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An Instanton-Invariant for 3-Manifolds

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Abstract. To an oriented closed 3-dimensional manifold M with $H_1(M,\mathbb{Z})=0$, we assign a $\mathbb{Z}_8$-graded homology group $I_*(M)$ whose Euler characteristic is twice Casson's invariant. The definition uses a construction on the space of instantons on $M\times\mathbb{R}$.

Contents

1. Instanton Homology

1a) Introduction

Let M be a closed connected oriented 3-manifold. As is well known (see e.g. [He]), every 3-dimensional topological manifold carries a unique differentiable structure, so that we can consider M in either of these two categories. For the sake of brevity, we will refer to M simply as a 3-manifold.

A strong algebraic invariant of M is its fundamental group $\pi_1(M)$. Unfortunately, as a satisfactory description of the set 3-manifolds, the fundamental group falls short in two crucial ways: First, the classification of manifolds with isomorphic fundamental groups depends on the well known and as yet unsettled question raised by Poincaré whether $\pi_1(M)=(1)$ implies that $M=S^3$. The second problem concerns the range of π_1, and here, not even a good question seems to be known yet. Despite the fact that there are several constructions and presentations that are characteristic for fundamental groups of 3-manifolds, no intrinsic characterization has been given.

In defining alternative invariants for 3-manifolds, one has come to study representations of $\pi_1(M)$ in certain nonabelian groups G. In fact, it is one of the characteristic features of $\pi_1(M)$ as a group that the set

$$\mathscr{R}=\mathscr{R}(M)=\mathrm{Hom}(\pi_1(M),G)/\mathrm{ad}(G) \tag{1a.1}$$

of equivalence classes of representations tends to be discrete (in a sense which will become more clear below). While $\mathscr{R}(M)$ itself clearly depends only on the homotopy type of M, the corresponding flat bundles together with the differentiable structure of M may lead to topological invariants, like the Reidemeister torsion (see e.g. [RS]). Recently, Casson [C] defined a new integer valued topological invariant of M in the case where $H_1(M,\mathbb{Z})=0$; i.e. for homology 3-spheres M. It can be described as follows: Consider $G=SU_2$. Given a Heegard splitting $M=M_+\cup_S M_-$ (see e.g. [He]), one can consider $\mathscr{R}(M)$ as the intersection of $\mathscr{R}(M_+)$ and $\mathscr{R}(M_-)$ in $\mathscr{R}(S)$. The resulting intersection number (ignoring the trivial representation) can be shown to be independent of the particular Heegard splitting. The intersection $a\in\mathscr{R}$ is transverse if and only if the twisted cohomology $H^1_a(M,su_2)$ vanishes. Hence if this is the case for all $a\in\mathscr{R}$ (we then say that $\mathscr{R}$ is regular), Casson's construction defines a sign for each $a\in\mathscr{R}$. The reason for the restriction to homology 3-spheres is that if $a\in\mathscr{R}$ is reducible, i.e. if the adjoint operation $pf\ SU_2$ on a in (1a.1) is not free up to the center, then a factors through a homomorphism $\bar{a}:\pi_1(M)\to S^1$. Hence reducible representations correspond to elements of the group $H^1(M,S^1)$, which is trivial for homology 3-spheres. For the same reason, we will also assume from now on that M is a homology 3-sphere. Then the set of irreducible representations of $\pi_1(M)$ is

$$\mathscr{R}^*=\mathscr{R}-(1).$$

We want to define a new invariant for homology 3-spheres, which takes the form of an Abelian group $I_*(M)$ carrying a natural grading by $\mathbb{Z}_8$. Casson's invariant will turn out to be one half of its Euler characteristic. Under the simplifying assumption that $\mathscr{R}$ is regular, it can be described as follows: Following Taubes [T4], we consider the space $\mathscr{M}$ of finite action instantons on the infinite cylinder $\mathbb{R}\times M$ with a product metric $1\times\sigma$. These are gauge equivalence classes of connections on $\mathbb{R}\times M\times SU_2$ which have selfdual curvature tensor F_A and finite Yang Mills action $\mathscr{A}(A)=\|F_A\|_2^2$. The first conditions means that $F_A=*_\sigma F_A$, where $*_\sigma$ is the Hodge duality isomorphism with respect to $1\times\sigma$. The condition of finite action forces each instanton to approach gauge equivalence classes a and b of flat connections at the ends, so that $\mathscr{M}$ decomposes into spaces $\mathscr{M}(a,b)$ of instantons "connecting" $a,b\in\mathscr{R}$. It turns out that if $\mathscr{R}$ is regular, then $\mathscr{M}(a,b)$ is a smooth manifold whenever the L^2-adjoint of the linearized instanton equation has no solutions in L^2. Let us assume that this is the case for all of $\mathscr{M}$. For $a,b\neq 1$, the dimension of $\mathscr{M}(a,b)$ can then be computed modulo 8 as the difference of a suitably defined "relative Morse index"

$$\mu:\mathscr{R}^*\to\mathbb{Z}_8$$

between a and b. Because of the translation invariance of $\mathbb{R}\times M$, the lowest possible dimension of a nontrivial component of $\mathcal{M}$ is 1. We can define an orientation on $\mathcal{M}$ (see Sect. 2d), so that each 1-dimensional component carries a sign defined by comparing its orientation with the orientation of $\mathbb{R}$. In this "nondegenerate case", our invariant is defined as follows:

Theorem 1. *Let M be a homology 3-sphere such that $H^1_a(M, su_2)=0$ for all $a\in\mathcal{R}(M)$. Moreover, let σ be a metric on M so that all finite action instantons are regular. Then for all $a, b\in\mathcal{R}^*$ with $\mu(a)-\mu(b)=1$, $\mathcal{M}(a,b)$ has finitely many one-dimensional components. If $\langle\partial a, b\rangle\in\mathbb{Z}$ is the sum over the signs of these components, and R_p, $p\in\mathbb{Z}_8$, is the free Abelian group over the set $\mathcal{R}_p=\{a\in\mathcal{R}^*\mid\mu(a)=p\}$, then the operators*

$$\partial: R_p\to R_{p-1}; \qquad \partial a=\sum_{\mu(b)=p-1}\langle\partial a, b\rangle b$$

satisfy $\partial\partial=0$. The homology

$$I_*(M)=\ker\partial/\operatorname{im}\partial$$

does not depend on the choice of the metric σ on M.

It is conceivable that the regularity assumption on $\mathcal{M}(a,b)$ is satisfied for "generic" metrics σ on M, as is the case for instantons on 4-manifolds by [FU]. However, because of the translational symmetry, this is by no means immediate. We do not pursue this question here, because the possible degeneracies in the set $\mathcal{R}$ force us to consider perturbations of the curvature equation itself, see Sects. 1b and 2c.

Theorem 1 follows from Theorem 2 in Sect. 1c. The idea behind this construction is the observation that instantons on $\mathbb{R}\times M$ can be considered as trajectories of the "gradient flow" of the Chern-Simons function

$$a(a)=\int_M \operatorname{tr}(\tfrac{1}{2}a\wedge da+\tfrac{1}{3}a\wedge a\wedge a)$$

on $\mathcal{A}(M)$. (In the integrand, the exterior derivatives are taken with respect to the matrix multiplication in su_2 and tr is the trace on complex 2×2-matrices.) This "variational principle," though deeply rooted in the "physical origin" of instantons, has received little attention in connection with topological applications, since it does not extend to general 4-manifolds. It should not be confused with the variational theory of the Yang Mills action $\|F_A\|_2^2$, see [T4], where instantons (i.e. trajectories of our gradient flow) are minima of the action. This double variational structure even occurs in finite dimensional gradient flows; one can easily define a functional on (infinite) paths in the manifold whose minima are precisely the trajectories, and which yields the Yang Mills action when applied to our case. For this and for an explanation of the construction of Theorems 1 and 2 in finite dimensional terms, the reader is referred to the first sections of Witten's paper [W]. The "quantum mechanical" derivation of the Morse inequalities in [W1] would lead in this case to the problem of quantum field theory, which was recently considered by Witten in [W2]. In fact, Witten also discusses in quantum field theoretical terms the relation between $I_*(M)$ and Donaldson's invariants [D4] on 4-manifolds, which was originally pointed out to the author by Atiyah and Donaldson, and which we will briefly describe next.

In studying compact 4-manifolds X with boundary $\partial X = M$, one often considers the space of instantons on the noncompact manifold $\bar{X} := X \cup_M M \times [0, \infty)$, see e.g. [FS, T2, Mc]. If M is a homology 3-sphere, one can define an invariant of (X, M) counting instantons (with respect to an appropriate conformal structure) labeled by the flat connections they approach at the end. The proof that the resulting element $[X]$ of R_* satisfies $\partial[X] = 0$ and that its class in $I_*(M)$ is independent of the conformal structure on X is similar to the proof of Theorem 1. The general construction would result in a polynomial invariant on $H_2(X, \mathbb{Z})$ taking values in $I_*(M)$.

The same point of view can be applied to cobordisms W between homology 3-spheres M and N. Here, a count of instantons would define an homomorphism between $R_*(M)$ and $R_*(N)$ which can be shown to commute with ∂. Its degree depends on the rational homology of W; in particular it is zero for a rational homology cobordism. Moreover, it is natural with respect to compositions of cobordisms, so that $I_*(M)$ might be considered as a functor on the category of homology 3-spheres and their cobordisms. (It is not a functor with respect to maps between 3-manifolds. Of course, diffeomorphisms of M define an automorphism of $I_*(M)$, which may be of independent interest.) The details are given in Sect. 1c.

As a final remark, we would like to point out a possible relation between the construction of Theorem 1 and Casson's construction. Note that for any Heegard splitting $M = M_+ \cup_S M_-$ as above, the representation space $\mathscr{R}(S)$ of the surface carries a symplectic structure $\omega \in \Omega^2(\mathscr{R}(S))$, see Goldman [G]. Moreover, $\mathscr{R}(M_\pm)$ are Lagrangian submanifolds in $\mathscr{R}(S)$, i.e. ω vanishes on $\mathscr{R}(M_\pm)$. Now the Lagrangian intersection theory of [F1] defines a similar (but more restrictive) situation of transversally intersecting Lagrangian submanifolds $L_\pm$ of a symplectic manifold P a chain operator ∂ on the free $\mathbb{Z}_2$-module over the intersection set $L_+ \cap L_-$. (The restriction to $\mathbb{Z}_2$-coefficients is not expected to be necessary here.) The matrix elements of ∂ are in this case given by counting the numbers of holomorphic discs (with respect to some almost complex structure on P) whose boundary lies half in L_+ and half in L_- and which therefore necessarily have "corners" at two intersection points. In fact, there exists a relation between instantons and holomorphic maps, see [At] and [D2], so that a relation between the two constructions is conceivable (an idea which I owe to M. Atiyah). However, at present none of these ideas have been carried out.

1b) Connections on M

Note that for topological reasons, every principal SU_2-bundle over a 3-dimensional manifold is topologically trivial, i.e. admits the product form $P = M \times SU_2$. Given such a trivialization, one can identify the space of connections of Sobolev type L^p_k with the space $\mathscr{A}^p_k = L^p_k(\Omega^1(M) \otimes su_2)$ of 1-forms on M with values in su_2 in such a way that the zero element of Ω^1_{ad} corresponds to the product trivial connection on P. The gauge group of bundle isomorphisms of P can be identified with $\mathscr{G}^p_k(M) = L^p_{k+1}(M, SU_2)$ acting on $\mathscr{A}^p_k(M)$ by the nonlinear transformation law

$$g(a) = gag^{-1} + (dg)g^{-1}. \tag{1b.1}$$

We will always assume that $k+1 > 3/p$ so that $\mathscr{G}(M)$ consists of continuous maps. In fact, we will mostly work with $\mathscr{A}(M) = \mathscr{A}^4_1(M)$ and $\mathscr{G}(M) = \mathscr{G}^4_2(M)$. The quotients $\mathscr{B}_k^p(M)=\mathscr{A}_k^p(M)/\mathscr{G}_k^p(M)$ and $\mathscr{B}(M)=\mathscr{B}_1^4(M)$ can be considered as infinite dimensional manifolds except near those connections a for which the group

$$G_a=\{g\in\mathscr{G}\mid g(a)=a\}$$

is larger than the center $\mathbb{Z}_2=\{\pm\mathrm{id}\}$ of $\mathscr{G}$. Such connections are called reducible. For example, the trivial connection is reducible by all constant maps $g: M\to SU_2$. However, the irreducible connections form an open and dense set $\mathscr{B}^*(M)$ in $\mathscr{B}(M)$. The tangent spaces of $\mathscr{B}^*(M)$ can be identified with

$$T_{[a]}\mathscr{B}_k^p=\{\alpha\in L_k^p(\Omega^1(M)\otimes su_2)\mid d_a^*\alpha=0\}. \tag{1b.2}$$

If no confusion can arise, we will identify a with its gauge equivalence class in our notation. In (1b.2), d_a^* is the L^2-adjoint (with respect to some metric σ on M) of the exterior derivative $d: \Omega^0(M)\to\Omega^1(M)$ extended to Ω_{ad}^* by means of the connection a.

