilon_f$ and $\pi = \pi_k$. By taking a measurable logarithm of f one may divide out the $\sim$-invariant part of ε to obtain a representation (f', g', k) in which $\tilde{f}' = f'^{-1}$. Then the extent to which the representation of a given multiplier is unique is as follows: $(f, g, k) \equiv 1$ if and only if $k \equiv 1, g$ is multiplicative, $\varepsilon_f = g\tilde{g}$ and is $\{\pm 1\}$-valued. As before we define equivalence for multipliers by equivalence of the corresponding 2-cocycles (modulo null sets), and denote the group of equivalence classes of multipliers on Γ_Z by $\mathcal{M}(\Gamma_Z)$. We remind the reader of our convention that a group valued function f on $Y \times Y$ is skew-symmetric if $f(t, s) = f(s, t)^{-1}$.

Lemma 4.5. *With the above notation let p and p' be multipliers on Γ_Z. Then $p \approx p'$ if and only if almost everywhere ξ/ξ' and γ/γ' are $\{\pm 1\}$-valued and $\eta = \eta'$.*

Proof. p and p' are equivalent if and only if ξ/ξ', γ/γ' and η/η' are symmetric outside a null set. But ξ/ξ' and γ/γ' are skew-symmetric and η/η' $(t, s) = \eta/\eta'$ (s, t) if and only if $\eta/\eta'(s, t) = \eta/\eta'(t, s) = 1$. The result follows. $\quad\square$

Combining this with Proposition 4.4, identifying η with its support $\eta^{-1}(\{-1\})$ and again accommodating the group isomorphism between $\mathbb{C}^{\times}/\{\pm 1\}$ and $\mathbb{C}^{\times}$, we have:

Theorem 4.6.

$$\mathcal{M}(\Gamma_Z) \cong (\mu(\Gamma_2(Z_0)), \Delta) \times \mathcal{A}_r(Z_+^{(2)}; \mathbb{C}^{\times}) \times (\mu(Z_0 \times Z_+), \Delta),$$

where $\mathcal{A}_r$ is the group of (equivalence classes of) skew-symmetric functions with relatively compact range.

Finally notice that if $p = (f, g, k)$ and $p' = (f', g', k')$ are equivalent multipliers on Γ_Z, then $\mathcal{M}_{hg/g'}$ is an isomorphism from $(\mathcal{K}(S, m, i, S_+), \tilde{*}_p)$ to $(\mathcal{K}(S, m', i, S_+), \tilde{*}_{p'})$, where m' is the measure given by $dm' = (f/f')dm$, if h is the almost everywhere $\{\pm 1\}$-valued function $\sigma \to \prod_{\{s, t\} = \sigma} (k'/k)(s, t)$.

5. Levy Fields and Bose–Fermi Bridges

In this section $X = (S, m)$ is a fixed (non-atomic, σ-finite) measure space, so we abbreviate $\mathcal{K}(X)$ to $\mathcal{K}$, $dm(s)$ to ds and so on. $L^2(X)$ is naturally included in $L^2(\Gamma)$, in fact in $\mathcal{K}$—we use this without further comment.

A field of operators $\{l_q(f) : f \in \mathcal{K}\}$, with common domain $\mathcal{K}$, is associated with each bounded 2-cocycle q. It is shown that $\delta_{\varnothing}$ is cyclic for $\{l_q(f) : f \in L^2(X)\}$ and that $\{l_q(f), l_q^{\dagger}(f) : f \in L^2(X)\}$ is irreducible. Local commutation relations are then obtained under various conditions to be satisfied by the cocycles on sets in $\Gamma^{(2)}$ determined by the supports of the test functions. In particular we obtain bridges from Boson to Fermion quantum fields, and also a continuum of inequivalent smooth Levy fields with common covariance. This highlights some of the obstacles in the problem of deducing commutation relations from martingale conditions which was raised in [AP].

Definition 5.1. *Define a map* $\hat{\ }$ *from* $F(\Gamma^{(2)})$ *to operators on* $F(\Gamma)$, *a product* $\times$ *on* $F(\Gamma^{(2)})$ *and a bilinear map* $F(\Gamma) \times F(\Gamma^{(2)})$ *to* $F(\Gamma^{(2)})$ *by*

$$\hat{v}g(\sigma) = \sum_{|\alpha|=\sigma} v(\alpha_1, \alpha_2)g(\alpha_2), \tag{5.1}$$

$$u \times v(\sigma, \tau) = \sum_{|\alpha|=\sigma} u(\alpha_1, \alpha_2 \cup \tau)v(\alpha_2, \tau), \tag{5.2}$$

$$g.v(\sigma, \tau) = g(\sigma)v(\sigma, \tau). \tag{5.3}$$

The map $\hat{\ }$ is well defined by (3.6) and is clearly injective so that associativity for the product $\times$ follows from the easily verified relation $\widehat{u \times v} = \hat{u}\hat{v}$. If $u = g.v$ where $g \in \mathcal{K}$ and v is bounded then, by (3.6), $\hat{u}$ leaves $\mathcal{K}$ invariant permitting the following

Definition 5.2. *For a bounded 2-cocycle* q *and function* f *in* $\mathcal{K}$, *define an operator on* $L^2(\Gamma)$ *with domain* $\mathcal{K}$ *by*

$$l_q(f) = \widehat{f.q}.$$

By the previous remarks each operator leaves $\mathcal{K}$ invariant and if (q_i, f_i) $i = 1, \ldots, n$ are as above

$$l_{q_1}(f_1)\cdots l_{q_n}(f_n) = \widehat{f_1 \cdot q_1 \times \cdots \times f_n \cdot q_n}.$$

Moreover

$$l_{q_1}(f_1)\cdots l_{q_n}(f_n)f_{n+1} = f_1 \circ_{q_1} f_2 \circ_{q_2} \cdots \circ_{q_n} f_{n+1}, \tag{5.5}$$

where the right-hand side is independent of bracketing, generalising the associativity of $\circ_q$ (2.8).

Lemma 5.3. *Let* $f \in \mathcal{K}$ *and let* q *be a bounded 2-cocycle. Then* $\mathcal{K} \subset \mathscr{D}(l_q(f)^*)$ *and*

$$[l_q^\dagger(f)g](\sigma) = \int \bar{f}(\omega)\bar{q}(\omega, \sigma)g(\sigma \cup \omega)d\omega, \tag{5.6}$$

where $l_q^\dagger(f)$ *denotes the restriction of* $l_q(f)^*$ *to* $\mathcal{K}$.

