Math. Z. 178, 77–82 (1981)

Mathematische Zeitschrift

An Inequality between the Exterior Diameter and the Mean Curvature of Bounded Immersions

Luquesio P. de M. Jorge[1] and Frederico V. Xavier[2]

[1] Departamento de Matemática, Universidade Federal do Ceará, 60.000 Fortaleza, CE, Brazil
[2] Departamento de Matemática, Universidade Federal de Pernambuco, 50.000 Recife, PE, Brazil

This paper is a natural outgrowth of [9], where the problem of studying the boundedness properties of complete minimal surfaces in $\mathbb{R}^n$ was approached via the consideration of a certain gradient flow. After the completion of [9] it was soon realized that the basic technique (Lemma 3 below) could be successfully employed to study boundedness of arbitrary complete submanifolds. In this regard, the method can be used to prove the following three theorems.

Theorem 1. *Let M be a complete Riemannian manifold whose scalar curvature is bounded from below and let B_R be a closed normal ball of radius R in a Riemannian manifold $\bar{M}$. Set $\bar{K}$ for the supremum of the sectional curvatures of $\bar{M}$ in B_R. Let $I: M \to B_R \subset \bar{M}$ be an isometric immersion with bounded mean curvature vector H (say, $|H| \leqq H_0$). Then the following holds*

a) $R \geqq \frac{1}{\delta} \arctan \left(\frac{\delta}{H_0}\right)$, *if* $\bar{K} \leqq \delta^2$ $(\delta > 0)$ *and* $R < \frac{\pi}{2\delta}$,

b) $R \geqq \frac{1}{H_0}$, *if* $\bar{K} \leqq 0$,

c) $R \geqq \frac{1}{\delta} \operatorname{arctanh} \left(\frac{\delta}{H_0}\right)$, *if* $\bar{K} \leqq -\delta^2 < 0$.

Theorem 1 was proved in [1] for M = surface and $\bar{M} = \mathbb{R}^n$. For arbitrary M and $\bar{M} = \mathbb{R}^n$, see [4] and [5]. In [3] the case $\bar{M} = S^n$ is discussed. Recently, a different proof of Theorem 1 was offered by Koutroufiotis and the first named author in [8], where other related problems are also considered. Except for [1], all these proofs rely, in one way or another, on a rather technical theorem of Omori [11]. The present authors feel that the present approach is more conceptual and lends itself better for generalizations. For instance, the proofs of the following two seemingly unrelated theorems on minimal submanifolds have much in common with Theorem 1.

Theorem 2. *Let M be a complete Riemannian manifold with bounded scalar curvature and $I: M \to \mathbb{R}^n - \{0\}$ be a minimal immersion. It is not possible to find*

0025-5874/81/0178/0077/$1.20

a unit vector $v \in \mathbb{R}^n$ *and* $\delta > 0$ *such that* $\left\langle v, \frac{I(x)}{\|I(x)\|} \right\rangle \geqq \delta$ *for all* $x \in M$ *(otherwise said,* $I(M)$ *does not lie inside of a non-degenerate cone of* $\mathbb{R}^N$*).*

When M is a surface and $n=3$ a stronger statement can be made:

Theorem 3. *Let* M^2 *be a complete surface with bounded Gaussian curvature and let* $f\colon \mathbb{R} \to \mathbb{R}^+$ *be a proper function. It is not possible to find a minimal immersion of* M *into*

$$A_f = \{(x, y, z) \in \mathbb{R}^3 z > f(y)\}.$$

Theorems 2 and 3 seem to be innacessible to techniques using the theorem of Omori quoted above. They are very much in the spirit of [9] where, as pointed out at the beginning, the main enterprise was to find out the extrinsic boundedness properties of minimal surfaces in $\mathbb{R}^n$. This type of question was raised by Calabi and Chern [2]. The reader may also want to look at [7] and [10] where examples are constructed of complete minimal surfaces entirely contained in balls of $\mathbb{R}^4$ and slabs of $\mathbb{R}^3$, respectively. Theorem 3 above should be compared with be result of [10]. It is also to be mentionned that, by letting $H_0=0$ in Theorem 1, several results scattered throughout the aforementionned papers can be recovered.