The curvature of a connection is defined as the su_2-valued 2-form

$$(F_a)_{ij}=\frac{\partial a_i}{\partial x^j}-\frac{\partial a_j}{\partial x^i}+[a_i,a_j].$$

It is gauge invariant in the sense that

$$F_{g(a)}=gF_ag^{-1}. \tag{1b.3}$$

It follows that the set of flat connections, i.e. the set of all a for which $F_a=0$, is invariant under $\mathscr{G}$. Moreover, it is well known that we have an identification

$$\mathscr{R}=\{[a]\in\mathscr{B}\mid F_a=0\}.$$

In fact, for a flat connection, the holonomy along a loop depends only on the homotopy class of the loop and defines an element of $\mathscr{R}$. Conversely, consider the universal covering $\tilde{M}$ of M. Then if we extend the operation of $\pi_1(M)$ on $\tilde{M}$ to $\tilde{M}\times\mathbb{C}^2$ by means of a representation $\pi_1(M)\to SU_2$, the quotient $(\tilde{M}\times\mathbb{C}^2)/\pi_1(M)$ is a bundle over M which inherits a flat connection from the trivial connection on $\tilde{M}\times\mathbb{C}^2$. Note that we can convert the assignment $a\mapsto F_a$ into a vector field on the infinite dimensional manifold $\mathscr{B}^*(M)$: Given a metric σ on M, one defines the Hodge duality isomorphism $*_\sigma: \Omega^p(M)\to\Omega^{3-p}(M)$. Extending it trivially to the su_2-factor, we define

$$\not{f}_\sigma(a)=*F_a\in\Omega^1(M)\otimes su_2.$$

Since $\not{f}_\sigma$ is gauge invariant and since it follows from the Bianchi identity $d_aF_a=0$ that

$$d^*\not{f}_\sigma(a)=*d_a*(*F_a)=*d_aF_a=0, \tag{1b.4}$$

we have $\not{f}_\sigma(a)\in T_{[a]}\mathscr{B}_{k-1}^p(M)$. Over $\mathscr{B}=\mathscr{B}_1^4(M)$, it is a section of the bundle $\mathscr{L}$ with fiber

$$\mathscr{L}_a=L(a)=T_{[a]}\mathscr{B}_0^4(M).$$

It was observed by Taubes [T3] that Casson's invariant can be interpreted as the "Euler characteristic" of $\not{f}_\sigma$ in the following way. The linearization $D\not{f}_\sigma(a)=*_\sigma d_a: T_{[a]}\mathscr{B}\to L_a$ can be combined with a gauge condition in (1b.2) to an elliptic operator on the bundle $\Omega_{\mathrm{ad}}=(\Omega^0\oplus\Omega^1)\otimes su_2$ defined by

$$\mathscr{D}_a(\phi,\alpha)=(d_a^*\alpha, d_a\phi+*d_a\alpha). \tag{1b.5}$$

In particular, the kernels of $\mathscr{D}_a$ and $D\not{f}_\sigma(a)$ are isomorphic for any irreducible connection a, so that $[a]\in\mathscr{B}^*$ is a nondegenerate zero of $\not{f}_\sigma$ if and only if the kernel of $\mathscr{D}_a$ is trivial. If this is the case for all $a\in\mathscr{R}^*$, then one can assign a relative sign to any pair $a, b\in\mathscr{R}^*$ as the mod-2-reduction of the spectral flow (see Sect. 2c) of any continuous family $c(\tau)$ in $\mathscr{A}(M)$ connecting a and b. Up to an ambiguity in the overall sign (which can be fixed using the trivial connection), this is the analog of the "degree" of a nondegenerate zero of a vector field. For each nondegenerate $a\in\mathscr{R}^*$, it yields a sign which turns out to be the same as the one obtained by Casson's construction. Of course, Taubes' construction can also be extended to general manifolds by a perturbation.

To understand and generalize the homology groups $I_*(M)$ of Theorem 1, we have to take into account an additional structure of the vector field $\not{f}_\sigma$. Under the L^2-inner product on $T\mathscr{B}(M)$ defined by the metric σ on M, it induces a family of linear forms α_a on $T_a\mathscr{A}=L_1^4(\Omega^1(M))$ by

$$\alpha_a(\xi)=\langle\not{f}_\sigma(A),\xi\rangle_\sigma=\int F_a\wedge\xi\,, \tag{1b.6}$$

where $\wedge$ denotes the exterior derivative extended by the standard positive definite inner product on su_2. Note that (1b.6) does not depend on the metric σ. Moreover, since the linearization of $\not{f}_\sigma$ is selfadjoint, it is closed and can be integrated to a function $a:\mathscr{A}(M)\to\mathbb{R}$. This function can be obtained by integrating the Chern-Simons form $cs_2\in\Omega^3(M)$ over M, see Sect. 1b or [CS]. With respect to gauge transformations $g: M\to SU_2$, it satisfies

$$a(g(a))=a(a)+2\pi\cdot\deg(g)\,,$$

where $\deg(g)$ is the degree of g as a map between 3-dimensional closed manifolds. Hence a is well defined on

$$\tilde{\mathscr{B}}(M)=\mathscr{A}(M)/\{g\in\mathscr{G}\mid\deg(g)=0\}\,, \tag{1b.7}$$

which is the universal covering of $\mathscr{B}(M)$. We will say that $\not{f}_\sigma$ is the "σ-gradient field" for a function on $\tilde{\mathscr{B}}(M)$. The crucial observation now is that trajectories of the vector field $\not{f}_\sigma$, i.e. of the σ-gradient flow of a, can be identified with instantons on $\mathbb{R}\times M$. In fact, the instanton equation for a connection $A\in\Omega^1_{\mathrm{ad}}(\mathbb{R}\times M)$ is given by $F_A=*F_A$, where $*:\Omega^2(\mathbb{R}\times M)\to\Omega^2(\mathbb{R}\times M)$ is the Hodge duality isomorphism with respect to the product metric on $\mathbb{R}\times M$. By applying a gauge transformation $g:\mathbb{R}\times M\to SU_2$ if necessary, we can always assume that the component of A in the $\mathbb{R}$-direction vanishes. In this case, the self-duality equation becomes

$$0=\frac{\partial A(\tau)}{\partial\tau}+*_\sigma F_{A(\tau)}(x)=\frac{\partial A(\tau)}{\partial\tau}+\not{f}_\sigma(A(\tau)) \tag{1b.8}$$

for $A(\tau)=i_\tau^*A\in\mathscr{A}(M)$. We can therefore consider the moduli space $\mathscr{M}_\sigma(a,b)$ of Theorem 1 as the space of trajectories of the vector field $\not{f}_\sigma$ connecting the two zeros a and b of $\not{f}_\sigma$. The hypothesis of Theorem 1 now means that this flow is of "Morse-Smale-type", i.e. that all zeros of $\not{f}_\sigma$ are nondegenerate and that their stable and unstable manifolds intersect transversally in smooth finite dimensional manifolds. It was proved in [S2] that infinite dimensions, this is a "generic property" for gradient flows. Moreover, it is known that in finite dimensions, the homology groups defined as in Theorem 1 generally do not depend on the choice of the gradient field. For example, in a compact manifold B, they coincide with $H_*(B,\mathbb{Z})$, see [Wi] and also [S1]. Hence in order to define the index homology of Theorem 1 for arbitrary homology 3-spheres, we define an appropriate set of perturbations of a (compare [D3] for a similar construction): Consider a map

$$\gamma: S(m)\times D^2 := \bigvee_{i=1}^{m} S_i^1 \times D^2 \to M,$$

which restricts to smooth embeddings $\gamma_\theta: S(m)\to M$ for each $\theta\in D^2$ and $\gamma_i: S_i^1\times D^2\to M$ for each i. Here, we consider on $S(m)$ the smooth structure of the union of smooth embedded circles in $\mathbb{R}^3$ which intersects precisely at the origin and have the same tangent there. It defines a family of holonomy maps

$$\gamma_\theta: \mathscr{B}(M)\to L_m := SU_2^m/\mathrm{ad}(SU_2).$$

Now choose a smooth compactly supported volume form $d^2\theta$ on the interior of D^2. Let $C^\infty(L_m,\mathbb{R})$ be defined as the set of smooth $\mathrm{ad}(SU_2)$-invariant real functions on SU_2^m. Then for each $h\in C^\infty(L_m,\mathbb{R})$, we can define the function

$$h_\gamma: \mathscr{B}(M)\to\mathbb{R}; \qquad h_\gamma(a) = \int_{D^2} h(\gamma_\theta(a))d^2\theta. \tag{1b.9}$$

Let us denote by Γ_m the set of maps γ as described above and by Π the entire set of perturbations

$$\Pi = \bigcup_{m\in\mathbb{N}} \Gamma_m \times C^\infty(L_m,\mathbb{R}).$$

Lemma 1b.1. *For any $\pi=(\gamma,h)\in\Pi$, h_γ is a smooth function on $\mathscr{B}(M)$. Moreover, for every smooth metric σ on M, there exists a smooth section $\mathrm{grad}_\sigma h_\gamma$ of $T\mathscr{B}(M)$ such that for every $\xi\in T_a\mathscr{B}(M)$,*

$$\langle \mathrm{grad}_\sigma h_\gamma(a),\xi\rangle = Dh_\gamma(a)\xi.$$

Proof. Parametrize S^1 by the unit interval $I=[0,1]$, with 0 and 1 corresponding to the base point in S^1. Then γ gives rise to maps $\gamma_i: [0,1]\times D^2\to M$, which are independent of i on the discs $\{0\}\times D^2$. Fix a trivialization Φ_0 of P over the image of this disc. Then for each $a\in\mathscr{B}(M)$, it can be extended in a unique way to trivializations $\Phi_i(a): \gamma_i^*\,\mathrm{ad}(P)\to[0,1]\times D^2\times su_2$ defined by parallel transport along $\gamma_i(\cdot,\theta)$ for $\theta\in D^2$. Here, we make use of the fact that $a\in C^0(\Omega^1\otimes su_2)$. Under this condition, we have a smooth family

$$\Phi_i: \mathscr{A}(M)\to \mathrm{Hom}(\gamma_i^*\,\mathrm{ad}(P), [0,1]\times D^2\times su_2) \tag{1b.10}$$

of continuous bundle isomorphisms. Since the holonomy $\gamma_\theta(a)$ is defined as the conjugacy class of the m-tuple,

$$(\gamma_{\theta i}(a))_{1\leq i\leq m} = (\Phi_i(a)(1,\theta)\Phi_i^{-1}(a)(0,\theta))_{1\leq i\leq m}\in SU_2^m,$$

this proves that γ_θ and hence h_γ is smooth. Its derivative is

$$Dh_\gamma(a) = \sum_{i=1}^{m} D_i h_\gamma(a).$$

where in terms of the parametrization γ_i and the trivialization Φ_i,

$$D_i h_\gamma(a)(\phi,\theta) = \nabla_i h(\gamma_\theta(a))d^2\theta.$$

Here, $\nabla_i h$ is the partial derivative of the lifting of h to SU_2^m in the direction of the i^{th} factor, identified with an element of su_2 by virtue of the canonical bilinear form on su_2. The σ-gradient of h_γ is now the sum of sections of $T\mathscr{B}$ which in terms of γ_i and Φ_i are given by

$$\operatorname{grad}_{i,\sigma}(\theta, \phi) = \nabla_i h(\gamma_\theta(a)) *_\sigma d^2\theta . \tag{1b.11}$$