Proof. For $g \in \mathcal{K}$ the integral (5.6) is well defined, and the identity is established by taking inner products and applying the integral-sum lemma. $\square$

Note. For operators c having domain $\mathcal{K}$ we shall consistently use the notation $c^\dagger$ to denote the restriction to $\mathcal{K}$ of the adjoint of c. Moreover a sentence or equation involving terms $c^\#$ is to be understood as two statements: one in which each $\#$ is replaced by $\dagger$ and another in which each $\#$ is deleted. Elements of $\mathcal{K}$ will sometimes be referred to as test functions, and those whose support is contained in Γ_1, Γ_n or $\Gamma_0 \cup \cdots \cup \Gamma_m$ (for some m) as 1-particle, n-particle or finite particle functions respectively.

Proposition 5.4. *Let* $q_1, \ldots, q_n$ *be bounded 2-cocycles and* $f_1, \ldots, f_n$ *belong to* $L^2(X)$, *then*

$$l_{q_1}(f_1)\cdots l_{q_n}(f_n)\delta_\varnothing = n! P_S Q_n f_1 \otimes \cdots \otimes f_n,$$

where $Q_n(s) = \prod_{i=1}^{n-1} q_i(s_i; s_{i+1}, \ldots, s_n)$, P_S *is the symmetrisation projection:* $(n!)^{-1}\sum_{\pi \in \mathscr{S}_n} U_\pi$ $((U_\pi f)(s) = f(s_{\pi^{-1}(1)}, \ldots, s_{\pi^{-1}(n)}))$ *and the left-hand side is considered as a symmetric function on* $S^{(n)}$.

In particular for any sequence $\{q_i : i \in \mathbb{N}\}$ of bounded 2-cocycles the linear span of

$$\{l_{q_1}(f_1) \cdots l_{q_n}(f_n)\delta_\varnothing : n \in \mathbb{N}, f_1, \ldots, f_n \in L^2(X)\}$$

is dense in $L^2(\Gamma)$.

Proof. A straightforward induction gives the first part and the second part follows from the fact that, for each $n, \{Q_n f_1 \otimes \cdots \otimes f_n : f_i \in L^2(X)\}$ spans $L^2(X^n)$ since each Q_n is non-vanishing. $\square$

Proposition 5.5. *Let q be a bounded 2-cocycle, then*

$$\{l_q(f), l_q^\dagger(f) : f \in L^2(X)\} \tag{5.8}$$

is irreducible in the sense that the only bounded operators which, together with their adjoint, leave $\mathscr{K}$ invariant and commute with this set are the multiples of the identity.

Proof. Suppose that T is an operator on $L^2(\Gamma)$ which commutes with the family (5.8) and, together with its adjoint, leaves $\mathscr{K}$ invariant. Then, since (5.8) is symmetric, $T^\dagger$ will commute with this class, and

$$T(f_1 \circ f_2) = Tl(f_1)f_2 = f_1 \circ Tf_2; \quad T^\dagger(f_1 \circ f_2) = f_1 \circ T^\dagger f_2. \tag{5.9}$$

Letting $h = T\delta_\varnothing, k = T^\dagger \delta_\varnothing$ and iterating (5.9) we have

$$Tf = f \circ h; \quad T^\dagger f = f \circ k$$

for f in $\mathscr{K}_0 :=$ linear span $\{f_1 \circ \cdots \circ f_n : n \in \mathbb{N}, f_i \in L^2(X)\}$. Thus for $f \in \mathscr{K}_0$,

$$\langle \delta_\varnothing, T^*g \rangle = \langle h, g \rangle = \int \bar{h}(\sigma)g(\sigma)d\sigma,$$

whereas

$$\langle \delta_\varnothing, T^\dagger g \rangle = \langle \delta_\varnothing, g \circ k \rangle = g(\varnothing)k(\varnothing),$$

so that $h = k(\varnothing)\delta_\varnothing$. In other words T is a multiple of the identity on $\mathscr{K}_0$ which, by the previous proposition, is dense in $L^2(\Gamma)$. Thus if T is bounded then it must be a multiple of the identity. $\square$

Proposition. 5.6. *Let $f, g \in \mathscr{K}$ and ξ, η be bounded product cocycles, then for $\varepsilon, \delta \in \mathbb{C}$,*

$$l_\eta(g)l_\xi(f) + \delta l_\xi(f)l_\eta(g) = \hat{q},$$

where

$$q(\sigma, \tau) = \sum_{\alpha \subset \sigma} g(\alpha)f(\bar{\alpha})\eta(\alpha, \tau)\xi(\bar{\alpha}, \tau)\{\eta(\alpha, \bar{\alpha}) + \delta\xi(\bar{\alpha}, \alpha)\},$$

and if f and g are one-particle functions, then

$$(\{l_\eta^\dagger(g)l_\xi(f) + \varepsilon l_\xi(f)l_\eta^\dagger(g)\}h)(\sigma)$$

$$= \Omega_{f,g;\xi,\eta}(\sigma)h(\sigma) + \int \sum_{s \in \sigma} f(s)\bar{g}(t)h(\{t\} \cup \sigma \setminus \{s\}) \frac{\xi(s; \sigma)\bar{\eta}(t; \sigma)}{\bar{\eta}(t; s)} \{\xi(s; t)\bar{\eta}(t; s) + \varepsilon\} dt,$$

where $\Omega(\sigma) = \int \bar{g}f(t)\bar{\eta}\xi(t, \sigma)dt$.

Proof. Remembering our convention that cocycles take the value 1 on the set diagonals, these identities are immediate. $\square$

For $\varepsilon \in \mathbb{C}$, define a bracket on the algebra of operators on $\mathcal{K}$ by

$$[X, Y]_\varepsilon = XY + \varepsilon YX.$$

For $\varepsilon = \pm 1$ this is the anti-commutator (Jordan product) and commutator (Lie bracket) respectively—in general it is a deformation of these.

