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Global Validity of the Boltzmann Equation for Two- and Three-Dimensional Rare Gas in Vacuum: Erratum and Improved Result

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Abstract. We point out an error in our earlier papers [1] and [2] and present a more direct and natural proof which, although based on the same physical ideas of the previous ones, saves and actually improves the validity results for the Boltzmann equation given in [1] and [2].

1. The Error

We refer to the numbers from [1] as [1, (1.1)] say. The mistake in [1] is in [1, (3.8)]. The right conclusion of the arguments [1, (3.5–7)] leading to [1, (3.8)] is

$$I(\phi_t^d X) = I(\phi_t X) + \sum_{i=1}^{k} (t - t_i)(y_i' - y_i)(p_i' - u_i'), \tag{1.1}$$

with the original meaning of the symbols. Therefore, the lower bound [1, (3.3)] $I(\phi_t^d X) \geqq I(\phi_t X)$ remains correct, but the upper bound [1, (3.4)] is wrong. In Sect. 2, we show how the BBGKY-hierarchy can be controlled in a different, actually more efficient way.

By a slight generalization of the arguments leading to (1.1) it is easy to prove the following inequality (Lemma 1 below) showing that the interacting flow is more dispersive than the free one. This property enables us to control the solutions to the BBGKY and the Boltzmann hierarchies by one estimate along the lines followed in [1].

Lemma 1.

$$I(\phi_s \phi_t^d X) \geqq I(\phi_{s+t} X) \tag{1.2}$$

for all $s, t \geqq 0$ *or* $s, t \leqq 0$.

2. Estimate of the BBGKY Hierarchy

The application for which we needed [1, (3.8)] was to bound (uniformly in the diameter $d > 0$) the solution of the BBGKY hierarchy given by Lanford's perturbation series [1, (2.17)] which we repeat:

$$f_t^d = S^d(t) f_0^d + \sum_{n \geqq 0} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n S^d(t-t_1) C^d \dots S^d(t_n) f_0^d . \tag{2.1}$$

Our hypothesis is that there are $\beta_0 > 0$ and $z_0 > 0$ such that

$$\sup |f_{0,j}^d(X_j)| \exp 2\beta_0 (I + T)(X_j) \leqq C z_0^j \tag{2.2}$$

for all $j = 1, 2, \dots$. The notation is as in [1] and we consider the Boltzmann-Grad limit in the three-dimensional case, i.e. $N \to \infty$, $d \to 0$, and $Nd^2 = \lambda^{-1}$. λ is a measure of the mean free path of the collisions in the gas.

Our target is to prove that for $z_0 \lambda^{-1}$ sufficiently small, we can bound the series on the right of (2.1) by a converging series of nonnegative terms not depending on d. In the sequel C will always denote a positive constant independent of d and t, but C can have different values in different formulas.

Choose j arbitrary but fixed in the n^{th} term of the right-hand side of (2.1),

$$S^d(t-t_1) C^d S^d(t_1 - t_2) \dots C^d S^d(t_{n-1} - t_n) C^d S^d(t_n) f_0^d(X_j) . \tag{2.3}$$

This can be written as

$$\begin{aligned} &(N-j)(N-j-1) \dots (N-j-n) d^{2n} \\ &\times \sum_{k_1=1}^{j} \sum_{k_2=1}^{j+1} \dots \sum_{k_n=1}^{j+n-1} \int dv_{j+1} \dots dv_{j+n} \int dn_1 \dots dn_n \\ &\times \left[\prod_{r=1}^{n} n_r (u_{k_r} - v_{j+r}) \right] f_{0,j+n}^d (Y_{j+r}), \end{aligned}$$

where

$$\begin{aligned} Y_j = \phi_{\tau_0}^d X_j , \qquad & Y_{j+r} = \phi_{\tau_r}^d (\xi_{j+r}^{k_r} \cup Y_{j+r-1}), \\ & \xi_{j+r}^{k_r} = (y_{k_r} + n_r d, v_{j+r}), \end{aligned} \tag{2.5}$$

$\tau_r = t_{r+1} - t_r$, $r = 0 \dots n$, $t_0 = t$, $t_{n+1} = 0$ and y_{k_r} and u_{k_r} are position and velocity of the k_r^{th} particle in the configuration $Y_{j+r-1} \cdot \xi_{j+r}^{k_r} \cup Y_{j+r-1}$ denotes the configuration in which we join the particle $\xi_{j+r}^{k_r}$ to Y_{j+r-1}.