One checks directly that $d_a^* \operatorname{grad} h_\gamma(a) = 0$ and that

$$\operatorname{grad} h_\gamma(a) \in L_1^4(\Omega^1(M) \otimes su_2),$$

so that it in fact defines a vector field on $\mathscr{B}(M)$. Smoothness follows from (1b.11) and from smoothness of Φ_i in (1b.10). □

1c) The General Construction

Given $\pi \in \Pi$, we define $a_\pi = a + h_\gamma$. We denote by $\mathscr{R}_\pi$ its critical set, i.e. the zero set of the section

$$f_{\sigma\pi}(a) = *_\sigma F_a + \pi(a) = f_\sigma(a) + \operatorname{grad} h_\gamma(a) \tag{1c.1}$$

of $\mathscr{L}$. It is nondegenerate if and only if the kernel of the "Hessian"

$$\begin{gathered} D_{\sigma\pi}(a): L_1^p(a) \to L^p(a), \\ D_{\sigma\pi}(a)(\phi, \alpha) = (d_a^* \alpha, d_a \phi + Df_{\sigma\pi}(a)\alpha) \end{gathered} \tag{1c.2}$$

consists only of elements of the form $(\phi, 0)$ with $d_a\phi = 0$, i.e. corresponds to the Lie algebra of the group G_a of gauge symmetries of a. We now define

$$\begin{aligned} \mathscr{M}_{\sigma\pi} = \Big\{ A \in L_1^p(p^*\Omega^1(M) \otimes su_2) \,\Big|\, & \frac{\partial A}{\partial \tau} + f_{\sigma\pi}(A(\tau)) \\ & = 0 \text{ and } l(A) < \infty \Big\} / \mathscr{G}(M), \end{aligned} \tag{1c.3}$$

where

$$l(A) = \left\| \frac{\partial A}{\partial \tau} \right\|_2^2 .$$

We omit the subscripts σ and π whenever possible. We may think of $\mathscr{M}$ as the "Morse complex" of bounded trajectories of the gradient flow, but also as the perturbed "moduli space" of instantons. In fact, if we extended $l(A)$ and

$$F_{\sigma\pi}(A) = \frac{\partial A}{\partial \tau} + f_{\sigma\pi}(A)$$

gauge equivariantly to $\mathscr{A}(\mathbb{R} \times M)$, then we have

$$\mathscr{M}_{\sigma\pi} = \{ A \in \mathscr{A}(\mathbb{R} \times M) \mid F_{\sigma\pi}(A) = 0 \text{ and } l(A) < \infty \} / \mathscr{G}(\mathbb{R} \times M) .$$

We will use either of the two terms for elements of $\mathscr{M}$, depending on which aspect is more important at a particular time. Note that if for $A \in \mathscr{M}$ with $\pi = 0$, $l(A)$ is proportional to the Yang-Mills action. For a general $[A] \in \mathscr{M}_{\sigma\pi}(M)$, it is equal to the difference of the perturbed Chern-Simons function along the path in $\mathscr{B}(M)$ defined by A. In fact,

$$\frac{d}{d\tau} (a_\pi(A(\tau))) = Da_\pi(A(\tau)) \frac{\partial A(\tau)}{\partial \tau} = \left\langle f_{\sigma\pi}, \frac{\partial A(\tau)}{\partial \tau} \right\rangle_\sigma = - \left\| \frac{\partial A(\tau)}{\partial \tau} \right\|_2^2 . \tag{1c.4}$$

Hence a_π decreases along trajectories in $\mathcal{M}_{\sigma\pi}$ and the trajectories of finite length l are precisely those along which a_π is bounded.

The linearization of the flow equation at $A \in \mathcal{M}_{\sigma\pi}$ is equivalent to the elliptic operator

$$D_A\alpha = \frac{\partial\alpha(\tau)}{\partial\tau} + D_{\sigma\pi}(A(\tau))\alpha \tag{1c.5}$$

on $\Omega^1(\mathbb{R}\times M) = p^*(\Omega^0(M)\oplus\Omega^1(M))$. We call A regular if the adjoint

$$D_A^+\alpha = -\frac{\partial\alpha(\tau)}{\partial\tau} + D_{\sigma\pi}^+(A(\tau))\alpha$$

has no bounded solutions. We call $\mathcal{M}_{\sigma\pi}$ regular if all $A \in \mathcal{M}_{\sigma\pi}$ are regular. In this case, it follows from the implicit function theorem that $\mathcal{M}_{\sigma\pi}$ is a smooth manifold. The following proposition summarizes the principal properties of $\mathcal{M}_{\sigma\pi}$, which we prove in Sect. 2 and 3.

Proposition 1c.1. *For a dense set of parameters $(\sigma,\pi)\in\Sigma\times\Pi$, $\mathcal{R}$ is nondegenerate and $\mathcal{M}$ decomposes into smooth oriented manifolds $\mathcal{M}(a,b)$ of regular trajectories connecting $a,b\in\mathcal{R}$. To each $a\in\mathcal{R}$ we can associate a Morse index $\mu(a)\in\mathbb{Z}_8$ so that*

$$\dim\mathcal{M}(a,b) = \mu(a) - \mu(b) - \dim G_b \quad (\operatorname{mod} 8)\,. \tag{1c.6}$$

Moreover, let $\hat{\mathcal{M}}(a,b)$ denote the (oriented) quotient by the translational symmetry. Then if $b_0,\ldots,b_n$ are irreducible, we have orientation preserving local diffeomorphisms

$$\#: \hat{\mathcal{M}}(b_0,b_1)\times\mathbb{R}_+\times\hat{\mathcal{M}}(b_1,b_2)\times\ldots\times\mathbb{R}_+\times\hat{\mathcal{M}}(b_{n-1},b_n)\supset\mathcal{O}\to\hat{\mathcal{M}}(b_0,b_n)$$

with the following properties:

(T1) *For each compact $K\subset\hat{\mathcal{M}}(b_0,b_1)\times\ldots\times\hat{\mathcal{M}}(b_{n-1},b_n)$, there exists $\varrho(K)\in\mathcal{R}_+$ such that $K\times[\varrho(k),\infty)^{n-1}\subset\mathcal{O}$.*

(T2) *For $1\leqq i\leqq n$ there exists a lifting $\#_i$ of $\#$ to $\mathcal{M}(b_0,b_n)$ such that $\#_i(A,\varrho)\to A_i$ locally with ϱ_i, $\varrho_{i+1}\to\infty$.*

(T3) *If $\operatorname{din}|(\mathcal{M}(a,b))|\leqq 4$, then the complement of all maps $\#$ for all possible configurations $a=b_0$, $b_1,\ldots,b_n=b$ is compact.*

The existence of the map $\#$ is well known in finite dimensions as "transitivity" of Morse-Smale flows: Trajectories ending at and originating from a nondegenerate fixed point x can be "connected" to yield a family of unbroken flow trajectories converging "in the image" to the components. In a compact finite dimensional manifold, this construction would cover all ends of the trajectory spaces. The dimensional restriction in (T3) is due to the singular critical point $0\in\mathcal{R}(M)$: It follows from (1c.6) that if $\hat{\mathcal{M}}(a,0)$ and $\hat{\mathcal{M}}(0,b)$ are both discrete, then $\mu(a)-\mu(b) = 5 \operatorname{mod} 8$. In fact, one can show that the appropriate gluing map in this case would be

$$\hat{\mathcal{M}}(a,0)\times SO_3\times\mathcal{R}_+\times\hat{\mathcal{M}}(0,b)\to\hat{\mathcal{M}}(a,b)\,, \tag{1c.8}$$

i.e. we would have an additional 3-dimensional gluing parameter. This is a problem which one also encounters in finite dimensional equivariant Morse theory. However, even if we took the map (1c.7) into account, we would still have a restriction in (T3): For $\dim\mathcal{M}(a,b)\geqq 8$, we may have sequences of instantons diverging by splitting off a "point-instanton" anywhere on $\mathbb{R}\times M$. In fact, Taubes' gluing procedure (see [T1]) allows us to "glue" instantons into $\mathcal{M}(a,b)$ at arbitrary points on $\mathbb{R}\times M$ and with arbitrary scaling and "gauge orientation," obtaining a local diffeomorphism

$$\mathcal{M}(a,b)\times SO_3\times\mathbb{R}_+\times\mathbb{R}\times M\to\mathcal{M}(a,b). \tag{1c.8}$$

Hence this phenomenon is closely related to the mod-8 ambiguity of $\dim\mathcal{M}(a,b)$, and has no analogon in finite dimensional Morse theory. In this paper, we will be concerned only with 1- and 2-dimensional trajectory spaces, so that neither (1c.7) nor (1c.8) will enter the discussion. In fact, let us first consider the 1-dimensional part of $\mathcal{M}$, i.e. the zero dimensional part of $\hat{\mathcal{M}}$, whose elements we call "isolated trajectories."

Theorem 2. *For (σ,π) as in Proposition 1c.1 the number of isolated instantons is finite. Moreover, let $o(A)=\pm1$ denote the orientation of an isolated instanton A and let $\langle\partial a,b\rangle$ denote the sum of $o(A)$ over all isolated trajectories in $\hat{\mathcal{M}}(x,y)$. For $p\in\mathbb{Z}_8$, let R_p denote the free Abelian group over the set $\mathcal{R}_p=\{a\in\mathcal{R}^*\mid\mu(a)=p\}$. Then the homomorphism*

$$\partial:R_p\to R_{p-1};\qquad \partial x=\sum_{y\in R}\langle\partial x,y\rangle y$$

satisfies $\partial\partial=0$. The group

$$I_p(M;\sigma,\pi)=\ker\partial_p/\operatorname{im}\partial_{p+1} \tag{1c.9}$$

does not depend on the choice of σ and π. In fact, for any two parameter sets (σ_i,π_i), $i=0,1$, there exists a canonical isomorphism

$$h:I^*(M;\sigma_0,\pi_0)\to I^*(M;\sigma_1,\pi_1).$$

Proof of Theorem 2. Part 1. It follows from Proposition 1c.1 that the zero dimensional part of $\hat{\mathcal{M}}$ is compact and hence finite. To prove the second assertion, note that the matrix elements

$$\langle\partial\partial a,c\rangle=\sum_{b\in\mathcal{R}^*}\langle\partial a,b\rangle\langle\partial b,c\rangle$$

correspond to the sum of the product signs $o^2(A,B)=o(A)o(B)$ over the zero dimensional part of

$$\hat{\mathcal{M}}^2(a,c):=\bigcup_{b\in\mathcal{R}^*}\hat{\mathcal{M}}(a,b)\times\hat{\mathcal{M}}(b,c).$$

Now note that by (T3) of Proposition 1c.1, the ends of the 1-dimensional part of $\hat{\mathcal{M}}(a,c)$ are in oriented $1-1$ correspondence with the zero dimensional part of $(\hat{\mathcal{M}}^2(a,c),o^2)$. This proves that $\partial\partial=0$. □

The proof of the invariance of I_* is related to the functorial properties of I_* mentioned in the Introduction. We will consider a cobordism $W:M\to N$ between closed oriented 3-manifolds as an oriented smooth 4-manifold W with open submanifolds $M\times\mathbb{R}_-$ and $N\times\mathbb{R}_+$ such that $W_0:=W-M\times\mathbb{R}_+-N\times\mathbb{R}_-$ is a compact smooth manifold with boundary $N\cup M$. Here, $\mathbb{R}_\pm\subset\mathbb{R}$ are oriented in the usual way. On W, we consider the set Σ_W of conformal structures σ extending the product conformal structures on the ends. Recall that a conformal structure defines (and is defined by) a 3-dimensional subbundle $\Omega_\sigma^+ \subset \Omega^2(W)$ and a canonical completement Ω_σ^-. We also define a class Π_W of perturbations of the selfduality equation $(F_A)_\pm = 0$ which extend the translationally invariant Hamiltonian perturbations on the ends and which are the following form: Let B be an open 3-dimensional manifold and consider maps

$$\gamma_W : \Gamma_m \times B \to W_0$$

restricting to embeddings on $S_i^1 \times B$ in complete analogy with the definition of the maps γ in Sect. 1b. On B, consider a compactly supported 2-form ω. Then for any smooth vector field k on L_m (i.e. for any smooth and invariant vector field on SU_2^m), we can define the adjoint-valued 2-form $k_\gamma \in \Omega^2_{\mathrm{ad}}(W)$ by superposition of forms $(k_\gamma)_i$ defined by

$$(k_\gamma)_i(\gamma_i(\phi, \beta)) = \omega k_i(\gamma(A))$$

on $\gamma(S_i^1 \times B)$. Here, k_i is the i^{th} component of k as a vector field on SU_2^m. This defines a set Π_W of $\mathscr{G}(W)$-equivariant maps

$$\pi : \mathscr{A}(W) \to L_1^p(\Omega^2_{\mathrm{ad}}(W)).$$

Note that $\pi(A) \in \Omega^2_{\mathrm{ad}}$ is neither selfdual nor antiselfdual in general, since $\frac{\partial}{\partial \phi} \lrcorner\, \pi(A) = 0$ whenever γ is injective. We now define a section of $\mathscr{L}(\mathscr{B}(W), a, a')$ as the quotient by $\mathscr{G}(W)$ of the function

$$F_{\sigma\pi} : \mathscr{A}(W) \to L^p(\Omega^\pm_{\mathrm{ad}}(W, \sigma)), \qquad F^\pm_{\sigma\pi}(A) = (F_A + \pi(A)).$$

By [D3], a homology orientation of a closed oriented manifold W is an orientation of the real line

$$\Lambda(W) = \det(H^1(W, \mathbb{R})) \times \det(H^2(W, \mathbb{R}) + H^2_-(W, \mathbb{R}).$$

Here, H^2_- is the negative part with respect to the intersection form and det denotes the highest nontrivial exterior power. This definition still makes sense for a manifold whose boundary consists of homology 3-spheres.