Corollary 5.7. *Let* $f, g \in \mathcal{K}$ *and let* ξ, η *be bounded product cocycles, then for* $\delta, \varepsilon \in \mathbb{C}$,

(i) *if* $\tilde{\eta} + \delta\xi = 0$ *on* $\operatorname{supp} f \times \operatorname{supp} g$, *then* $[l_\eta(f), l_\xi(g)]_\delta = 0$,
(ii) *if* f *and* g *are one-particle functions and* $\xi\tilde{\bar{\eta}} + \varepsilon = 0$ *on* $\operatorname{supp} f \times \operatorname{supp} g$, *then* $[l_\eta^\dagger(g), l_\xi(f)]_\varepsilon = M_{\Omega_{f,g;\xi,\eta}}$, *the multiplication operator.*

In particular, if f is a one-particle function and $\eta\tilde{\bar{\eta}} + \varepsilon = 0$ on $\operatorname{supp} f \times \operatorname{supp} f$ where $\varepsilon > 0$, then $l_\eta(f)$ is bounded.

Corollary 5.8. *Let* q *be the product cocycle determined by a function of the form* k_F *(defined in (3.18)) and let* $E = F \cup \tilde{F}$, *then for* $f, g \in L^2(X)$,

(i) $\operatorname{supp} f \times \operatorname{supp} g \subset \begin{cases} E \\ E^c = S \backslash E \end{cases} \Rightarrow [l(f), l(g)]_\pm = 0; \ [l^\dagger(g), l(f)]_\pm = \langle g, f \rangle$,

(ii) *If* $\operatorname{supp} f \subset \sum_{j=1}^{n} A_j$ *for a finite collection of measurable subsets* $\{A_j\}$ *of* S *for*

which $\bigcup_{j=1}^{n} A_j \times A_j \subset E$, *then* $l_q(f)$ *is bounded.*

Proof. On $E, q + \tilde{q} = 0$ and $q\tilde{q} + 1 = 0$, whereas on $E^c, q = \bar{q} = \tilde{q} = 1$. The result follows from the linearity of $l_q(\cdot)$:

$$l_q(f) = \sum_j l_q(f\chi_{A_j}), \quad \text{each } l_q(f\chi_{A_j}) \text{ is bounded.} \quad \square$$

An interesting class of examples where the above applies is given by $X = (\mathbb{R}^k,$ Lebesgue) and F such that $E = \{(x, y) : |x - y| < \lambda\}$, $\lambda \in (0, \infty)$, then $l(f)$ is bounded whenever the test function f has compact support. We shall return to this example shortly.

Right $\circ$-multiplication may be similarly defined:

$$r_q(f)g = g \circ_q f \quad (f, g \in \mathcal{K}).$$

By the generalised associativity (5.5), the left and right $\circ$-multiplication operators commute. When the cocycle takes particular form we can say more:

Proposition 5.9. *Again let* q *be the product cocycle determined by a function of the form* k_F, *then there is a* $\{\pm 1\}$-*valued function* ψ *on* Γ *such that*

$$M_\psi l(f) M_\psi^{-1} = r(\psi f), \quad f \in \mathcal{K}. \tag{5.10}$$

Proof. Let $f, g \in \mathcal{K}$, then

$$[r(f)g](\sigma) = \sum_{\alpha \subset \sigma} q(\bar{\alpha}, \alpha) f(\alpha) g(\bar{\alpha}) = \sum_{\alpha \subset \sigma} q(\alpha, \bar{\alpha}) q^2(\bar{\alpha}, \alpha) f(\alpha) g(\bar{\alpha}),$$

but q^2 is skew-symmetric and $\{\pm 1\}$-valued—it is therefore symmetric and so equal to τ_ψ for some $\{\pm 1\}$—valued function ψ. Since $\psi = \psi^{-1}$ (5.10) follows. $\quad \square$

For each multiplier p on Γ and test function f in $\mathcal{K}$, let $L_p(f), R_p(f)$ be the left and right $*$-multiplication operators on $\mathcal{K}$:

$$L_p(f):g \to f*_p g; \quad R_p(f):g \to g*_p f.$$

Proposition 5.10. *Let p be a multiplier on $\Gamma, f, g \in \mathcal{K}$.*

(i) $L_p^\dagger(f)g = \bar{f} *_{\hat{p}} g$, *where* $\hat{p}:(\omega, \alpha, \beta) \to \bar{p}(\alpha, \omega, \beta)$.
(ii) *If* $p(\omega, \alpha, \beta) = \eta(\alpha, \beta)/\eta(\alpha, \omega)\eta(\omega, \beta)$, *where η is a skew-symmetric, unit modulus 2-cocycle, then*

$$L_p^\dagger(f) = L_p(\bar{f}).$$

(iii) *If further f is a one-particle function, then*

$$L_p(f) = l_\eta(f) + l_\eta^\dagger(\bar{f}),$$
$$R_p(f) = l_{\bar{\eta}}(f) + l_{\bar{\eta}}^\dagger(\bar{f}) = L_{\tilde{p}}(f),$$

where

$$\tilde{p}:(\omega, \alpha, \beta) \to p(\omega, \beta, \alpha).$$

Proof.

(i) Follows from the applications of the integral-sum lemma and (ii) and (iii) are immediate. $\quad\square$

Proposition 5.11. *Let $f, g, h \in \mathcal{K}$ and ξ, η be bounded 2-cocycles, then*

$$\| [l_\xi^\dagger(f) - l_\eta^\dagger(g)]h \| \leq \| \sqrt{3^N}h \| \{ \|f\| \, \|\xi - \eta\|_\infty + \|\eta\|_\infty \|f - g\| \},$$
$$\| [l_\xi(f) - l_\eta(g)]h \| \leq \| \sqrt{3^N}h \| \{ \| \sqrt{3^N}f \| \, \|\xi - \eta\|_\infty + \|\eta\|_\infty \| \sqrt{3^N}(f - g) \| \}. \quad (5.11)$$

Proof. Several applications of the integral-sum lemma give the identities

$$\langle h, l_\xi(f)l_\eta^\dagger(g)h \rangle = \iiint \bar{h}(\alpha \cup \beta) f(\alpha)\xi(\alpha, \beta)\bar{g}(\gamma)\bar{\eta}(\gamma, \beta)h(\beta \cup \gamma)d\alpha d\beta d\gamma,$$
$$\langle h, l_\xi^\dagger(f)l_\eta(g)h \rangle = \iiiint \bar{h}(\alpha \cup \beta)\bar{f}(\gamma \cup \delta)\bar{\xi}(\gamma \cup \delta, \alpha \cup \beta)\eta(\alpha \cup \gamma, \beta \cup \delta)$$
$$\cdot g(\alpha \cup \gamma)h(\beta \cup \delta)d\alpha d\beta d\gamma d\delta.$$