In order to stress the conceptual simplicity of the methods the authors have refrained from proving a few other minor results that spring naturally from the given ones.

The authors would like to thank Koutroufiotis for his critical reading of the first draft of this paper.

Preliminary Results

Lemma 1. *Let* M *be a complete Riemannian manifold and* $f\colon M \to \mathbb{R}^+$ *a smooth function satisfying*

$$\operatorname{Hess} f(p)(X, X) \geqq c\,|X|^2 \qquad (c>0)$$

for all $p \in M_\varepsilon = f^{-1}([0, \varepsilon])$, *and for all* $X \in T_p(M)$. *Then each connected component of* M_ε *is compact.*

Proof. Let $\bar{M}_\varepsilon$ be a connected component of M_ε and $\mathscr{C}$ be the set of critical points of f in $\bar{M}_\varepsilon$. The above condition on the Hessian of f implies that $\mathscr{C}$ contains only non-degenerate points of minimum. Let $x \in \bar{M}_\varepsilon$ and consider $\varphi_x\colon (T_{(\bar{x})}, 0] \to \bar{M}_\varepsilon$, the maximal semi-orbit of $df^{\#}$ ending at x (i.e., $\varphi_x(0)=x$). Suppose, by way of contradiction, that φ_x has infinite length. Let then $\beta\colon [0, \infty) \to \bar{M}_\varepsilon$ be its reparametrization by arc-length. A straightforward computation gives $(f \circ \beta)'(s) = |df(\beta(s))|^2$ and $(f \circ \beta)''(s) \geqq c$. It follows from this inequality that $f \circ \beta$ is unbounded, thus contradicting the fact that $\beta(s) \in \bar{M}_\varepsilon$. Therefore φ_x has bounded length. In particular, $\lim_{T \to T_{(\bar{x})}} \varphi_x(T)$ exists and belongs to $\mathscr{C}$. Let

$$\mathcal{O}_y = \{x \in \bar{M}_\varepsilon \lim_{T \to T_{(\bar{x})}} \varphi_x(T) = y\}.$$

It is easy to see that each $\mathcal{O}_y$ is open ($\mathscr{C}$ contains only points of strict minimum). Since $\bar{M}_\varepsilon = \bigcup_{y\in\mathscr{C}} \mathcal{O}_y$ is connected, $\mathscr{C}$ consists of a single point say y_0. Suppose now that $\bar{M}_\varepsilon$ is noncompact. Since M is complete there are points $p_n \in \bar{M}_\varepsilon$ with $d(p_n, y_0) \to \infty$. Consider $\varphi_{p_n}\colon [T_{(\bar{p}_n)}, 0] \to \bar{M}_\varepsilon$ (notation as above) and let $\beta_n\colon [0, s_n]$ be its reparametrization by arc-length. It follows from the relations given above for $(f\circ\beta_n)'$ and $(f\circ\beta_n)''$ that

$$f(\beta_n(s_n)) \geqq \frac{c s_n^2}{2} + f(y_0) = \frac{c}{2} d(p_n, y_0)^2 + f(y_0).$$

Hence $f(\beta_n(s_n)) \to \infty$ as $n\to\infty$, contradicting the boundedness of f on $\bar{M}_\varepsilon$.

The proof of next lemma uses standard comparison theorems and may be found in [8] and [13].

Lemma 2. *Let N be an m-dimensional Riemannian manifold with sectional curvature K, x_0 and x points of N so that x does not lie in the cut locus of x_0. Let $\gamma\colon [0, \ell]$ be the minimizing geodesic segment connecting x_0 with x, parametrized by* arc-*length. Take a positive number δ. For any unit vector $X\in T_x N$, perpendicular to $\gamma'(\ell)$, the Hessian of the function $f(x) = \frac{1}{2} d(x_0, x)^2$ satisfies* $\operatorname{Hess} f(X, X) \geqq \mu(\ell)$, *where*

$$\mu(\ell) = \begin{cases} \ell\delta \cot(\ell\delta) & \text{if } \max_\gamma K = \delta^2 \text{ and } \ell < \dfrac{\pi}{\delta} \\ 1 & \text{if } \max_\gamma K = 0 \\ \ell\delta \coth(\ell\delta) & \text{if } \max_\gamma K = -\delta^2. \end{cases}$$