Proposition 1c.2. *For a dense set of parameters $(\sigma, \pi) \in \Sigma_W \times \Pi_W$ extending a given set of regular parameters on M and N, the zero set of $F_{\sigma\pi}$ on $\mathscr{B}(W, a, a')$ is regular for all $a, a' \in R(M) \times R(N)$ with*

$$\dim \mathscr{M}(W, a, a') = \mu(a) - \mu(a') + 3(\beta_1(W) - \beta_2^-(W)).$$

Here, $\beta_i = \dim(H^i(W, \mathbb{R}))$ and β_2^- is the dimension of the negative definite part $H^2_-(W, \mathbb{R})$ of $H^2(W, \mathbb{R})$ with respect to the cup product and the orientation of W. Moreover, given a homology orientation of W and orientations of $\mathscr{M}(N)$ and $\mathscr{M}(N)$, there exists a natural orientation on $\mathscr{M}(W)$ with orientation preserving local diffeomorphisms

$$\hat{\mathscr{M}}(a, b) \times \mathbb{R}_- \times \mathscr{M}(b, b') \to \mathscr{M}(a, b'),$$

$$\mathscr{M}(a, a') \times \mathbb{R}_+ \times \mathscr{M}(a', b') \to \mathscr{M}(a, b').$$

As an immediate consequence of Proposition 1c.2, the zero dimensional part of $\mathcal{M}(W)$ is compact again, and hence finite for "regular" (σ, π). Hence it defines a homomorphism

$$\gamma = \gamma(W, \sigma, \pi): R_*(M) \to R_*(N). \tag{1c.10}$$

Theorem 3. *For each (W, σ, π) as described above, γ is a chain map. The induced map*

$$W_* := \gamma_* : I_*(M) \to I_*(N)$$

depends only on the smooth cobordism $W: M \to \mathbb{N}$. It is the identity for the product cobordism. For a composite cobordism $W = UV$, we have $W_* = U_* V_*$.

Proof. We first prove Theorem 3 for the groups $I_*(M, \sigma, \pi)$ of (1c.9), without assuming invariance. That γ is a chain homomorphism is proved in a similarway the chain property of ∂ itself. For $a \in \mathcal{R}(M, \pi_M)$ and $b' \in \mathcal{R}(N, \pi_N)$, the matrix element $\langle (\partial_\gamma - \gamma\partial)a, b' \rangle$ is oriented cardinality of the set

$$\mathcal{M}^2(a, a') = \bigcup_{b \in \mathcal{R}^*(M)} (-\hat{\mathcal{M}}(a, b)) \times \mathcal{M}(b, b') \cup \bigcup_{a' \in \mathcal{R}(N)} \mathcal{M}(a, a') \times \hat{\mathcal{M}}(a', b').$$

This proves that $\partial\gamma - \gamma\partial = 0$, i.e. we associate to a regular (W, σ, π) a chain map $R_*(M, \pi_M) \to (R_*(N, \pi_N)$. Now consider two regular cobordisms $(u, \sigma_U, \pi_U): M \to N$ and $(V, \sigma_V, \pi_V): N \to L$ so that (σ_U, π_U) and (σ_V, π_V) coincide on N. Then for each compact set in $K \subset \mathcal{M}(a, b; U) \times \mathcal{M}(b, c; V)$ there exists $\varrho_K \in \mathbb{R}_+$ and for all $\varrho > \varrho_K$ a local diffeomorphism

$$\mathcal{M}(a, b; U) \times \mathcal{M}(b, c; V) \supset K \to \mathcal{M}(a, c; U \#_\varrho V),$$

constructed by the same method as the maps in Propositions 1c.1 and 1c.2. It follows that γ is functorial with respect to composition of cobordisms in the sense that for ϱ large enough, $(U \#_\varrho V, \sigma_\varrho, \pi_\varrho)$ is regular if (U, σ_U, π_U) and (V, σ_V, π_V) are, and that

$$\gamma(U \#_\varrho V, \sigma_\varrho, \pi_\varrho) = \gamma(U, \sigma_U, \pi_U) \circ \gamma(V, \sigma_V, \pi_V).$$

Now let $(\bar{\sigma}, \bar{\pi}) = (\sigma_\lambda, \gamma_\lambda, h_\lambda)_{\sigma \leq \lambda \leq 1}$ be a smooth family of parameters on W which are constant in λ outside W_0, and which are generic for $\lambda = 0, 1$ in the sense of (1). Then applying an arbitrarily small perturbation $\bar{\sigma}\bar{\pi}$ if necessary, the sets

$$\bar{\mathcal{M}}(a, a') := \{(u, \lambda) \mid u \in \mathcal{M}_{\sigma_\lambda \pi_\lambda}(a, a')\} \subset \mathcal{B}(W; a, a') \times [0, 1]$$

are regular zero sets of $F^-_{\bar{\sigma}\bar{\pi}}(A, \lambda) = F^-_{\sigma_\lambda \pi_\lambda}(A)$ and are therefore smooth manifolds of dimension

$$\dim \bar{\mathcal{M}}(a, b) = \mu(a) - \mu(b) + 1 \qquad (\mathrm{mod}\, 8).$$

It follows now from the obvious parametrized version of Proposition 1c.2 that the zero dimensional part of $\bar{\mathcal{M}}$ defines a $\mathbb{Z}$-module homomorphism

$$\bar{\gamma} := \gamma(W, \bar{\sigma}, \bar{\pi}): R_*(M, \pi_M) \to R_*(N, \pi_N)$$

with $\deg(\bar{\gamma}) = \deg(\gamma_i) + 1$ such that

$$\gamma_0 - \gamma_1 = \partial_M \bar{\gamma} - \bar{\gamma}\partial_N. \tag{1c.11}$$

This proves that $(\gamma_0)_* = (\gamma_1)_* : \mathrm{I}_*(M, \sigma_M, \pi_M) \to I_*(N, \sigma_N, \pi_N)$. □

Proof of Theorem 2. Part 2. Consider $W = M \times \mathbb{R}$ with $\pi(A)(\tau) = \operatorname{grad} h_\gamma(A(\tau); \tau)$. Then (W, π) is a product for τ outside $[0,1]$, and hence defines a chain homomorphism $\gamma(W, \pi): R_*(M, \pi_0) \to R_*(M, \pi_1)$. Reversing the τ-direction in $\bar{W}$, we obtain a chain homomorphism $\bar{\gamma} = \gamma(\bar{W}, \pi_+): R_*(M, \pi_1) \to R_*(M, \pi_0)$. Now the compositions $\bar{W}W$ and $W\bar{W}$ can be deformed (with fixed ends) to the products $(M, \pi_0) \times \mathbb{R}$ and $(M, \pi_1) \times \mathbb{R}$, respectively. Since the product induces the identity on R_*, this proves that γ_* and $\bar{\gamma}_*$ are mutually inverse in homology. □

2. Local Properties of $\mathcal{M}(\mathbb{R} \times \mathcal{M})$

2a) Connections on $\mathbb{R} \times M$

For $\delta > 0$, let $e_\delta : \mathbb{R} \times M \to \mathbb{R}$ be a smooth positive function with $e_\delta(\tau, x) = e^{\delta|\tau|}$ for $|\tau| \geqq 1$. Let E be a bundle over $\mathbb{R} \times M$ with a translationally invariant metric $|\ |$ and metric preserving connection ∇. Following [K] and [LM], we define the norms

$$\|\xi\|_{k,p;\delta} = \|e_\delta \cdot \xi\|_{k,p}$$

on sections ξ of E. Here, $\|\ \|_{k,p}$ is the usual Sobolev norm. To define suitable Banach manifolds $\mathcal{B}(a, b)$ of paths connecting a and b in $\mathcal{B}(M)$, choose any smooth representation of $a, b \in \mathcal{A}(M)$ and a connection A on $\mathbb{R} \times M$ whose first component vanishes and coincides with a for $\tau \leqq -1$ and with b for $\tau \geqq 1$. Then the set

$$\mathcal{A}_\delta(a, b) = A + L^4_{1;\delta}(\Omega^1_{\mathrm{ad}}(\mathbb{R} \times M)) \tag{2a.1}$$

obviously does not depend on the choice of A. By virtue of (1b.1), it is acted upon by

$$\begin{aligned} \mathcal{G}_\delta = \{ g \in L^4_{2,\mathrm{loc}}(\mathbb{R} \times M, SU_2) \mid & \text{ there exists } R > 0 \\ & \text{and } \xi \in L^4_{2;\delta}(\Omega^0_{\mathrm{ad}}(\mathbb{R} \times \Sigma)) \text{ so that } g = \exp \xi \text{ for } |\tau| \geqq R \}. \end{aligned} \tag{2a.2}$$

Proposition 2a.1. *For $\delta \geqq 0$, $\mathcal{G}$ is a Banach Lie group with Lie algebra*

$$\mathcal{G} = L^4_{2,\delta}(\Omega^0_{\mathrm{ad}}(\mathbb{R} \times M)).$$

It acts smoothly on $\mathcal{A}_\delta(a, b)$. The quotient space $\mathcal{B}_\delta(a, b) = \mathcal{A}_\delta(a, b)/\mathcal{G}_\delta$ is a smooth Banach manifold with tangent spaces

$$T_{[A]}\mathcal{B}(a, b) = \{\alpha \in L^4_{1;\delta}(\Omega^1_{\mathrm{ad}}(\mathbb{R} \times M) \mid e_\delta d_A^* e_\delta^{-1} \alpha = 0\}$$

and charts

$$T_{[A]}\mathcal{B}(a, b) \to \mathcal{B}(a, b); \qquad \alpha \mapsto [A + \alpha].$$

Proof. For $\delta > 0$, this is essentially Lemma 7.3 of [T2]. The extension to $\delta = 0$ in the case where a and b are irreducible is possible because the central ingredient in the proof is the fact that

$$(e_\delta d_A^* e_\delta^{-1}) d_A : L^p_{2,\delta}(\Omega^0_{\mathrm{ad}}(\mathbb{R} \times \Sigma)) \to L^p_{0,\delta}(\Omega^0_{\mathrm{ad}}(\mathbb{R} \times \Sigma))$$

is an isomorphism. If we set $\delta = 0$, then it follows from Theorem 1.3 of [LM] that $d_A^* d_A$ is Fredholm if and only if A is irreducible. In this case, an integration by parts proves that it is also positive, i.e. an isomorphism. □

Consider now the linear operation $g(\xi)=g\xi g^{-1}$ for a section $\xi\in L^q_{l;\delta'}(E_{ad})$, where E is some metric bundle over $\mathbb{R}\times M$ and $q\geqq 1$, $l\geqq 0$, and δ' are real numbers. The operation is continuous if and only if $l\leqq \min(1,4/q)$ and $\delta'\leqq\delta$. For such l, q, and δ', it follows with the methods of Proposition 2a.1 that the quotient

$$\mathscr{L}^q_{l;\delta'}(E)=(\mathscr{A}_\delta\times L^q_{l;\delta'})/\mathscr{G}_\delta\,, \tag{2a.3}$$

with respect to the diagonal operation is a smooth Banach space bundle with trivializations

$$T_{[A]}\mathscr{B}\times L^q_{l;\delta'}(E_{ad})\to\mathscr{L}^q_{l;\delta'}(E);\qquad(\alpha,\beta)\mapsto[(\alpha,\beta)]\,.$$

Here $[(\alpha,\beta)]$ denotes the orbit in (2a.3). A (smooth) section s of $\mathscr{L}^q_{l;\delta'}(E)$ is given by any (smooth) G_δ-invariant function from $\mathscr{A}_\delta$ to $L^q_{l;\delta'}(E)$. As its "derivative," we then define the ordinary derivative $Ds(A)$ restricted to $\ker d^\delta_A$. We will be mainly concerned with $\mathscr{L}_\delta=\mathscr{L}^4_{0,\delta}(\Omega^-_\sigma)$, where Ω^-_σ is the bundle of anti-self-dual 2-forms on $\mathbb{R}\times M$.