Applying the estimate $\int \|k^\alpha\|^2 d\alpha \leq \| \sqrt{3^N}k \|^2$ $(k \in \mathcal{K})$, which follows from (3.7), then gives the inequalities (5.11). $\quad\square$

Corollary 5.12. *Let $\xi = \pi_{k_1}, \eta = \pi_{k_2}$ be bounded product cocycles and f, g be one-particle functions. For h in the linear span of the exponential vectors $\{\varepsilon_f : f \in L^2(X)\}$,*

$$\max\{ \| [l_\xi(f) - l_\eta(g)]h \|, \| [l_\xi^\dagger(f) - l_\eta^\dagger(g)]h \| \} \leq c\{ \|f\| \, \|k_1 - k_2\|_\infty + \|f - g\| \},$$

where c is a constant dependent only on h.

Let $F \subset S^{(2)}$ be such that $F \cup \tilde{F}$ is a measurable partition of $S^{(2)}$ and consider the family of 2-cocycles $\{\xi_v : v \in [0, 1]\}$, where ξ_v is the product cocycle determined by the function $k := e^{iv\pi/2}\chi_F + e^{-iv\pi/2}\chi_{\tilde{F}}$, then $\|\xi_v - \xi_\psi\| < ((\pi/2)e^{\pi/2})|v - \psi|$, so $\{l^v : f \mapsto l_{\xi_v}(f) | v \in [0, 1]\}$ is a homotopy between the boson and fermion Fock representations over $L^2(X)$—the first *Bose-Fermi bridge*.

In the notation of (3.7) the following estimates are easily obtained by more applications of the integral-sum identity.

Proposition 5.13. *Let $f, g, h \in \mathcal{K}$ and let ξ, η be bounded 2-cocycles, then*

$$K(\xi, \eta)^{-1} \| [l_\xi^\dagger(f) - l_\eta^\dagger(g)] h \|^2 \leq \iint_{E_1(f,\xi,\eta)} |f(\alpha)| \, \|h^\alpha\| \, |f(\beta)| \, \|h^\beta\| \, d\alpha \, d\beta$$

$$+ c(f, g, h) \|f - g\|,$$

$$K(\xi, \eta)^{-1} \| [l_\xi(f) - l_\eta(g)] h \|^2 \leq \iint_{E_2(f,\xi,\eta)} \|f^\alpha\| \, \|h^\alpha\| \, \|f^\beta\| \, \|h^\beta\| \, d\alpha \, d\beta$$

$$+ c(\sqrt{3^N} f, \sqrt{3^N} g, h) \| \sqrt{3^N}(f - g) \|, \quad (5.12)$$

where

$$E_1 = \{(\alpha, \beta) \in \operatorname{supp} f \times \operatorname{supp} f : (\xi - \eta)(\alpha), (\xi - \eta)(\beta) \not\equiv 0 \text{ as functions of } \gamma\},$$
$$E_2 = S_f \times S_f \cap S_{\xi - \eta} \cap S_{\xi - \eta},$$
$$c(f, g, h) = \| \sqrt{3^N} h \|^2 \{ \|f - g\| + 2\|f\| \}; \quad K(\xi, \eta) = 5 \max\{ \|\xi\|_\infty^2, \|\eta\|_\infty^2 \},$$

and $S_q = \{(\alpha, \beta) : \alpha \subset \sigma, \beta \subset \tau, q(\sigma, \tau) = 0\}$ with similar definition for one argument functions.

Let X be a finite dimensional Euclidean space with Lebesgue measure and $\{\eta_\lambda : \lambda \in \mathbb{R}_+\}$ be the product cocycles determined by functions k_{F_λ} (see 3.18) $(F_\lambda = \{(s, t) : |s - t| < \lambda, s_1 < t_1\}$, then for functions f of compact support the sets $E_i(f, \eta_\lambda, \eta_\mu)$ $(i = 1, 2)$ tend to zero, as λ approaches μ, in measure. Since, for $\lambda = 0$, $l_\lambda^\# : f \mapsto l_\lambda^\#(f)$ is the boson Fock representation and as $\lambda \to \infty$, $l_\lambda^\#$ approaches the Fermion Fock representation, we have a second Bose-Fermi bridge. In fact, if f and g are one particle functions and $\{|s - t| : f(s), g(t) \neq 0\} \subset [0, \lambda]$, then $l_\lambda^\#(f), l_\lambda^\#(g)$ satisfy the canonical anti-commutation relations (Corollary 6.8) whereas if $\{|s - t| : f(s), g(t) \neq 0\} \subset (\lambda, \infty)$ they satisfy the canonical commutation relations. If f and g have compact support then the former condition will eventually be satisfied (for large enough λ).

Quasi-free, or positive temperature, Bose-Fermi bridges may also be constructed; either via fields of operators $\{L_p(f) : f \in L^2(S, m)\}$ defined through suitable multipliers p on a twisted finite power set Γ_z, or by employing linear/conjugate linear operators R, T on $L^2(S, m)$ and defining

$$L(f) = l_\xi(Rf) + l_\eta^\dagger(Tf) \quad f \in L^2(S, m)$$

for well-chosen pairs of 2-cocycles (ξ, η).

The central result of [A P] (see also [P]) is that two smooth Levy boson (fermion) fields with a given covariance kernel are equivalent, and that smooth Levy boson (fermion) fields may be constructed for any covariance density. The question of whether other smooth Levy fields exist or not is left open but has been a source of lively speculation. We now demonstrate the existence of a plentiful supply of such smooth Levy fields in which the past and future satisfy no fixed (anti-) commutation relations. A Levy-type theorem characterising quantum Brownian motion thus remains beyond our reach at present. The reader is referred to [A P] for the definitions of Levy-field, covariance kernel etc.

Lemma 5.14. *Let ξ be a bounded 2-cocycle and let $f, g, h \in \mathcal{K}$ be such that $\operatorname{supp} g \subset \Gamma_I$ and $\operatorname{supp} g \cup \operatorname{supp} h \subset \Gamma_J$, where $I \cap J = \varnothing$, then*

$$\langle f, [l_\xi^\#(g) - g^\#(\varnothing)]h\rangle = 0, \tag{5.13}$$

where $g^\dagger := \bar{g}$.

