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Trace Class Perturbations and the Absence of Absolutely Continuous Spectra

Barry Simon[1] and Thomas Spencer[2]

[1] Division of Physics, Mathematics, and Astronomy, California Institute of Technology, 253-37, Pasadena, CA 91125, USA; Research partially funded under NSF grant number DMS-8801918
[2] Institute for Advanced Study, Princeton, NJ 80309, USA

Dedicated to Roland Dobrushin

Abstract. We show that various Hamiltonians and Jacobi matrices have no absolutely continuous spectrum by showing that under a trace class perturbation they become a direct sum of finite matrices.

1. Introduction

One of the most versatile tools in the study of scattering theory is the trace class theory which goes back to the basic work of Kato, Kuroda, Rosenblum and Birman, and which was raised to a high art by Pearson. A summary of the basic results can be found in Reed-Simon [13].

We will apply these ideas to the study of stochastic Schrödinger operators and Jacobi matrices to show that, typically, there is no absolutely continuous spectrum (at least in one dimension). At first sight, this seems an unlikely tool since there are no scattering states if σ_{ac} is empty. The point is that the trace class theory is ideal for showing that two operators have the same absolutely continuous spectrum so if we can show that under some kind of trace class perturbation h (or its equivalent) can be transformed to an operator without any absolutely continuous spectrum, we are done. In a different context, this idea has recently been used by Howland [4, 5]. Obviously direct sums of finite matrices have no absolutely continuous spectrum and it is these operators which we will show to be equivalent to the original ones.

A simple example concerns one dimensional Jacobi matrices of the form

$$(hu)(n)=u(n+1)+u(n-1)+v(n)u(n) \tag{1}$$

on $l^2(\mathbb{Z})$. If $v(n)=\lambda\cos(\pi\alpha n)$, then for small λ and suitable α, h has absolutely continuous spectrum [1] but if $v(n)=\lambda\tan(\pi\alpha n)$, there is not absolutely continuous spectrum for any $\lambda\neq 0$ if α is irrational [16]. There have been spectulations that this is due to the fact that tan is unbounded and we will prove that this is so in Sect. 2. So long as

$$\overline{\lim_{n\to\infty}}\,|v(n)| = \overline{\lim_{n\to-\infty}}\,|v(n)| = \infty\ ,$$

h has no absolutely continuous spectrum. Thus, for example, if $v(j)$ are Gaussian random variables, then $\sigma_{ac}(h) = \emptyset$.

The idea will be to pick a subsequence of the set of points where $|v|$ is large and consider the operator where it is taken to infinity (i.e. a Dirichlet boundary condition is put in). We will show the difference of resolvents is trace class.

The intuition behind why there is no absolutely continuous spectrum concerns tunnelling. High barriers make tunnelling difficult and the trace class theory mostly concerns that.

In Sect. 5, we consider potentials having a sequence of intervals, I_k, on which $v \geqq 0$ and whose width $|I_k|$ goes to infinity. Such a potential should produce a Hamiltonian with no absolutely continuous spectrum at negative energies by the same intuition. The barriers here are broad rather than high. To realize this we need to localize in energy the trace class analysis and the key is that if n is in the middle of an interval of length l and Δ is a closed interval in $(-\infty, 0)$, then (with E_Δ the spectral projection of h) $\|E_\Delta \delta_n\|$ is exponentially small in l. Such a priori results are presented in Sect. 4.

Klaus [7] has studied models like those we consider in Sect. 5 although he looked at the simpler question of identifying the essential spectrum (see also Cycon et al. [3]).

It isn't only high barriers that force tunneling. Forbidden energies due to gaps in the spectrum for a Dirichlet operator are also effective. The results in Sect. 4 are stated in this framework. We then apply them in Sect. 6 to obtain a new proof of absence of absolutely continuous spectrum in certain Anderson models. We settle for a result under rather strong conditions on the potential distribution because this section is intended for illustration purposes only.

The conditions under which the ideas of Sect. 6 apply are rather close to those for conditions where Kirch et al. [6] apply ideas of Kotani [9] to prove that σ_{ac} is empty. The methods are rather different, and extend easily to operators on a strip or to where h_0 is replaced by a matrix Δ obeying

$$|\Delta_{ij}| \leqq C\{|i-j|^{4+\varepsilon} + 1\}^{-1}\ .$$

This extension is discussed in Sect. 7.

While we focus on the discrete Jacobi matrix case, these ideas apply to the continuum Schrödinger case also. We illustrate this in Sect. 3 which is an analog of Sect. 2. Analogs of Sect. 4–6 are possible also.