Proposition 2a.2. *For $(\sigma,\pi)\in\Sigma\times\Pi$, the function*

$$F_{\sigma\pi}:\mathscr{A}_\delta\to\mathscr{L}_\delta\,,\qquad F_{\sigma\pi}(A)=\tfrac{1}{2}(F_A-*_\sigma F_A)+\pi(A)\,, \tag{2a.4}$$

with $\pi(A)(\tau)=i_\tau\pi(A(\tau))$ is smooth and $\mathscr{G}_\delta$-equivariant. The first order expansion

$$F_{\sigma\pi}(A+\alpha)=F_{\sigma\pi}(A)+DF_{\sigma\pi}(A)\alpha+N(\alpha) \tag{2a.5}$$

satisfies

$$\|N(\xi)-N(\zeta)\|_p\leqq C(A,\pi)\,(\|\xi\|_{1,p}+\|\zeta\|_{1,p})\,\|\xi-\zeta\|_{1,p}\,, \tag{2a.6}$$

$$\|N(\xi)\|_{p;\delta}\leqq C(A,\pi)\,\|\xi\|_\infty\|\xi\|_{1,p;\delta}\,. \tag{2a.7}$$

Proof. Equivariance follows from (1b.3) for $F-*F$ and from equivariance of the restriction $A(\tau)=i^*_\tau A$. To show that $F_{\sigma\pi}(A)\in L^4_{0,\delta}(\Omega^-_\sigma)$, note that $f_{\sigma\pi}$ is a C^2 section of L^p and that, for R large enough,

$$\begin{aligned}\|F_{\sigma\pi}(A)\|^p_{p,\delta}&=\int e^{\tau\delta p}\|f_{\sigma\pi}(A(\tau))\|^p_p d\tau\\ &\leqq C\int_{-R}^{R}e^{\tau\delta p}\|A(\tau)\|^p_{1,p}+C_1\int_R^\infty e^{\tau\delta p}\|A(\tau)-a\|^p_{1,p}\\ &\quad+C\int_{-\infty}^{-R}e^{\tau\delta p}\|A(\tau)-b\|^p_{1,p}<\infty\,.\end{aligned}$$

Now consider the linear part of the expansion (2a.5). Note that π is C^1-bounded. Moreover, the operator d_A-*d_A is bounded with respect to the norms $\|\ \|_{1,p}$ and $\|\ \|_p$. The estimates on the nonlinear part follow from the definition of the curvature and a C^2-bound on π. □

2b) Fredholm Theory

Note that for any $a\in\mathscr{B}(M)$, the operator $D_a=D_{\sigma\pi}(a)$ of (1c.2) is selfadjoint and elliptic up to a part which is compact on the domain. It follows that the real part

$$\sigma_{\mathbb{R}}(a;\sigma,\pi)=\{\operatorname{Re}z\mid z\in\sigma(Da)\} \tag{2b.1}$$

of the spectrum of D_a is a discrete subset of $\mathbb{R}$. Roughly speaking, purely imaginary eigenvalues of D_a correspond to "rotational modes" of the flow, which complicate the structure of the trajectory spaces $\mathcal{M}(a,b)$. This is reflected in the Fredholm property of the linearization of $F_{\sigma\pi}$, i.e. of $DF_{\sigma\pi}(A)$ restricted to the kernel of d_A^{δ}. As in the case of the linearization of $f_{\sigma\pi}$, it is convenient to combine the operator and the gauge to one essentially elliptic operator. Define therefore

$$W_\delta = L^p_{1;\delta}(\Omega^1_{\mathrm{ad}}(\mathbb{R}\times\Sigma)), \qquad L_\delta = L^p_{0;\delta}(\Omega^-_{\mathrm{ad}}\oplus\Omega^0_{\mathrm{ad}}(\mathbb{R}\times\Sigma)). \tag{2b.2}$$

Proposition 2b.1. *If $\delta>0$ is smaller than the first positive element of $\sigma_{\mathbb{R}}(a)$ and the absolute value of the first negative element of $\sigma_{\mathbb{R}}(b)$, then for any zero of $F_{\sigma\pi}$ on $\mathcal{B}(a,b)$, the operator*

$$D_A: W_\delta \to L_\delta; \qquad D_A\alpha = (d_A^\delta\alpha, DF_{\sigma\pi}(A)\alpha) \tag{2b.3}$$

is Fredholm. If a and b are irreducible and nondegenerate zeros of $f_{\sigma\pi}$, then $D_A: W_0 \to L_0$ is Fredholm.

Proof. Representing A in the temporal gauge and identifying both $\Omega^1(\mathbb{R}\times M)$ and $\Omega^2_-(\mathbb{R}\times M)\oplus\Omega^0(\mathbb{R}\times M)$ with $\Omega^0_M\oplus\Omega^1_M$ in the canonical way, we have for $\tau<-1$,

$$\begin{aligned} D_A\alpha(\tau) &= \left(d^*_{A(\tau)}\alpha_1 + \left(\frac{\partial}{\partial\tau} \pm \delta_-\right)\alpha_0, * d_{A(\tau)}\alpha_1 + \frac{\partial}{\partial\tau}\alpha_0\right) + D\pi(A(\tau))\alpha \\ &= \left(\frac{\partial}{\partial\tau} + D_{A(\tau)}\right)\alpha \pm \delta_-\alpha_0 . \end{aligned} \tag{2b.4}$$

Now Proposition 2b.1 essentially Theorem 1.3 of [LM]. To recall the main steps of the proof, note first that by standard elliptic Fredholm theory on compact manifolds, one can reduce the assertion to the case of arbitrarily small paths in $\mathcal{B}(a,a)$ for $a\in\mathcal{R}$. Here we use the fact that the perturbation $D\pi$ factors through a continuous operator from $L^4_{2/3}$ to L^4 and is therefore "locally" a compact perturbation of the elliptic linearization of the instanton equation. Moreover, since the set of Fredholm operators is open in the operator topology, we only have to consider the case where A is a "constant" connection $\hat{a}$. By hypothesis, there exists for each $\lambda\in\mathcal{R}$ a resolvent

$$R_\lambda = (D_x + \delta_- + i\lambda)^{-1} \cdot L^p(a)\otimes\mathbb{C} \to L^p_1(a)\otimes\mathbb{C}$$

bounded independently of λ. Define for $\xi\in L^p(p^*(\Omega^0_M\oplus\Omega^1_M))$,

$$\hat{\xi}(\lambda,x) = \int_{-\infty}^{+\infty} e^{i\lambda\tau}\xi(\tau,x)\,d\tau .$$

Then

$$\zeta(\tau) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\lambda\tau} R_\lambda\hat{\xi}(\lambda)\,d\lambda$$

satisfies $D_{\hat{a}}\zeta = \left(\frac{\partial}{\partial\tau} + D_a\right)\zeta = \xi$. As in [MP]. Theorem 4.1 (see also [K] for the case $p=2$), we now conclude that

$$\|\zeta\|_{1,p;\delta} \leqq C\cdot\|\xi\|_{p;\delta} = C_i\cdot\|D_{\hat{a}}\zeta\|_{p;\delta} . \tag{2b.5}$$

To determine the index of D_A, we consider A as connecting two elements $\tilde{a}, \tilde{b}$ of the zero set $\tilde{\mathscr{R}}$ of $\not f$ in the covering space $\tilde{\mathscr{B}}(M)$ of (1b.7). Then we have

Proposition 2b.2. *To each nondegenerate $\tilde{a} \in \tilde{\mathscr{R}}$ we can associate an integer $\mu(\tilde{a})$ such that $\mu(g(\tilde{a})) = \mu(\tilde{a}) + 8 \cdot \deg(g)$, and so that for $A \in \mathscr{B}(\tilde{a}, \tilde{b})$,*

$$\operatorname{Index} D_A = \mu(\tilde{a}) - \mu(\tilde{b}) - \dim G_b .$$

Proof. Assume first that a and b are irreducible and $\delta = 0$. As in the proof of Proposition 2b.1, write

$$D_A = \frac{\partial}{\partial \tau} + D_\tau ,$$

$D_\tau = D_{A(\tau)}$. Then the Fredholm index of D_A is given by the "spectral flow" (see [AS2]) of D_τ through the imaginary axes. A construction of this integer quantity is given as follows: Choose $\tau_0 < \ldots < \tau_N$ and $\lambda_1 \ldots \lambda_N$ in $\mathbb{R}$ so that $\lambda_i \notin \sigma_{\mathbb{R}}(\tau)$ for $\tau_{i-1} \leqq \tau \leqq \tau_i$ and so that the same is true for $\lambda_0 := 0$ for $\tau \leqq \tau_0$ and for $\lambda_{N+1} := 0$ for $\tau \geqq \tau_N$. Define the integer n_i, $1 \leqq i \leqq N$, as the dimension of the eigenspace of D_{τ_i} corresponding to the eigenvalues in the strip $\lambda_{i+1} \leqq \operatorname{Re} z \leqq \lambda_i$ in $\mathbb{C}$. Otherwise, we define n_i to be the negative of the dimension of the eigenspace of D_{τ_i} in $\lambda_i \leqq \operatorname{Re} z \leqq \lambda_{i+1}$. We claim that

$$\operatorname{Index} D_A = \sum_{i=0}^{N} n_i . \tag{2b.6}$$

This also proves that the right-hand side is independent of the construction and that it is continuous in A. To prove (2b.6), consider for each $\varrho \in \mathbb{R}$ the operator

$$D(\varrho) = \frac{\partial}{\partial \tau} + D_{\bar{\beta}_\varrho(\tau)} ,$$

where $\bar{\beta}_\varrho(\tau) = \beta(\tau - \varrho)\tau + (1 - \beta(\tau - \varrho))\varrho$. Then $D(\varrho)$ is an asymptotically "constant" operator so that we can apply Theorem 1.3 of [LM] as in the proof of Proposition 2b.1. In particular, $D(\tau_i)$ is a Fredholm operator with respect to the exponential weight $e^{\beta(\tau)\lambda_i \tau}$ as well as to $e^{\beta(\tau)\lambda_{i+1}\tau}$ by the choice of λ_i. The difference of the first and the second Fredholm index is n_i by Theorem 1.4 of [LM]. On the other hand, $D(\varrho)$ for $\tau_{i-1} \leqq \varrho \leqq \tau_i$ is a continuous family of Fredholm operators for the weight $e^{\beta(\tau)\lambda_i \tau}$, so that the Fredholm index does not change through this deformation. Finally, it follows from the proof of Proposition 2b.1 that for large negative ϱ, $D(\varrho)$ is an isomorphism. This proves (2b.6).