Proof. Under the above asumptions on f, g, h and ξ,

$$\langle f, l_\xi(g)h\rangle = \int \bar{f}(\sigma) \sum_{\alpha \subset \sigma} \xi(\alpha, \bar{\alpha})g(\alpha)h(\bar{\alpha})d\sigma$$

$$= \iint \bar{f}(\alpha \cup \beta)\xi(\alpha, \beta)g(\alpha)h(\beta)d\alpha d\beta = g(\varnothing)\langle f, h\rangle$$

by the integral sum identity and (5.13) follows. $\square$

Proposition 5.15. *Let q be a unit modulus 2-cocycle on $\Gamma(\mathbb{R}_+)$, then the family*

$$\mathfrak{X} = \{X(f) := l_q(f) + l_q^\dagger(f) \mid f \in L^2(\mathbb{R}_+)\}$$

is a smooth Levy field, with cyclic vector $\delta_\varnothing$ and covariance kernel $(f, g) \to \int \bar{f}g$, whenever $\mathfrak{X}$ consists of bounded operators.

Proof. Since $l_q(f) = \frac{1}{2}\{X(f) - iX(if)\}$, $\delta_\varnothing$ is cyclic for the linear span of $\{X(f_1) \cdots X(f_n)\delta_\varnothing : n \in \mathbb{N}, f_i \in L^2(\mathbb{R}_+), i = 1, \ldots, n\}$ by Proposition 5.4. $(f_1, \ldots, f_n) \mapsto X(f_n)\delta_\varnothing$ is clearly real multilinear, and for fixed $f^{(1)}, \ldots, f^{(n)}$ the continuity of the map $t \mapsto X(f_{t]}^{(1)}) \cdots X(f_{t]}^{(n)})\delta_\varnothing$ is ensured by the estimates (5.11). The martingale property: $\langle u, X(f_{[t})v\rangle = 0$ whenever $\operatorname{supp} u, v \subset \Gamma_{[0,t]}$, is an instance of Lemma 5.14; in fact $l_q^\dagger(f_{[t})u = l_q^\dagger(f_{[t})v = 0$ and

$$\langle u, X(f_{[t})X(g_{[t})v\rangle = \langle u, l_q^\dagger(f_{[t})l_q(g_{[t})v\rangle$$

$$= \int_{\Gamma_{[0,t]}} \int_t^\infty d\beta \, ds \, \bar{u}(\beta)\bar{f}(s)g(s)v(\beta) = \langle u, v\rangle \int_t^\infty \bar{f}g$$

in other words $\mathfrak{X}$ is a Levy field with the given cyclic vector and covariance kernel. Now suppose that $u \in L^2(\Gamma)$ has support in $\Gamma_{[0,s]}$ and $f, g \in L^2(\mathbb{R}_+)$ have support in $[s, t]$, then it is easy to see that $l^\dagger(f)l(g)u = \langle f, g\rangle u$, so that

$$\|\{X(f)X(g) - \langle f, g\rangle\}u\|^2 = \|l(f \circ g)u\|^2$$

$$= \iint d\beta \, d\gamma \, \bar{u}(\beta)\overline{f \circ g}(\gamma)f \circ g(\gamma)u(\beta)$$

$$= \|u\|^2 \|f \circ g\|^2 \leq 4\|u\|^2 \|f \otimes g\|^2.$$

The smoothness condition now follows from this inequality by the dominated convergence theorem. $\square$

Theorem 5.16. *For $\lambda \geq 0$ let $\mathfrak{X}^\lambda$ be the Levy field (5.14) in which the 2-cocycle is the (skew-symmetric) product cocycle determined by the function*

$$(s, t) \to \begin{cases} i & \text{on } F_\lambda; \\ -i & \text{on } \tilde{F}_\lambda; \\ 1 & \text{elsewhere,} \end{cases}$$
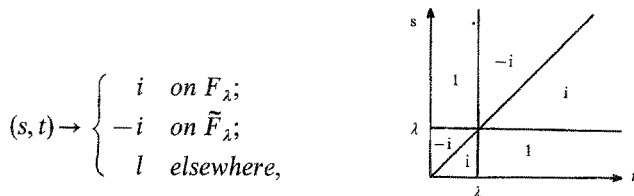
where $F_\lambda = \{(s, t): s < t < \lambda \text{ or } \lambda < s < t\}$. Then the family $\{\mathfrak{X}^\lambda : \lambda \geq 0\}$ consists of inequivalent smooth Levy fields with common covariance kernel $(f, g) \to \int \bar{f}g$.

Proof. By Corollary 5.8 (ii) $\mathfrak{X}^\lambda$ consists of bounded operators for each λ ($\mathbb{R}_+ = [0, \lambda) \cup [\lambda, \infty)$, $(0, \lambda) \times [0, \lambda) \cup [\lambda, \infty) \times [\lambda, \infty) = F_\lambda \cup \widetilde{F}_\lambda$). Since these fields satisfy distinct local commutation relations, they must be inequivalent.

Remark. The Levy field $\mathfrak{X}^0$ is equivalent to the Levy fermion field. Smooth Levy fields with more general covariance kernels (inequivalent to the Bose and Fermi fields) may be constructed from operators $l_q(f)$, where q is a 2-cocycle on $\Gamma(S, m, i, S_+)$ in which $S_+ = J \times \{0\}$, $S_0 = I, S_- = \{0\} \times J$, where $I \cup J$ is a partition of $\mathbb{R}_+$, i interchanges $(s, 0)$ and $(0, s)$ and m_0, m_+, m_- are absolutely continuous measures on the line.