The main results of this paper may be summarized as follows. Let h be given as in (1).

a) If $\overline{\lim_{n\to\infty}}\,|v(n)| = \overline{\lim_{n\to-\infty}}\,|v(n)| = \infty$, then $\sigma_{ac}(h) = \emptyset$.

b) If I_k are intervals of width $l_k \to \infty$ as $|k| \to \infty$ and $v(j) \geqq 0$ if $j \in I_k$, then

$$\sigma_{ac}(h) \cap (-\infty, 0) = \emptyset\ .$$

c) Let I_k be as in (b) and let v_P be a periodic potential. If

$$\max_{j\in I_k} |v(j)-v_P(j)|\to 0$$

as $|k|\to\infty$ then, with probability 1 $\sigma_{ac}(h)\subset\sigma(h_0+v_P)$.

While we have stated results in terms of the absence of absolutely continuous spectrum, it follows by results of Kotani [8] and Simon [17] that when the absolutely continuous spectrum is empty then the Lyaponov exponent is a.e. positive.

2. High Barriers-Jacobi Case

Theorem 2.1. *Let h be the operator*

$$(hu)(n)=u(n+1)+u(n-1)+v(n)u(n)\equiv[(h_0+v)u](n)$$

on $l^2(\mathbb{Z})$. Suppose that

$$\overline{\lim_{n\to\infty}}\,|v(n)|=\overline{\lim_{n\to-\infty}}\,|v(n)|=\infty\ .$$

Then

$$\sigma_{ac}(h)=0\ .$$

Remark and Example. v can be mainly zero or small. This result says that so long as there are arbitrarily high barriers, h has no absolutely continuous spectrum. As an example let f be an arbitrary $\mathbb{R}\cup\{\infty\}$ valued function on S^1 continuous in extended sense with f unbounded. Let α be irrational and let

$$v(n)=f(\alpha n+\theta)\ ,$$

then $\sigma_{ac}(h)=0$.

Lemma 2.2. *Let $\tilde{h}_0$ be an arbitrary matrix with*

$$(\tilde{h}_0)_{i,j}=0\ ,\quad \textit{if}\ \|j-i\|\neq 1\ ,$$

$$|(\tilde{h}_0)_{i,i\pm 1}|\leqq 1\ .$$

Then

$$\|(\tilde{h}_0+v+i)^{-1}\delta_n\|\leqq 3|v(n)+i|^{-1}\ .$$

Proof. Write

$$(\tilde{h}_0+v+i)^{-1}=[1-(\tilde{h}_0+v+i)^{-1}\tilde{h}_0](v+i)^{-1}\ .$$

Since $\|(\tilde{h}_0+i)^{-1}\|\leqq 1$ and $\|\tilde{h}_0\|\leqq 2$, we have that

$$\|(\tilde{h}_0+v+i)^{-1}\delta_n\|\leqq 3|v(n)+i|^{-1}\ .$$

Proof of Theorem 2.1. Let t_n be the matrix with ones in the $(n,n+1)$ and $(n+1,n)$ position and zeros elsewhere so that

$$h_0=\sum_{n=-\infty}^{\infty} t_n\ ,$$

and (h_0-t_n) is a direct sum of an operation on $(-\infty,n]$ and one on $[(n+1),\infty)$.

Pick $\{n_j\}]_{j=-\infty}^{\infty}$ so that $\pm n_{\pm j}\to\infty$ as $j\to\infty$ and so that

$$\sum_j |v(n_j)|^{-1} < \infty \ .$$

Define h_n inductively by

$$h \equiv h_1 = h_0 + v \ , \qquad h_{j+1} \equiv h_j - t_{n_j} - t_{n_{-j}} \ .$$

By Lemma 2.2 and the resolvent identity

$$\|(h_j+i)^{-1} - (h_{j+1}+i)^{-1}\| \leqq 6[|v(n_j)|^{-1} + |v(n_{-j})|^{-1}] \ .$$

Since that operator is rank 4, we have that the trace norm obeys

$$\|(h_j+i)^{-1} - (h_{j+1}+1)^{-1}\|_1 \leqq 24[|v(n_j)|^{-1} + v(n_{-j})|^{-1}] \ .$$

By the choice of the n_j, $(h_j+i)^{-1}$ is Cauchy in trace norm, so since s-$\lim h_j \equiv h_\infty$ exists we have that

$$(h+i)^{-1} - (h_\infty + i)^{-1}$$

is trace class and thus (see e.g. [13, Theorem XI.9])

$$\sigma_{\mathrm{ac}}(h) = \sigma_{\mathrm{ac}}(h_\infty) \ .$$

But h_∞ is a direct sum of finite matrices and so $\sigma_{\mathrm{ac}}(h_\infty) = \phi$. □

3. High Barriers-Schrödinger Case

Theorem 3.1. *Let $V(x)$ be a function on $(-\infty, \infty)$ so that there exist points $\{x_n\}_{n=-\infty}^{\infty}$ with $x_n \to \pm\infty$ and sequence $\{l_n\}_{n=-\infty}^{\infty}$ and $\{h_n\}_{n=-\infty}^{\infty}$ of positive numbers (the half-widths and heights of barriers) and v_0 such that*

(i) $V(x) \geqq -v_0$ *all* x,
(ii) $V(x) \geqq h_n$ *if* $|x-x_n| \leqq l_n$,
(iii) $h_n \to \infty$ *as* $|n| \to \infty$,
(iv) $h_n l_n^2 \to \infty$ *as* $|n| \to \infty$.