It follows immediately from the definition that (2b.6) is additive with respect to composition of paths in $\tilde{\mathscr{B}}(M)$. Since each $\mathscr{B}(a, b)$ is simply connected, it follows from continuity of the Fredholm index that (2b.6) depends only on the end points. Hence fixing an arbitrary value of $\mu(a)$ for some irreducible nondegenerate $a \in \tilde{\mathscr{R}}$, we can define $\mu(b) = \mu(a) + \operatorname{Index} E_A$ for any $A \in \mathscr{B}(a, b)$.

The formula for reducible nondegenerate zeros is obtained in the same way, with $\lambda_0 = -\varepsilon$ and $\lambda_N = \varepsilon$. Due to this modification, the index is not additive any more with respect to compositions of paths $A, B \in \mathscr{B}(a, b) \times \mathscr{B}(b, c)$ to $A \# B \in \mathscr{B}(a, c)$ but satisfies

$$\operatorname{Index} D_{A \# B} = \operatorname{Index} D_A + \operatorname{Index} D_B + \dim G_b . \tag{2b.7}$$

This accounts for the correction term in Proposition 2b.2 for a suitable extension of μ. Clearly, the "relative Morse index" μ is well defined only up to an additive constant. Fixing a representative a of the trivial connection in $\tilde{\mathscr{B}}$ (which is always nondegenerate for a rational homology sphere), we can fix this additive constant by requiring that $\mu(0)=0$. □

2c) Transversality

The aim of this section is to prove the density of regular parameters $\sigma\pi \in \Sigma \times \Pi$ of Proposition 1c.1. The method is essentially the one of [FU]. The crucial point is to produce "enough" perturbations:

Lemma 2c.1. (1) *For every compact subset $K \subset \mathscr{B}^*(M)$, there exists $m \in \mathbb{N}$ and $\gamma: \Gamma_m \times D^2 \to M$, as described in Sect. 1b, inducing injective maps $\gamma_\theta: K \to L_m$.*

(2) *For any $a \in \mathscr{B}^*(M)$, the set $\Pi(a) = \{\pi(a) \mid \pi \in \Pi\}$ is dense in $T_a\mathscr{B}(M)$.*

Proof. The proof of (1) is obvious since two connections are equal in $\mathscr{B}^*(M)$ if and only if their holonomies coincide on all graphs in M. The second statement follows from the first, since every finite dimensional subspace of $T_a\mathscr{B}(M)$ is the tangent space of a suitably defined finite dimensional and hence locally compact submanifold of $\mathscr{B}(M)$. □

In the following, we would rather work with Banach spaces of perturbations than with the Frechet spaces $C^\infty(L_m, \mathbb{R})$. We therefore follow [F2] and define for every sequence $\underline{\varepsilon} = (\varepsilon_i)_{i \in \mathbb{N}}$ of positive real numbers a Banach space

$$C^{\underline{\varepsilon}}(L_m, \mathbb{R}) = \{h \in C^\infty(L_m, \mathbb{R}) \mid \|h\|_{\underline{\varepsilon}} < \infty\}$$

of smooth functions on L_m with norm

$$\|h\|_{\underline{\varepsilon}} = \sum_{i=1}^{\infty} \varepsilon_i \max_{x \in L_m} |D^i h(x)|.$$

Then by Lemma 5.1 of [F2], $\underline{\varepsilon}$ can be chosen small enough so that $C^{\varepsilon}(L_m, \mathbb{R})$ is dense in $L^p(L_m, \mathbb{R})$ for all $1 \leqq p < \infty$. In particular, we can approximate step functions in all L^p-norms. Fixing such an $\underline{\varepsilon}$, we can associate to each graph γ a Banach space Π_γ. Now recall that a subset is said to be of first category if it is the countable intersection of open and dense sets. It is then always dense by Baire's theorem. We are now ready to state

Proposition 2c.1. *For every $\pi \in \Pi$, there exists $\gamma \in \Gamma$ such that the set of all $\hat{\pi} \in \pi + \Pi_\gamma$ for which $\mathscr{R}_{\hat{\pi}}$ is nondegenerate is of first category near π.*

Proof. Note first that by elliptic regularity, the set

$$\mathscr{C}_\pi = \{(a, \xi) \subset \mathscr{L} \mid a \in R_\pi, \|\xi\|_2 = 1, \text{ and } \xi \in \ker D_{\sigma\pi}(a) \text{ for any } \sigma \in \Sigma\}$$

is compact. By Lemma 2c.1, we can therefore choose γ such that γ_0 is injective on $\mathscr{R}_\pi$ and $\Pi_\gamma(a)$ for $a \in R_\pi$ approximates $\ker D_{\sigma\pi}(a)$ arbitrarily well. For each $(a, \xi) \in \mathscr{G}_\pi$ we can then define $h \in C^{\varepsilon}(L_m, \mathbb{R})$ by

$$h(\exp(e)) = \beta(|l|/\varepsilon) \cdot \langle \hat{\xi}, e \rangle,$$

where β is some cutoff function, $l \in T_{\gamma(a)}L_m$, and $\hat{\xi}$ is the best approximation of ξ in $T_{\gamma(a)}L_m$. Now consider the section

$$\tilde{f}: \Pi_\gamma \times \mathscr{B}(\Sigma) \to \mathscr{L}, \quad \tilde{f}(\hat{\pi}, a) = f_{\sigma\pi}(a) + \hat{\pi},$$

where we identify in notation $\mathscr{L}$ with its pullback under the projection onto $\mathscr{B}(M)$. Denote by $\tilde{\mathscr{R}}$ the zero set of $\tilde{f}$. From the above, we conclude that $\tilde{\mathscr{R}}$ is a smooth Banach submanifold of $\Pi_\gamma \times \mathscr{B}(M)$ near $(0) \times \mathscr{R}_\pi$. Now the proposition follows from the Sard Smale Theorem ([Sm], see also [Q]) applied to the projection $\tilde{\mathscr{R}} \times \hat{\Pi}$. □

Let us now call $\mathscr{M}_{\sigma\pi}$ regular if for all $[A] \in \mathscr{M}_{\sigma\pi} \cap \mathscr{B}_\delta$, so that D_A is Fredholm with respect to δ, D_A is surjective. Of course, this implies by the implicit function theorem that all such sets are smooth manifolds, whose dimensions are given by Proposition 2b.2.

Proposition 2c.2. *For any $(\sigma, \pi) \in \Sigma \times \Pi$, the set of all $\pi \in \pi + \Pi$ for which $\mathscr{M}_{\sigma\pi}$ is regular of first category near π.*

Proof. First note that it follows from Arondzajn's theorem [An] that $\iota: \mathscr{M}(M) \to \mathscr{B}(M)$ is injective, and that the same is true for the "linearization"

$$\tilde{\iota}: \tilde{\mathscr{M}} = \{(A, \xi) \in \mathscr{L}(\mathbb{R} \times M) \mid A \in \mathscr{M}(M) \text{ and } \xi \in \operatorname{cok} D_A\} \to \mathscr{L}(M),$$

$$\tilde{\iota}(A, \xi) = (A(0), \xi(0)).$$

Moreover, for subsets $\mathscr{N} \subset \mathscr{M}(M)$ such that $\iota\mathscr{N}$ is compact, it follows from elliptic regularity that $\tilde{\iota}(\tilde{\mathscr{N}})$ is compact. Now choose m, γ so that D_γ is injective on $\tilde{\iota}(\mathscr{C}_{\sigma\pi})$. For every $(A, \xi) \in \tilde{\mathscr{M}}$ we construct $h_{A,\xi} \in C^\varepsilon(L_m, \mathbb{R})$ such that

$$\nabla h_{A,\xi}(A(\tau)) = \beta(\tau/\varepsilon) D_\gamma(A(\tau)) \xi(\tau).$$

This defines h on the image of the trajectory A under γ; it can be extended in an arbitrary way to the rest of L_m. Using these functions, it is easy to see that the section

$$\tilde{F}: \hat{\Pi} \times \mathscr{P}_\delta \to \mathscr{L}_\delta; \quad \tilde{F}(\hat{\pi}, A) = F_{\sigma, \pi + \hat{\pi}}(A)$$

has surjective linearizations at $(0) \times \tilde{\mathscr{N}}$. Now Proposition 2c.s follows like Proposition 2c.1 from the Sard Smale Theorem. □

2d) Transitivity

It is the aim of this section to construct the gluing map # of Proposition 1c.1. The general method used here was introduced by Taubes, see e.g. [T1]. In the present form, though in a different context, it was previously used in [F1]. Consider $a, b, c \in \mathscr{R}_{\sigma\pi}$ and choose ϱ large enough so that in the temporal gauge, we have $A(\tau) = [b + \xi(\tau)]$ for $\tau \geqq \varrho - 1$ and $B(\tau) = [b + \zeta(\tau)]$ for $\tau \leqq \varrho + 1$ (see Proposition 3b.1 below). Define

$$(A \mathbin{\hat{\#}_\varrho} B)(\tau) = \begin{cases} A(\tau + \varrho) & \text{for} \quad \tau \leqq 1 \\ [b + \beta(-\tau)\xi(\zeta + \varrho) + \beta(\tau)\zeta(\tau - \varrho)] & \text{for} \quad -1 \leqq \tau \leqq 1 \\ B(\tau - \varrho) & \text{for} \quad \tau \geqq 1 . \end{cases} \tag{2d.1}$$

More generally, define the set $\hat{\mathcal{M}}^k_{\sigma\pi}(a,b)$ of k-trajectories as the set of k-tuples $(A_i)_{1\leqq i\leqq k}$ with $A_i\in\hat{\mathcal{M}}_{\sigma\pi}(c_{i-1},c_i)$, where $c_0=a$ and $c_k=b$. Then we can extend the above construction to a map

$$\hat{\#}\,\hat{\mathcal{M}}^k(x,y)\times\mathbb{R}^{k-1}_+\supset K\times[\varrho(k),\infty)^{k-1}\to\mathcal{B}(x,y),$$
$$\chi=(\underline{u},\underline{\varrho})\mapsto A_\chi=(u_1\,\hat{\#}_{\varrho_1}u_2)\ldots\hat{\#}_{\varrho_{k-1}}u_k.$$

It follows from Proposition 3b.1 below that for any compact $k\subset\hat{\mathcal{M}}^k(x,y)$ there exists a constant $C_K\in\mathbb{R}_+$ with

$$\|F_{\sigma\pi}(A_\chi)\|_p\leqq C_K\sum e^{-\varrho_i/C_K} \tag{2d.2}$$

The main result of this section is

Proposition 2d. *Let K be as above and assume that all A_i for $\underline{A}\in K$ are regular. Then there exist positive constants ϱ_K, C_K and a C^2 map*

$$K\times[\varrho_K,\infty)^{k-1}\to\hat{\mathcal{M}}(a,c),\qquad \chi\mapsto[A_\chi+\xi],$$

with $\|\xi\|_{1,p}\leqq C_K\sum e^{-\varrho_i/C_K}$.

The proof of Proposition 2d uses an iterative procedure known as Picard's method. It can be summarized as follows.

Lemma 2d.1. *Let $f:E\to F$ be a C^1 map between Banach spaces E and F. Assume that in the first order expansion*

$$f(\xi)=f(0)+Df(0)\xi+N(\xi),$$

$Df(0)$ has a finite dimensional kernel and a right inverse G so that for $\xi,\zeta\in E$

$$\|GN(\xi)-GN(\zeta)\|_E\leqq C(\|\xi\|_E+\|\zeta\|_E)\,\|\xi-\zeta\|_E$$

for some constant C. Set $\varepsilon=(5C)^{-1}$. Then if $\|Gf(0)\|_E\leqq\frac{\varepsilon}{2}$, there exists a C^1-function

$$\phi:K_\varepsilon:=\{\xi\in\ker Df(0)\mid\|\xi\|_E<\varepsilon\}\to GF,$$

with $f(\xi+\phi(\xi))=0$ for all $\xi\in K_\varepsilon$. Moreover, we have estimates

$$\|\phi(0)\|_E\leqq 2\|Gf(0)\|_E,\qquad \|D\phi(0)\zeta\|_E\leqq 8\|Gf(0)\|_E\,\|\zeta\|_E.$$

The proof of Lemma 2d.1 is an elementary application of the contraction principle. We apply it to the family of functions

$$\begin{gathered}\Phi_\chi: W(A_\chi)\to L(A_\chi),\\ \Phi_\chi(\xi)=(F_{\sigma\pi}(A_\chi+\xi),d^*_{A_\chi}\xi)=F_{\sigma\pi}(A_\chi)+D_{A_\chi}\xi+N_\chi(\xi),\end{gathered} \tag{2d.3}$$

where $L=L_0$ and $W=W_0$, see (2*b*.2). The crucial step is to invert the linear part.