6. Differential Equations with Mixed Noise

In the final section we solve linear stochastic differential equations driven by mixed noise, giving the action of solutions on vectors from $\mathscr{K}$ explicitly. This extends results from [H P 1, 2, Maa, ApH, Me 1 and L M 2]. Our formula (6.2) for the solution is inspired by Theorem 5.3 of [Maa]. The equations we treat are those having the form

$$dX_t = \left\{ \sum_i L_i(t) \Phi(\phi^{(i)}(\cdot, t)) dA_i^*(t) + \sum_{i,j} M_{ij}(t) \Phi(\lambda^{(i,j)}(\cdot, t)) d\Lambda_{ij}(t) \right.$$

$$\left. + \sum_j N_j(t) \Phi(\psi^{(j)}(\cdot, t)) dA_j(t) + K(t) dt \right\} X_t \qquad (6.1)$$

on $h_0 \otimes L^2(\Gamma(\mathbb{R}_+)^n) = L^2(\Gamma(\mathbb{R}_+)^n; h_0)$. The reader is referred to [HP 1] for the definitions of the creation, gauge and annihilation processes: A_i^*, Λ_{ij} and A_j respectively, and for the meaning of Eq. (6.1) and a solution for (6.1). For a linear operator T on $\oplus^{(n)} L^2(\mathbb{R}_+)$, $\Phi(T)$ denotes the operator defined on $\mathscr{E} :=$ linear span $\{v \otimes \varepsilon_f : v \in h_0, \mathbf{f} \in \oplus^{(n)} L^2(\mathbb{R}_+)\}$ by $\Phi(T) v \otimes \varepsilon_f = v \otimes \varepsilon_{Tf}$. L_i, M_{ij}, N_j and K are locally bounded, strongly measurable $\mathscr{B}(h_0)$-valued maps on $\mathbb{R}_+$ and $\phi^{(i)}, \lambda^{(i,j)}$ and $\psi^{(j)}$ are locally bounded, measurable diagonal $M_n(\mathbb{C})$-valued maps on $\mathbb{R}_+^2$ which are adapted in the sense that they take the value I on $\{(s, t) : s > t\}$. Here $\mathscr{B}(h_0)$ is the algebra of bounded linear operators on the complex separable Hilbert space h_0, $M_n(\mathbb{C})$ is the algebra of $n \times n$ complex matrices and the indices i, j run from 1 to n.

To state the theorem we need a little more notation. $\boldsymbol{\beta} = (\beta_i^j)$ will denote a matrix of elements of $\Gamma(\mathbb{R}_+)$, while $\boldsymbol{\beta}^{\cdot}$ will denote the n-tuple of sets whose i^{th} component is $\beta_i^1 \cup \beta_i^2 \cup \cdots \cup \beta_i^n$ with $\boldsymbol{\beta}_{\cdot}$ similarly defined. For n-tuples of sets $\boldsymbol{\alpha}, \boldsymbol{\gamma}$, the n-tuple whose i^{th} component is $\alpha_i \cup \gamma_i$ will be denoted $\boldsymbol{\alpha} \cup \boldsymbol{\gamma}$ and $d\mu^n(\omega)$ will be abbreviated to $d\omega$.

Theorem 6.1. *The quantum stochastic differential equation (6.1) has a unique solution, with initial condition $X_0 = I$, given by: for $t \in \mathbb{R}_+, |\boldsymbol{\sigma}| \subset [0, t]$ and $k \in \mathscr{E}$ with support in $\Gamma([0, t])^n$,*

$$(X_t k)(\boldsymbol{\sigma}) = \int \sum_{\boldsymbol{\alpha} \cup \boldsymbol{\beta}^{\cdot} \cup \boldsymbol{\gamma} = \boldsymbol{\sigma}} P(\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \omega) x_t(\boldsymbol{\alpha}, \boldsymbol{\beta}, \omega) k(\omega \cup \boldsymbol{\beta}_{\cdot} \cup \boldsymbol{\gamma}) d\omega, \qquad (6.2)$$

where, if $|(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})| = \{t_1 < t_2 < \cdots < t_N\}$, the kernel x_t is given by

$$x_t(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) = P(\boldsymbol{\alpha} \cup \boldsymbol{\beta}^{\cdot}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) r_t(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}),$$

$$r_t(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) = R_t(R_{t_N}^{-1} G(t_N) R_{t_N}) \cdots (R_{t_1}^{-1} G(t_1) R_{t_1}),$$

R being the solution of the ordinary differential equation

$$\frac{dR}{dt} = KR; \quad R_0 = I,$$

G being given by

$$G(t) = \begin{cases} L_i(t) & \text{if} \quad t \in \alpha_i \\ M_{ij}(t) & \text{if} \quad t \in \beta_i^j, \\ N_j(t) & \text{if} \quad t \in \gamma_j \end{cases}$$

and the multiplier P being given by

$$P(\omega, \underline{\alpha}, \underline{\beta}, \underline{\gamma}) = P_1(\omega, \underline{\alpha}) P_2(\omega, \underline{\beta}) P_3(\omega, \underline{\gamma})$$

$$= \prod_{1 \le i,j,k \le n} \prod_{\substack{a \in \alpha_i, b \in \beta_i^j \\ c \in \gamma_j, t \in \omega_k}} \phi_k^{(i)}(t, a) \lambda_k^{(i,j)}(t, b) \psi_k^{(j)}(t, c).$$

Proof (sketch). In the above notation consider the family of functions $\mathscr{F} = \{F(u, v; f, g, \cdot) : u, v \in h_0, \mathbf{f}, \mathbf{g} \in \oplus^{(n)} L^2(\mathbb{R}_+)\}$ given by

$$t \mapsto \int_{\Gamma([0,t])^n} \sum_{\alpha \cup \beta \cup \gamma = \omega} \langle V(\underline{\alpha}, \underline{\beta}, \underline{\gamma}) u \otimes \varepsilon_{\mathbf{f}}, r_t(\underline{\alpha}, \underline{\beta}, \underline{\gamma}) U_{\beta \cup \gamma} v \otimes \varepsilon_{\mathbf{j}} \rangle \, d\omega,$$

where $V(\underline{\alpha}, \underline{\beta}, \underline{\gamma})$ is the product $V_1 V_2 \cdots V_N$ in which

$$V_k = \begin{cases} \Phi(\overline{\phi}^{(i)}(\cdot, t_k)) U_{t_k}^i & \text{if} \quad t_k \in \alpha_i \\ \Phi(\overline{\lambda}^{(i,j)}(\cdot, t_k)) U_{t_k}^i & \text{if} \quad t_k \in \beta_i^j \\ \Phi(\overline{\psi}^{(j)}(\cdot, t_k)) & \text{if} \quad t_k \in \gamma_j \end{cases}$$

and U is defined by $(U_\alpha k)(\omega) = k(\omega \cup \alpha)$, $(U_t^i k)(\omega) = k(\omega \cup (\varnothing, \dots, \varnothing, t, \varnothing, \dots, \varnothing))$ with t in the ith place.