Lemma 3. *Let M be complete and $I\colon M \to \bar{M}$ an isometric immersion with bounded second fundamental form. Then, for each $p\in\bar{M}$ there exists a closed ball $B(p)$ such that all connected components of $I^{-1}(B(p))$ are compact.*

Proof. It is well-known that the function $g\colon \bar{M}\to\mathbb{R}$, $g(q) = \frac{1}{2} d(p, q)^2$ is strictly convex on a sufficiently small neighborhood of p, say $\operatorname{Hess} g(q)\,(X, X) \geqq c|X|^2$ $(c>0)$ for $q\in B(p)$. Let then $f = g|_M$. A straightforward computation yields

$$\operatorname{Hess} f(q)(X, X) = \operatorname{Hess} g(q)(X, X) + \langle \nabla g(q), \alpha(q)(X, X)\rangle,$$

where α represents the second fundamental form. On the other hand

$$|\langle \nabla g(q), \alpha(q)(X, X)\rangle| \leqq |\nabla g(q)|\,|\alpha(q)|\,|X|^2 \leqq d(p, q)\,|\alpha(q)|\,|X|^2$$

and this in turn can be made less than $\frac{c}{2}|X|^2$ by taking $d(p, q)$ small enough (i.e., by shrinking $B(p)$). In particular,

$$\operatorname{Hess} f(q)(X, X) \geqq \frac{c}{2}|X|^2.$$

The result now follows from Lemma 1.

Proofs of the Theorems

Proof of Theorem 1. Suppose that $I\colon M \to B_R(P_0)$ and that $P \in \partial B_R(P_0)$ is an accumulation point of $I(M)$. By Lemma 3 the set $M_\varepsilon = I^{-1}(B_\varepsilon(P))$ has only compact components, provided $\varepsilon > 0$ is small enough. Let γ be the radius joining P and P_0, extended somewhat. Take a sequence $\{P_j\} \subset \gamma$ converging to P_0 and such that P_0 lies between P_j and P. For $x \in S_\varepsilon = B_R(P_0) \cap \partial B_\varepsilon(P)$ one has

$$d(P_j, x) \leqq d(P_j, P_0) + d(P_0, x) \leqq d(P_j, P_0) + R = d(P_j, P).$$

Since broken geodesis do not realize distance the first inequality is strict. In particular, $R_j = \max_{x \in S_\varepsilon} d(P_j, x) < d(P_j, P)$. Let $f_j\colon M \to \mathbb{R}$ be given by $f_j(x) = \frac{1}{2} d(P_j, I(x))^2$. Since P is an accumulation point of $I(M)$ and $I(\partial M_\varepsilon) \subset S_\varepsilon$ there exists points x in M_ε for which $R_j < f_j(x) < d(P_j, P)$. In particular, the maximum of f_j at each (compact) component of M_ε is attained at some interior point, say x_j. The formula given in the proof of Lemma 3 together with the estimate of Lemma 2 yields

$$0 \geqq \Delta f_j(x_j) = \operatorname{tr} \operatorname{Hess} f_j(x_j) \geqq (n-1)(\mu(T_j) - H_0 T_j), \qquad \text{where } T_j = d(P_j, P).$$

The result now follows from $\mu(T_j) \leqq H_0 T_j$ by leting $j \to \infty$.