Lemma 2d.2. *There exist constants C and ϱ so that if $\chi\in K\times[\varrho,\infty)$, then there exists a continuous right inverse $G_\chi:L\to W$ of L_χ with*

$$\|G_\chi\xi\|_{1,p}\leqq C_G\|\xi\|_p.$$

Proof. For the sake of clarity, we consider only the case $k=2$. The proof is completely analogous to the proof of Lemma 4.3 of [F1]. We therefore only sketch the main idea. First, we identify the domain on which we want to invert L_χ to be an L^2-orthogonal complement of the set of all sections of $\Omega^1_{\mathrm{ad}}(\mathbb{R}\times\Sigma)$ of the form

$$(\xi \mathbin{\hat{\#}} \zeta)(\tau)=\begin{cases}\beta(\tau)\xi(\tau) & \text{for} \quad \tau\geqq 1\\ \beta(-\tau)\zeta(\tau) & \text{for} \quad \tau\leqq -1\end{cases}, \tag{2d.4}$$

for any $\xi\in T_{[A]}\mathcal{M}(a,b)$ and $\zeta\in T_{[B]}\mathcal{M}(b,c)$. Then since the Fredholm index of D_χ restricted to $W_\chi^\perp$ is zero, it suffices to show that for all $\xi\in W_\chi^\perp$,

$$\|\xi\|_{1,p}\leqq C\|D_\chi\xi\|_p,$$

with C depending only on K. This is done directly: Assume there exists a sequence $\chi_\alpha=(A_\alpha,B_\alpha,\varrho_\alpha)\in K\times[\alpha,\infty)$ and $\xi_\alpha\leqq W_\alpha^\perp$ satisfying

$$\|\xi_\alpha\|_{1,p}=1\,,\qquad \|D_\alpha\xi_\alpha\|_p\to 0\,.$$

Here we abbreviate a double index χ_α by α. As in the proof of Lemma 4.3 of [F1], one now proves first that $\xi_\alpha\to 0$ uniformly on $[-3,3]\times M$. Then one can show that $\|\xi_\alpha\|_{1,p}\to 0$ separately on $\mathbb{R}_+\times\Sigma$ and $\mathbb{R}_+\times\Sigma$ using the invertibility of D_A and D_B. This yields a contradiction to the first assumption and hence proves Lemma 2d.2. □

Now the proof of Proposition 2d is completed by (2a.6). To define the orientations of Proposition 1c.1, fix $a_0\in\mathcal{R}_\pi$ and consider for each $a\in\mathcal{R}_\pi$ the space $\mathcal{B}(a_0,a)$. By [D1], there exists a lifting of the structure map

$$D:\mathcal{B}(a_0,y)\to BO\,,\qquad A\mapsto D_A \tag{2d.5}$$

to BSO. It defines an orientation on $\mathcal{B}(a_0,a)$ in the sense that for each $A\in\mathcal{B}(a_0,a)$ and for each finite dimensional subspace $E\subset L^p(u)$ which is not annihilated by any element of $\operatorname{cok}D_A$, the manifold

$$\mathcal{M}_E(a)=\{\xi\in T_A\mathcal{B}\mid F_{\sigma\pi}(A+\xi)\in E\} \tag{2d.6}$$

can be naturally oriented given an orientation on E. In fact, we only have to choose a lifting on one component of $\mathcal{B}(x_0,x)$; it is then defined on the other components through the constriction (1c.8) extended to $\mathcal{M}_E$ (see also [D3] in the context of determinant bundles). We will consider such a choice of lifting as an "orientation of x." (In finite dimensional Morse theory, it corresponds to an orientation of the negative subspace of the Hessian at x.) We now extend the (unreduced) gluing map $\#$ to incorporate the spaces $\mathcal{M}_E$. Assuming that the elements of E have compact support, it defines a bundle $\hat{E}$ over $\hat{\#}\,(\hat{\mathcal{M}}_E(x_0,x)\times\mathbb{R}_+\times\hat{\mathcal{M}}^k(x,y))$. Now the proof of Proposition 2d can be easily adapted to yield local diffeomorphisms

$$\#:\hat{\mathcal{M}}_E(x_0,x)\times\mathbb{R}_+\times\hat{\mathcal{M}}^k(x,y)\to\mathcal{M}_E(x_0,y)\,. \tag{2d.7}$$

Now there exists a unique orientation on $\mathcal{M}(x,y)$ making this map oriented. The orientations $\mathcal{M}(W)$ are constructed by combining the above constriction with the one in [D3] in the obvious way.

3. Compactness

3a) Local Convergence

In order to formulate compactness properties, recall the set $\mathcal{MC}_{\sigma\pi}$ of cusp trajectories of Definition

Definition 3a.1. A cusp trajectory $(A, \underline{I}) \in \mathcal{MC}_{\sigma\pi}$ is a trajectory $A \in \mathcal{M}_{\sigma\pi}$ together with a finite collection $\underline{I}$ of instantons on Euclidean $\mathbb{R}^4$. We say that A_k converges (locally) to $(A, \underline{I}) \in \mathcal{MC}_{\sigma\pi}$ if there exist representations of A_k and sequences x_{ik} in $\mathbb{R} \times M$ and $\varepsilon_k \to 0$ so that

(1) $A_k \to A$ in the L_1^4-norm on $K - \bigcup_i U_{\varepsilon_k}(x_{ik})$ for every compact $K \subset \mathbb{R} \times M$.

(2) $I_{ik} := (\exp_{x_k} \circ \varepsilon_k)^* A_k$ converges to I_i in $L_1^p(\Omega^1(B_R))$ for every $R > 0$.

Here, $\exp_x$ for $x \in \mathbb{R} \times M$ denotes the Gauss normal chart. Now define for any $I \subset \mathbb{R}$

$$\mathcal{B}|_I = \mathcal{A}(I \times M)/\mathcal{G}(I \times M)$$

and write $\mathcal{B}|_\varrho = \mathcal{B}|_{[-\varrho, \varrho]}$. With respect to the "weak convergence" of Definition 3a.1, we have the following compactness result:

Proposition 3a. *Let $A_\varrho \in \mathcal{B}|_\varrho$, $\varrho \in \mathbb{N}$, be a sequence so that $l^2(A_\varrho)$ is bounded. Moreover, assume that $\sigma_\varrho \pi_\varrho \to \sigma\pi$ in $\Sigma \times \Pi$ and that $\lim_{\varrho \to \infty} \|F_{\sigma_\varrho \pi_\varrho}(A_\varrho)\|_p = 0$. Then there exists a subsequence A_ϱ converging to some $(A, \underline{I}) \in \mathcal{MC}_{\sigma\pi}$. We have*

$$\sum_{i=1}^{N} \|F_{B_i}\|_2^2 + l^2(A) \leqq \frac{1}{2} \limsup_{\varrho \to \infty} l^2(A_\varrho).$$

In particular if $l^2(A_\varrho) \leqq 3\pi$, then $\underline{I}$ is empty and A_ϱ converges locally in L_1^4 to A.

Proof. The essential ingredient of the proof is Uhlenbeck's elliptic regularity results for the selfduality equation, see [U1]. It remains valid for the operator $F_{\sigma\pi}$ due to the "compactness" of the perturbation π.

Lemma 3a.1. *Let A_ϱ be a sequence of connections on $[-R, R] \times M$, so that $\|F_{A_k}\|_p$ is bounded and $\|F_{\sigma\pi}(A_k)\|_p \to 0$ for some $p > 2$. Then A_ϱ contains a subsequence converging on M_{R-1} to a connection A with $F_{\sigma\pi}(A) = 0$.*

Proof. By [U1] the bound on $\|F_{A_k}\|_p$ implies that there exist representatives A_k so that $\|A_k\|_{1,p}$ is bounded on $M_{R-1/2}$. Hence it contains a subsequence A_k converging weakly in $L_1^p(\Omega^1(M_{R-1/2}))$ to some limit A. That $F_{\sigma\pi}^-(A) = 0$ follows from the weak compactness of the nonlinear term of $F_{\sigma\pi}^-$, see (2a.N2). Similarly, applying (2a.N2) and ellipticity of D_A to $\xi_k = \beta(A_k - A)$ for a suitable cutoff function β proves that $\|\xi_k\|_{1,p} \to 0$. □

Now define for any compact $K \subset \mathbb{R} \times M$ the sequence of numbers

$$\varepsilon_k(K) = \inf\{\varepsilon > 0 \mid \text{there exists } \theta \in K \text{ so that } \|F_{A_k}\|_{p, U_\varepsilon(\theta)} = \varepsilon^{2/p-1}\}. \tag{3a.1}$$

Here, $U_\varepsilon(\theta)$ is the ε-ball around θ. If $\varepsilon_\alpha(K)$ is bounded from below, then the hypothesis of Lemma 3a.1 is satisfied for each $\varrho \in \mathbb{R}_+$. Hence let us assume that

$\varepsilon_k(K)\to 0$ for a subsequence and choose $\theta_k\in\times M$ so that with $U_k=U_{\varepsilon_k}(\theta_k)$,

$$\int_{U_k} |F_A(\theta_k)|^p d\theta=\tfrac{1}{2}\varepsilon^{2-p}.$$

Let $\exp_\alpha:\mathbb{R}^4\to\mathbb{R}\times M$ be a Gauss normal chart centered at θ_α preceded by multiplication with ε_α. Then the connection $B_\alpha:=\exp_\alpha^* A_\alpha$ is defined on the ball of radius $\varepsilon_\alpha^{-1}r$, if r is the injectivity radius on $\mathbb{R}\times M$. It satisfies

(1) $\|p_-F_{B_k}\|_4\to 0$,

(2) $\limsup\|F_{B_k}\|_2^2\leqq l$,

(3) $\|F_{B_k}\|_{4,U_1(0)}=\tfrac{1}{2}$,

(4) $\|F_{B_k}\|_{4,U_1(\theta)}\leqq 1$ for all $\theta\in U_{R-1}(0)$.

The projection in (1) is taken with respect to the standard Euclidean conformal structure on $\mathbb{R}^4$. To prove (1), note that the pullback conformal structure under $\exp_\alpha$ converges to the Euclidean one uniformly on B_R, and that the perturbation π is L^4-bounded. (2) follows for the same reason from the conformal invariance of the "Yang-Mills action" $\|F_A\|_2^2$. (3) and (4) follow by direct calculation. By [U1], it follows from (1) and (4) that B_k contains a subsequence converging in $L_1^p(\Omega^1(U_{R-1}(0))$ to a selfdual connection I. Now (3) implies that I is nontrivial, so that $\|F_B\|_2^2\geqq 8\pi$ (see e.g. [ADHM]). We now repeat the above argument $\mathbb{R}\times M-\bar{U}_k$ with $\bar{U}_k=B_{\sqrt{\varepsilon_k}}(\theta_k)$. That is, we consider the sequence $\varepsilon'_k\geqq\varepsilon_k$ defined as in (3a.1) with $\bar{U}_k$ removed from K. If $\varepsilon'_k\times 0$, then the corresponding reparametrized sequence B'_k converges as before to a positively charged instanton I'. We claim that the action $l(A_k)=\mathscr{a}(a)-\mathscr{a}(b)$ of the trajectory must be at least the sum of the actions of I and I'. To see this, note that l_k tends to infinity in the reparametrized chart around θ'_k, since $\operatorname{dist}(\theta_k,\theta'_k)\geqq\varepsilon_k^{1/2}$. If also $\operatorname{dist}(\theta_k,\theta'_k)/\varepsilon'_k\to\infty$, then the converse is also true, so that there exist disjoint neighborhoods V_k of θ_k and v'_k of θ'_k on which the action of A_k approaches that of I and I', respectively. On the other hand, if $\operatorname{dist}(\theta_k,\theta'_k)/\varepsilon'_k$ is bounded, then $\varepsilon'_k/\varepsilon_k\to\infty$, and we can choose V_k and V'_k so that the actions of A_k restricted to V_k and V'_k-V_k approach the action of I and I', respectively. By induction, we conclude that there exists a finite set $\{\theta_1,\ldots,\theta_N\}$ with $N\leqq 8\pi^{-1}(\mathscr{a}(a)-\mathscr{a}(b))$ so that a subsequence of A_k converges locally on $\mathbb{R}\times M-\{\theta_1,\ldots,\theta_N\}$ to an element of $\mathscr{M}(a,b)$. This completes the proof of Proposition 3a. □

3b) Global Convergences

Proposition 3a allows us to prove

Proposition 3b.1. *For each $[A]\in\mathscr{M}_{\sigma\pi}$ there exist $a,b\in\mathscr{R}_\pi$ so that $[A(\tau)]$ converges in $\mathscr{B}(M)$ to a for $\tau\to\infty$ and to b for $\tau\to-\infty$. Moreover, if $\mathscr{R}_\pi$ is nondegenerate, then for some $\delta>0$, the canonical map induces a bijection*

$$\bigcup_{a,b\in R}\mathscr{M}_\delta(a,b)/(G_a\times G_b)\to\mathscr{M}. \tag{2b.1}$$

Here, $G_a\times G_b$ acts on $\mathscr{M}_\delta(a,b)$ by means of gauge transformations which are constant outside a compact set. Restricted to each $\mathscr{M}_\delta(a,b)$, (b_{ij}) is a homeomorphism with respect to the local topology in $\mathscr{M}$ and the relative topology of $\mathscr{M}_\delta(a,b)$ in $\mathscr{B}_\delta(a,b)$.