By extracting the highest component $s = t_N$ from $|\omega|$, taking into account each of the possibilities $s \in \alpha_i, \beta_i^j$ or γ_j, applying the integral-sum identity and differentiating, it may be seen that the family $\mathscr{F}$ satisfies the system of ordinary differential equations

$$\frac{dF}{dt}(u, v; \mathbf{f}, \mathbf{g}, t) = \sum_i f_i(t) F(L_i^*(t) u, v; \overline{\phi}^{(i)}(\cdot, t) \mathbf{f}, \mathbf{g}; t)$$

$$+ \sum_{i,j} \overline{f}_i g_i(t) F(M_{ij}^*(t) u, v; \overline{\lambda}^{(i,j)}(\cdot, t) \mathbf{f}, \mathbf{g}; t)$$

$$+ \sum_j g_j(t) F(N_j^*(t) u, v; \overline{\psi}^{(j)}(\cdot, t) \mathbf{f}, \mathbf{g}; t)$$

$$+ F(K^*(t) u, v; \mathbf{f}, \mathbf{g}, t),$$

$$F(u, v, \mathbf{f}, \mathbf{g}, 0) = \langle u \otimes e_{\mathbf{f}}, v \otimes e_{\mathbf{g}} \rangle.$$

On the other hand, applying the commutation relation

$$U_s^k \Phi(\phi) = \phi_k(s) \Phi(\phi) U_s^k \quad \text{for} \quad \phi = \text{diag}[\phi_1, \dots, \phi_n]$$

and the integral-sum identity, it may be seen that

$$F(u, v; \mathbf{f}, \mathbf{g}; t) = \langle u \otimes \varepsilon_{\mathbf{f}}, X_t v \otimes \varepsilon_{\mathbf{g}} \rangle, \tag{6.4}$$

where X_t is given by (6.2). Combining (6.3) and (6.4) we see that X is a solution of (6.1). Uniqueness for solutions of (6.1) follows by minor modifications of the arguments employed in [H P 1].   □

Remark. Equation (6.2) makes good sense when $k \in \mathcal{K}$, moreover $X_t k \in \mathcal{K}$ for each t. Here $\mathcal{K}$ has to be redefined to take into account the initial Hilbert space h_0:

$$\mathcal{K} := \{k \in L^2(\Gamma; h_0): a^N k \in L^2(\Gamma; h_0) \ \forall a > 0\}.$$

The quantum Itô formula [H P 1] yields sufficient conditions for a process X satisfying (6.1) to be isometry valued and necessary conditions for it to be co-isometry or unitary valued. The following are necessary conditions for X to be unitary valued:

(i) $M_{ij} = U_{ij} - \delta_{ij}$, where $\{U_{ij}\}$ is unitary $M_n(\mathscr{B}(h_0))$-valued;

(ii) $N_j = \sum_i L_i^* U_{ij}$;

(iii) $K = iH - \frac{1}{2} \sum_i L_i^* L_i$, where H is self-adjoint valued;

(iv) $\phi^{(i)}$ and $\psi^{(i)}$ are unitary (diagonal) $M_n(\mathbb{C})$-valued;

(v) $\{U_{ij} \otimes \Phi(\lambda^{ij}) + \delta_{ij} \otimes [I - \Phi(\lambda^{ii})]\}$ is unitary $M_n(\mathscr{B}(h_0 \otimes L^2(\Gamma^n)))$-valued;

(vi) $N_j \Phi(\psi^{(j)}) + L_j^* [\Phi(\bar{\phi}^{(j)}) - \Phi(\bar{\phi}^{(j)} \lambda^{jj})] + \sum_i L_i^* U_{ij} \Phi(\bar{\phi}^{(i)} \lambda^{ij}) = 0;$

(vii) $L_j^* \Phi(\bar{\phi}^{(j)}) + N_j [\Phi(\psi^{(j)}) - \Phi(\bar{\lambda}^{jj} \psi^{(j)})] + \sum_i N_i U_{ji}^* \Phi(\bar{\lambda}^{ii} \psi^{(i)}) = 0;$

(6.5)

they are also sufficient conditions for X to be isometry valued. We leave the non-trivial question of whether (6.5) implies unitarity for the solution of (6.1), and indeed the more general question of whether reasonable conditions may be found on a kernel x for the corresponding operator:

$$(Xk)(\sigma) = \int \sum_{\alpha \cup \beta \cup \gamma = \sigma} x(\omega, \alpha, \underline{\beta}, \gamma) k(\omega \cup \underline{\beta} \cup \gamma) d\omega$$

to be unitary.

Remark. The following special cases of isometry-valued processes satisfying a quantum stochastic differential equation of the type (6.1) may be noted:

(i) $\psi^{(j)} = \bar{\phi}^{(j)}; \lambda^{i,j} = \phi^{(i)} \bar{\phi}^{(j)};$

(ii) $\psi^{(j)} = \bar{\phi}^{(j)}; \lambda^{ij} = \delta_{ij} I_n$ on $\{(s,t): s < t\}; N_j = -L_j^* U_{jj}$ (i.e. $\sum_{i \neq j} L_i^* U_{ij} = 0$).;

(iii) $\psi^{(j)} = \bar{\phi}^{(j)}; \lambda^{ij} = 0$ on $\{(s,t): s < t\}; N_j = -L_j^*;$

and when $n = 1$:

(iv) $\psi = \bar{\phi}; \lambda = \chi_E$; for each t *either* $\lambda_t \equiv 1$ or $N(t) = -L^*(t)$, where $E \subset \mathbb{R}_+^2$ is measurable and contains $\{(s,t): s > t\};$

(v) $(X_t k)(\sigma) = \sum_{\{\beta \subset \sigma: \sigma \times \beta \subset E\}} \prod_{t \in \beta} (U_t - I) k(\sigma);$