By repeated application of (1.2):

$$\begin{aligned} I(Y_{j+n}) &= I(\phi_{-t_n}^d [\xi_{j+n}^{k_n} \cup Y_{j+n-1}]) \geqq I(\phi_{-t_n} [\xi_{j+n}^{k_n} \cup Y_{j+n-1}]) = (y_{k_n} + n_n d - v_{j+n} t_n)^2 \\ &+ I(\phi_{-t_n} Y_{j+n-1}) \geqq \dots \geqq \sum_{r=1}^{n} (y_{k_r} + n_r d - v_{j+r} t_r)^2 + I(\phi_{-t} X_j) . \end{aligned} \tag{2.6}$$

By (2.2), (2.6) and the energy conservation law $\left(\text{which implies } T(Y_{j+r}) = T(X_j) + \frac{1}{2} \sum_{r=1}^{n} v_{j+r}^2 \right)$, the right-hand side of (2.4) is bounded by (we put $\beta_0 = 1$)

$$\begin{aligned} & C z_0^{j+n} [\exp - I(\phi_{-t} X_j)] \sup_{y_1 \dots y_n} \sum_{k_1=1}^{j} \sum_{k_2=1}^{j+1} \dots \sum_{k_n=1}^{j+n-1} \int dv_{j+1} \dots dv_{j+n} \\ & \times \exp - \left[\sum_{r=1}^{n} v_{j+r}^2 + 2T(X_j) \right] \exp - \sum_{r=1}^{n} (y_r - v_{j+r} t_r)^2 \prod_{r=1}^{n} (|u_{k_r}| + |v_{j+r}|) . \end{aligned} \tag{2.7}$$

Proceeding as in the paper, we have

$$\sup_y \int dv \exp -(y-vt)^2 - \frac{v^2}{2} \leqq \frac{C}{1+t^3} \tag{2.8}$$

and

$$\exp -\left[T(X_j) + \frac{1}{2}\sum_{r=1}^{n} v_{j+r}^2 \right) \Bigg] \left(\sum_{k_1=1}^{j} |u_{k_1}| + |v_{j+1}| \right) \dots \left(\sum_{k_n=1}^{j+n-1} |u_{k_n}| + |v_{j+n}| \right) \leqq C^n (j+n)^n. \tag{2.9}$$

We explain (2.9) in detail. By the conservation of the energy:

$$\sum_{k_r=1}^{j+r-1} |u_{k_r}| \exp - \left[T(X_j) + \frac{1}{4}\sum_{r=1}^{n} v_{j+r}^2 \right] \Big/ n \leqq \sum_{k_r=1}^{j+r-1} |u_{k_r}| \exp - \frac{1}{4n} \sum_{k_r=1}^{j+r-1} u_{k_r}^2$$
$$\leqq C(j+n)^{1/2} n^{1/2} . \tag{2.10}$$

Furthermore

$$\sum_{k_r=1}^{j+r-1} |v_{j+r}| (\exp - \tfrac{1}{4} v_{j+r}^2) \leqq C(j+r-1) . \tag{2.11}$$

Thus the left-hand side of (2.9) is bounded by

$$C^n \left\{ \prod_{r=1}^{n} (j+r-1) + (j+n)^{n/2} n^{n/2} \right\}, \tag{2.12}$$

which easily implies (2.9).

Therefore (2.4) is bounded by

$$z_0^j [\exp - I(\phi_{-t} X_j)] \, (j+n)^n \left(\frac{C z_0}{\lambda} \right)^n \prod_{i=1}^{n} \left(\frac{1}{1+t_i} \right)^3 . \tag{2.13}$$

Ordering the times, we can easily prove the absolute convergence of the series (2.1), obtaining also the estimate:

$$f_{j,t}^d(X) \leqq (z_0 C)^j \exp \{ - I(\phi_{-t} X) \} . \tag{2.14}$$

In conclusion the following statement is proven

Theorem. *Assume* (2.2) *and suitable convergence at time zero* ((2.14) *of* [1]) *and let* $z_0 \lambda^{-1}$ *sufficiently small. Then:*

$$\lim_{d \to 0} f_{j,t}^d = f_{j,t} \quad \text{a.e.}, \tag{2.15}$$

where $f_{j,t}$ *solves uniquely the Boltzmann hierarchy.*

Remarks. This estimate also applies to the Boltzmann hierarchy and essentially is similar to the analysis presented in [1]. Actually the wrong upper bound was not used in proving the convergence of the Boltzmann hierarchy.

The present analysis removes the condition $\beta_0 \geqq 2 \exp(\lambda^{-1})$ which was made in [1] and, what is more relevant, is valid in any dimension.

References

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