Proposition 3b.1 implies that whenever the local limit of Proposition 3a happens to lie in $\mathscr{M}(a,b)$, the convergence is global. Of course, this assumption is very strong. To define the set of all possible "global" limits, we have to combine the notion of a k-trajectory of Sect. 2 with the notion of a cusp trajectory. We therefore define the set $\hat{\mathcal{M}}\mathcal{C}^k_{\sigma\pi}(a,b)$ of k-cusp-trajectories joining a and b as the set of $(A_i, \underline{I}_i)_{1\leq i\leq k}$ with $(A_i, \underline{I}_i)\in\hat{\mathcal{M}}\mathcal{C}_{\sigma\pi}(c_{i-1}, c_i)$ with $c_0=a$, $c_{2k}=b$, and $c_{2i+1}=c_{2i}$ for $0\leq i\leq k$. We say that A_ϱ converges to $(A_i, \underline{I}_i)_{1\leq i\leq k}\in\mathcal{M}\mathcal{C}^k_\sigma(a,b)$ if there exist sequences $\tau_{i\varrho}$ so that $\tau_{ik} * A_\varrho = A_\varrho(\cdot - \tau_{i\varrho})$ converges weakly to $(A_i, \underline{I}_i)$. Then we have

Proposition 3b.2. *Let $\sigma_\varrho\pi_\varrho$ converge to $\sigma\pi$ in $\Sigma\times\Pi$ and let $a_\varrho, b_\varrho\in\mathcal{R}_{\pi_\varrho}$ converge to a, b in $\mathcal{B}(M)$. Then for any sequence $[A_\varrho]\in\mathcal{M}_{\sigma_\varrho\pi_\varrho}(a_\varrho, b_\varrho)$ with constant index $I(A_\varrho)=I$, there exists a subsequence converging to some $(B_i, \underline{I}_i)\in\mathcal{M}\mathcal{C}^k_{\sigma\pi}(a,b)$ for some $k\geq 0$. Moreover, we have*

$$l^2(A_\varrho)\geq \tfrac{1}{2}\sum l^2(B_i) + 8\pi\sum_j n(I_j)\,.$$

If a and b are nondegenerate, then

$$I=\sum_i I(B_i) + 8\sum_j n(I_j)\,.$$

If I is trivial, then there exists $N>0$ so that for $\varrho>N$, A_ϱ is contained in the ε-tube

$$\mathcal{U}_\varepsilon(\underline{B}) = \bigcup_i \bigcup_{\tau\in\mathbb{R}} \mathcal{U}_\varepsilon(B_i(\tau); \mathcal{B}(M))\,.$$

Finally, if each B_i is a regular zero of $F_{\sigma\pi}$, and all representations involved are irreducible, then there exists $\varepsilon>0$ so that each $A\in\mathcal{M}(a,b)$ which is fully contained in $\mathcal{U}_\varepsilon(\underline{B})$ is the image of the gluing map $\#$ of Proposition 2d.

Proof of Proposition 3b.1. To prove the first assertion, we assume the contrary. Then there exists $\varepsilon>0$ and a sequence $\tau_k\to\infty$ so that the sequence $[A(\tau_k)]$ in $\mathcal{B}(\Sigma)$ does not accumulate at $\mathcal{R}$. Now consider the sequence

$$A_k(\tau, x) = A(\tau - \tau_k/2))$$

on $[-\tau_k/2, \tau_k/2]\times M$. It satisfies the hypothesis of Proposition 3a with $\lim_{k\to\infty} l(A_k)=0$. Hence A_k converges locally to constant element of $\mathcal{M}$, which contradicts the assumption. By the same method, one can prove that for any $\varepsilon>0$ there exists $r>0$ so that on $[r,\infty)\times M$, we have $[A]=[a+\xi]$ with $\|\xi\|_\infty<\varepsilon$ and $\|\xi|_{[\varrho,\varrho+1]}\|_{1,p}<\varepsilon$ for all $\varrho>r$. To prove an exponential decay estimate, we work in the "temporal" gauge, i.e. we choose a representative $A\in\Omega^1_{\mathrm{ad}}(\mathbb{R}\times M)$ which vanishes on the vector $\frac{\partial}{\partial\tau}$. Now define $\beta_\sigma(\tau)=\beta(\tau-\theta+1)\beta(-\tau+\sigma)$ and consider the function $f(\sigma)=\|\beta_\sigma(A-\hat{a})\|_2^2$. We prove an estimate of the form

$$f''(\tau)\geq\mu f(\tau)\,, \tag{2b.2}$$

for some $\mu>0$. It then follows e.g. from the maximum principle that $\|A-a\|_{2;\delta}<\infty$ on the half cylinder on which $A(\tau)$ converges to a. To prove (2b.2), we calculate

$$\begin{aligned} f''(\tau) &= \int\beta^2(\tau)\{\|A'(\tau)\|_2^2 + \langle A''(\tau), A(\tau)-a\rangle\}d\tau \\ &= \int\beta^2(\tau)\,\|\not{A}A(\tau))\|_2^2 d\tau \\ &\quad + \int\beta^2(\tau)\langle A(\tau)-a), D\not{A}A(\tau))\not{A}A(\tau))\rangle d\tau\,. \end{aligned} \tag{2b.3}$$

Note that $\frac{d}{d\tau} d_a^*(A(\tau)-a)=d_a^* f(A(\tau))=0$, so that $A(\tau)-a\in T_a\mathscr{B}(M)$ for all $\tau\in\mathbb{R}$. To estimate the second term, note that with $D_\tau=Df(A(\tau))$,

$$\langle(A(\tau)-a), D_\tau \not{f}(A(\tau))\rangle=\langle D_\tau^+(A(\tau)-a), D_\tau(A(\tau)-a)\rangle + \langle D_\tau^+(A(\tau)-a), N(A(\tau)-a)\rangle,$$

where N is the nonlinear term in the first order expansion of $\not{f}$ around a. Since D_τ has no purely imaginary eigenvalues, the first term is positive. The nonlinear term can be estimated as follows.

$$\langle D_\tau^+(A(\tau)-a), N(A(\tau)-a)\rangle \leqq C\|(A(\tau)-a)\|_\infty \|A(\tau)-a\|_{1,2}\|D_\tau^+(A(\tau)-a)\|_2^2 \leqq \varepsilon(\tau)\,\|A(\tau)-a\|_{1,2}^2$$

with $\lim_{\tau\to\infty}\varepsilon(\tau)=0$.

Since a is a nondegenerate zero of $\not{f}$, we have $\|\not{f}(A(\tau))\|_2\geqq\gamma\|(A(\tau)-a\|_{1,2}^2$ for some $\gamma>0$. Hence for τ large enough, the nonlinear term is small compared with the first term of (2b.3). This implies the second order inequality (2b.2).

Using the invertibility of $D_{a\leftarrow}$ [see (2b.5)], we can strengthen this result to conclude that $\lim_{\tau\to\infty} e^{\delta\tau}\|\beta_\tau(A-\hat{a})\|_{1,p}=0$ for some $\delta>0$. Now define $\beta_{rR}(\tau)=\beta(\tau-R)\beta(-\tau-r)$ and $A_{rR}=\hat{a}+\beta_{rR}(A-\hat{a})=\hat{a}+\xi_{rR}$. Consider the expansion (2a.5) around $\hat{a}$:

$$F_{\sigma\pi}(A_{rR})=DF_{\sigma\pi}(\hat{a})\xi_{rR}+N(\beta_{rR}(A-\hat{a}))\,.$$

Then since $d_{\hat{a}}^*(A-\hat{a})=0$, we have by (2a.7) and (2b.5),

$$\|\xi_{rR}\|_{1,p;\delta}\leqq C\|D_{\hat{a}}\xi_{rR}\|_{p;\delta}\leqq C(\|N(\xi_{rR})\|_{p;\delta}+\|F_{\sigma\pi}(A_{rR})\|_{p;\delta} \leqq \varepsilon(r)\,\|\xi_{rR}\|_{1,p;\delta}+C$$

with $\lim_{r\to\infty}\varepsilon(r)=0$ by the above. For r large enough, this yields an estimate on $\|\xi_{rR}\|_{1,p;\delta}$ which does not depend on R. Hence $A\in\mathscr{A}_\delta(x,y)$.

The same method can be used to compare the topology on $\mathscr{B}_\delta(a,b)$ with the local topology on $\mathscr{M}(a,b)$. To prove injectivity, let us assume that $A,B\in\mathscr{A}_\delta$ are selfdual and that $B=g(A)$ for some $g\in L^4_{2;\mathrm{loc}}(\mathbb{R}\times M, SU_2)$. We have to show that $g\in\mathscr{G}_\delta\times G_a\times G_b$. Note that on the positive half cylinder, we have $A-\hat{a}$, $B-\hat{a}\in L^4_{1;\delta}$, where $\hat{a}$ is the constant connection. Then

$$g\hat{a}-ag+dg=\{g(A)-g(A-\hat{a})g^{-1}\}g$$

is in $L^p_{0;\delta}(\Omega^1)$. If we consider the left-hand side as an (overdetermined) translationally invariant elliptic operator applied to a function $g:\mathbb{R}\times M\to\mathrm{End}(\mathbb{C}^2)$, the theory of [LM] and [MP] (see Sect. 2b) implies that $g-g_0\in L^p_{1;\delta}(\mathbb{R}\times M, \mathrm{End}(\mathbb{C}^2))$ for some $g_0\in G_a$. Repeating the procedure proves that $g\in\mathscr{G}_\delta\times G_a\times G_b$ and therefore completes the proof of Proposition 3b.1. □

Proof of Proposition 3b.2. The first assertion follows from an iterative argument, applying Proposition 3a and 3b.1 to suitably rescaled sequences. The index formula follows from Proposition 2b.2. The estimate on l^2 follows from (1c.4). The assertion about the ε-tube is proved indirectly: Assume there exists $\varepsilon>0$ and a subsequence A_ϱ so that $A_\varrho(\tau_\varrho) \notin U_\varepsilon(\underline{B})$ for some $\tau_\varrho \in \mathbb{R}$. Then by Proposition 3a, there exists a subsequence A_ϱ converging to some $A_\infty \in \mathcal{M}_{J\pi}$ which is not identical with any of the B_i. This contradicts the assumption.

To prove the last assertion, we define numbers $s_i \in \mathbb{R}$ so that $a(A(s_i)) = a(b_i)$ for $1 \leq i < k$. It then follows from Proposition 3a that if $A \in \mathcal{M}(a, b)$ is contained in $U_\varepsilon(\underline{B})$ and ε is small enough, A is L_1^4-close to $\hat{\#}\,(B_i, \frac{1}{2}(\sigma_i + \sigma_{i+1}))$. Now the last assertion of Proposition 3b.2 follows from the uniqueness property of Lemma 2d.1. □

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