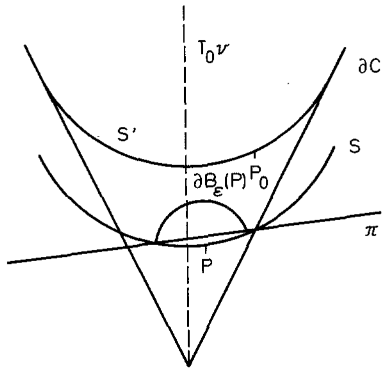
Proof of Theorem 2. There is no loss of generality in supposing that $v = (0, 0 \dots 1)$ and $\overline{I(M)} \cap \partial C = \varnothing$, where

$$C = \left\{ x \in \mathbb{R}^n \,\middle|\, \left\langle v, \frac{x}{\|x\|} \right\rangle \leqq \delta \right\}.$$

Let $P_0 \in I(M) \cap \operatorname{int} C$. It is possible to choose a sphere $S_{R_0}(T_0 v)$ in $\mathbb{R}^n$ which is tangent to the cone and so that P_0 is between the hyperplanes $\langle v, x \rangle = T_0$ and $\langle v, x \rangle = 0$, with $P_0 \in S_{R_0}(T_0 v)$. In particular the set $S_{R_0}(T_0 v) \backslash \partial C$ has exactly two components. Let S' be the one that contains P_0. Set

$$T' = \sup \{T\colon (S' - Tv) \cap I(M) \neq \varnothing\}, \qquad S = S' - T' v.$$

Let $P \in \overline{I(M)} \cap S$. Suppose that $T' > 0$ (if $T' = 0$, set $P = P_0$ in the sequel), so that $\partial S \cap \partial C = \varnothing$ (recall that $S_{R_0}(T_0 v)$ is tangent to C). In particular, $P \in \operatorname{int} S \cap \overline{I(M)}$. By Lemma 3 it is possible to chose $\varepsilon > 0$ sufficiently small so that the components of $I^{-1}(\overline{B_\varepsilon(P)})$ are compact submanifolds with boundary (it follows from the equation of Gauss and the minimality of $I(M)$ that boundedness of the second fundamental form is equivalent to boundedness of the scalar curvature). Let M_0 be such a component and let π be the hyperplane containing $S \cap \partial_\varepsilon B(P)$. Observe that $\partial I(M_0) \subset \partial B_\varepsilon(P)$ and that $\partial I(M_0)$ lies above S, in the obvious sense (actually, $I(M)$ lies above S). In particular $\partial I(M_0)$ lies above π. It is well-known that compact minimal submanifolds of $\mathbb{R}^n$ are contained in the convex hull of their boundaries. It follows from this that $I(M_0)$ lies above π. Since $d(P, \pi) > 0$ this implies that $P \notin I(M)$, a contradiction.



Proof of Theorem 3. Suppose that $I\colon M\to A_f$ is minimal. By passing to the universal covering it may be assumed, in view of the uniformization theorem for Riemann surfaces, that M is conformally the unit disc D or $\mathbb{C}$. Since the coordinate functions of a minimal immersion are harmonic and $I_3>0$ ($I=(I_1,I_2,I_3)$) the first alternative must prevail. Therefore, it may be supposed that (see [9] for details)

$$I\colon (D, \lambda|dz|^2)\to A_f, \qquad \lambda=\left(\sum_{i=1}^{3}|\varphi_i|^2\right),$$