(vi) $(W_t^{f, \phi} k)(\sigma) = \exp\left\{-\frac{1}{2} \int_0^t |f|^2\right\} \sum_{\alpha \subset \sigma} \pi_\phi(\sigma, \alpha) \int \pi_{\bar{\phi}}(\sigma, \omega) \varepsilon_f(\alpha_1) \varepsilon_{\bar{f}}(\omega) k(\omega \cup \alpha) d\omega;$

in which conditions (i)–(iv) of (6.5) are understood in each case. (iv) is the general solution to (6.5) in the case $n = 1$. (v) is the special case of (iv) in which $K = L = N = 0$, and is unitary valued by the characterisation of unitary valued martingales in [HLP]. (vi) is the special case of (iv) in which $K = M = 0$ and $h_0 = \mathbb{C}$—these are analogues of Weyl operators and satisfy

$$\frac{d}{dx}\bigg|_{x=0} W_t^{xf,\phi} = l_{\bar\phi}(f\chi_{[0,t]}) - l_{\bar\phi}^\dagger(f\chi_{[0,t]}).$$

We end with a few remarks on quantum stochastic integrals. The following extensions may be defined:

$$[A^*(F)k](\sigma) = \sum_{\alpha \subset \sigma} [F(\alpha)k](\bar\alpha), \qquad (6.6)$$

$$A(F)k = \int F(\omega)U_\omega k\, d\omega, \qquad (6.7)$$

$$\Lambda(F) = A^*(FU), \qquad (6.8)$$

where $(U_\omega k)(\sigma) = k(\sigma \cup \omega)$, (6.7) is a Bochner integral and the following identity for the singular operators $\{U_\omega\}$ is pertinent:

$$\int \|a^N U_\omega k\|^2 d\omega = \|(1 + a^2)^{N/2}k\|^2 \quad (a > 0) \qquad (6.9)$$

from which, in particular, we see that for each $k \in \mathcal{K}$, $U_\omega k \in \mathcal{K}$ for almost all ω. Thus $k \in \mathscr{D}(A(F))$, the domain of the operator $A(F)$, if and only if $\omega \mapsto F(\omega)U_\omega k \in L^1(\Gamma; \mathcal{H})$ $(\mathcal{H} = h_0 \otimes L^2(\Gamma))$ and, if F is 1-particle, the condition: $\int \|F(s)\varepsilon_f\|^2 ds < \infty \forall f$ (from [HP1]) amounts to $\mathscr{E} \subset \mathscr{D}(A(F))$. Sufficient conditions for k to lie in $\mathscr{D}(A^*(F))$ or $\mathscr{D}(\Lambda(F))$ (and also for $A^*(F)k, \Lambda(F)k$ to belong to $\mathcal{K}$) are obtained from the estimate

$$\|a^N A^*(F)k\|^2 \le \int \|(a\sqrt{2})^{N+\#\omega}F(\omega)k\|^2 d\omega \quad (a > 0). \qquad (6.10)$$

Equations (6.9) and (6.10) follow from the integral-sum identity, as does the fact that the definitions $(6.6 - 6.8)$ do extend those of [HP1] and that

$$A(F)^\dagger = A^*(F^*); \quad \Lambda(F)^\dagger = \Lambda(F^*).$$

An advantage of these definitions is that no adaptedness conditions are required, but we shall not pursue this further here.

References

[AP] Acardi, L., Parthasarathy, K. R.: A martingale characterisation of canonical commutation and anti-commutation relations. J. Funct. Anal. **77**, 211–231 (1988)
[ApH] Applebaum, D. B., Hudson, R. L.: Fermion Itô's formula and stochastic evolutions. Commun. Math. Phys. **96**, 473–496 (1984)
[Bar] Bargmann, V.: On unitary ray representations of continuous groups. Ann. Math. **59**, 1–46 (1954)

[BaW]   Bargmann, V., Wigner, E. P.: Group theoretical discussion of relativistic wave equations. Proc. Natl. Acad. Sci. USA **34**, 211–223 (1948)
[Gui]   Guichardet, A.: Symmetric Hilbert spaces and related topics. Lecture Notes in Mathematics vol. 261. Berlin, Heidelberg, New York: Springer 1972
[HLP]   Hudson, R. L., Lindsay, J. M., Parthasarathy, K. R.: Stochastic integral representation of some quantum martingales in Fock space. In: From local times to global geometry, control and physics. Elworthy, K. D. (ed.) Research Notes in Mathematics vol. **150**, pp. 121–131. London: Pitman 1986
[HP1]   Hudson, R. L., Parthasarathy, K. R.: Quantum Itô's formula and stochastic evolutions. Commun. Maths. Phys. **93**, 301–323 (1984)
[HP2]   Hudson, R. L., Parthasarathy, K. R.: Unification of fermion and boson stochastic calculus, Commun. Math. Phys. **104**, 457–470 (1986)
[LM1]   Lindsay, J. M., Maassen, H.: The stochastic calculus of bose noise, Preprint, 1988
[LM2]   Lindsay, J. M., Maassen, H.: An integral kernel approach to noise. In: Quantum probability and applications III (Proceedings, Oberwolfach 1987). Accardi, L., von Waldenfels, W. (eds.) Lecture Notes in Mathematics, pp. 192–208. Berlin, Heidelberg, New York: Springer 1988
[LP]   Lindsay, J. M.: Parthasarathy, K. R.: Rigidity of the Poisson convolution. In: 2nd Heidelberg workshop on quantum probability. Accardi, L., von Waldenfels, W. (eds.) Lecture Notes in Mathematics. Berlin, Heidelberg, New York: Springer to appear, 13pp.
[Maa]   Maassen, H.: Quantum Markov processes on Fock space described by integral kernels. In: Quantum probability and applications II. Accardi, L., von Waldenfels, W. (eds.) Lecture Notes in Mathematics vol. **1136**, pp. 361–374. Berlin, Heidelberg, New York: Springer 1985
[Mac]   Mackey: G. W., Unitary representations of group extensions. I, Acta. Math. **99**, 265–311 (1958)
[Me1]   Meyer, P. A.: Eléments de probabilités quantiques I–V: Sém Prob. XX. Lecture Notes in Mathematics, vol. **1204**, pp. 186–312. VI–VIII Sém Prob XXI. Lecture Notes in Mathematics, vol. **1247**, pp. 34–80. Berlin, Heidelberg, New York: Springer 1986, 1987
[Me2]   Meyer, P. A.: Fock space and probability theory. Bielefeld Preprint, 1985
[P]   Parthasarathy, K. R.: A unified approach to classical, bosonic and fermionic brownian motion. In: Proc. Lévy Centenary Col., 1987 (Astérisque, Soc. Math. de France)
[Var]   Varadarajan, V. S.: Geometry of quantum theory. New York: van Nostrand, 1968, 1970