where the φ_i's are holomorphic functions satisfying $\sum_{i=1}^{3}\varphi_i^2=0$. The immersion can be recovered by the formulas $I_k(z)=\operatorname{Re}\int_a^z\varphi_k(\xi)\,d\xi$, for a suitable choice of $a\in D$ (see [12] for general facts on minimal surfaces). For the completion of the proof it is necessary to use several classical results on the boundary behavior of holomorphic functions on the unit disc, associated to the names of Fatou, Herglotz, Lusin, Privalov, Marcinkiewiez and Zygmund. Since their proofs (and in some cases their statements) are fairly elaborate, it seems wise just to quote them. An excellent source is the treatise [14]; for those who want to get acquainted with some of the techniques involved, a convenient reference is [6]. Coming back to the proof, denote by $T_\theta\subset D$ an open equilateral triangle of side $\frac{1}{4}$ with a vertex at $e^{i\theta}$. Since $I_3>0$, Herglotz's theorem ([6], p. 38) implies that $\lim\limits_{z\in T_\theta, z\to e^{i\theta}} T_3(z)$ exists for almost all $e^{i\theta}\in S^1$. Pick $e^{i\theta}$ for which this limit exists. Since $0<f(I_2(z))<I_3(z)$ for all $z\in T_\theta$ and f is proper, the set $\{I_2(z)|z\in T_\theta\}$ is bounded. Theorem 1.1 of [14], p. 199, implies that $\lim\limits_{z\in T_\theta, z\to e^{i\theta}} I_2(z)$ exists almost everywhere. By the remark ii) following Theorem 1.10 of [14], p. 204, the limits of $\int_a^z\varphi_k(\xi)\,d\xi$ in T_θ as $z\to e^{i\theta}$ exist almost everywhere ($k=2,3$). By Theorem 2.2i of [14], p. 207, $\int_{T_\theta}|\varphi_2|^2+\int_{T_\theta}|\varphi_3|^2<\infty$ for almost all $e^{i\theta}\in S^1$. In view of the relation $\sum_{i=1}^{3}\varphi_i^2=0$ this implies that $\int_{T_\theta}|\varphi_1|^2<\infty$ almost everywhere. The converse of Theorem 2.2ii above can then be applied to show that

$\lim\limits_{z\in T_\theta, z\to e^{i\theta}} \int\limits_a^z \varphi_1(\xi)\,d\xi$ exists almost everywhere. In particular, $\lim\limits_{z\in T_\theta, z\to e^{i\theta}} I_1(z)$ exists almost everywhere. Let then $e^{i\alpha}\in S^1$ be such that the limits of I_k $(k=1,2,3)$ over T_α as $z\to e^{i\alpha}$ exist. Denote them by a_1, a_2 and a_3, respectively. Consider the function $f\colon D\to\mathbb{R}$, $f(z)=\sum\limits_{k=1}^{3}(I_k(z)-a_k)^2$. As before, for $\varepsilon>0$ small enough, the set $f^{-1}(0,\varepsilon)$ has only compact connected components. In particular it has a component C containing a segment of the form $\{re^{i\alpha},\ b\leqq r<1\}$. Since this path is divergent, C must be non-compact, a contradiction.

References

1. Aminov, Ju.: The exterior diameter of an immersed Riemannian manifold. Mat. Sb. **92 (134)**, 456–460 (1973) [Russian]. Engl. Transl.: Math. USSR-Sb. **21**, 449–454 (1973)
2. Chern, S.: The geometry of G-structures. Bull. Amer. Math. Soc. **72** (1966)
3. Hasanis, Th.: Isometric immersions into spheres. J. Math. Soc. Japan (to appear)
4. Hasanis, Th., Koutroubiotis, D.: Immersions of bounded mean curvature. Arch. Math. (Basel) **33**, 170–171 (1979) and "Addendum" (to appear)
5. Hasanis, Th., Koutroubiotis, D.: Addendum to [4]
6. Hoffman, K.: Banach spaces of Analytic functions. Englewood Cliffs, New Jersey: Prentice-Hall 1962
7. Jones, P.: A complete bounded complex submanifold of $\mathbb{C}^3$. Proc. Amer. Math. Soc. **76**, 305–306 (1979)
8. Jorge, L., Koutroubiotis, D.: An estimate for the curvature of bounded submanifolds. Preprint
9. Jorge, L., Xavier, F.: On the existence of complete bounded minimal surfaces. Bol. Soc. Brasil. Mat. **10**, No 2 (1979)
10. Jorge, L., Xavier, F.: A complete minimal surface in $\mathbb{R}^3$ between two paralel planes. Ann. of Math. (to appear)
11. Omori, H.: Isometric immersions of Riemannian manifolds. J. Math. Soc. Japan **19**, 205–214 (1967)
12. Ossermann, R.: A survey of minimal surfaces. New York-Toronto-London: Van Nostrand Reinhold 1969
13. Wu, H.: On a problem concerning the intrinsic characterization of $\mathbb{C}^n$. Math. Ann. **246**, 15–22 (1979)
14. Zygmund, A.: Trigonometric series (vol. II). Cambridge: At the University Press 1968

Received January 12, 1981