Then $\sigma_{\mathrm{ac}}\left(-\dfrac{d^2}{dx^2} + V(x)\right)$ *is empty.*

Remarks. 1. Without much effort one could presumably replace (i) with a condition of the negative part being uniformly locally L^1 or even a condition allowing $\int_n^{n+1} |V| dx$ to diverge but no faster than cn^2.

2. In terms of the intuition of Sect. 1, the conditions (iii), (iv) are quite natural. (iii) says the barriers are high compared to any finite energy and (iv) says that the barriers are effective [since tunnelling probabilities for energies small compared to h_n go as $\exp(-2h_n^{1/2} l_n)$].

Lemma 3.2. *Let $\tilde{H}_0$ be $-d^2/dx^2$ with Dirichlet boundary conditions at some set of points. Let $\tilde{H}_{0,D}$ be the some operator with an additional Dirichlet boundary condition at $x=0$. Let W be a potential obeying*

(i) $W(x) \geqq 0$,
(ii) $W(x) \geqq \lambda^2$ *if* $|x| \leqq l$,

where $\lambda > 10$, $\lambda l > 10$. *Then (with* $\|\cdot\|_1 =$ *trace norm)*

$$\|(\tilde{H}_0 + W + 1)^{-1} - (\tilde{H}_{0,D} + W + 1)^{-1}\|_1 \leqq O(\lambda^{-2}) + O(e^{-2\lambda l}) .$$

Proof. By general principles about quadratic forms, the operator in $\|\cdot\|_1$ is positive so the trace norm is just the trace. By writing the resolvant as an integral of semigroups and writing a path integral for the semigroup kernel (see e.g. [15]) (or by the maximum principle), one sees that this trace only goes up if we replace $\tilde{H}_0$ by $H_0 = -d^2/dx^2$ (on all of $(-\infty, \infty)$) and decrease W. Thus, we can restrict ourselves to $\tilde{H}_0 = H_0$ and $W(x) = \lambda \chi_{(-l,l)}(x)$. Let $G(x, y)$ be the integral kernel of $(H_0 + W + 1)^{-1}$. By the method of images

$$\mathrm{Tr}((H_0 + W + 1)^{-1} - (H_{0,D} + W + 1)^{-1}) = \int_{-\infty}^{\infty} G(x, -x)dx .$$

Let ϕ_+ solve $(H_0 + W + 1)\phi_+ = 0$, l^2 at infinity and normalized so that $\phi_-(x) \equiv \phi_+(-x)$ obeys

$$\phi_+(0)\phi'_-(0) - \phi'_+(0)\phi_-(0) = 1 ,$$

$$2\phi'_+(0)\phi_+(0) = -1 ,$$

with say $\phi_+(0) > 0$. Then

$$\int_{-\infty}^{\infty} G(x, -x)dx = 2\int_0^{\infty} |\phi_+(x)|^2 dx .$$

Straightforward matching analysis shows that

$$\begin{aligned} \phi_+(x) &= ae^{-\lambda x} + be^{-\lambda(x-l)} \quad & 0 \leqq x \leqq l \\ &= ce^{-x} & x \geqq l \end{aligned}$$

with

$$a = (2\lambda)^{-1/2} + O(\lambda^{-1}) + O(e^{-\lambda l}) ,$$

$$b = ae^{-\lambda l}[l + O(\lambda^{-1})] ,$$

$$c = 2b[1 + O(\lambda^{-1})] ,$$

from which it follows that

$$\int_0^{\infty} |\phi_+|^2 dx = O(\lambda^{-2}) + O(e^{-2\lambda l}) ,$$

as required. □

Proof of Theorem 3.1. Pass to a subsequence, also called x_n, with $x_{\pm n} \to \infty$ as $n \to \pm\infty$ and so that

$$\sum |h_n|^{-1} < \infty \quad \text{and} \quad \sum e^{-2h_n\sqrt{l_n}} < \infty .$$

By sucessively adding Dirichlet boundary conditions at $x_0, x_1, x_{-1}, \ldots, x_n, x_{-n}, \ldots$ and using the lemma

$$\|(H_0 + V + E)^{-1} - \tilde{H}_0 + V + E)^{-1}\|_1 < \infty ,$$

where $E=-v_0+1$ and $\tilde{H}_0$ has Dirichlet boundary conditions at $\{x_n\}$. Since $\tilde{H}_0+V$ is a direct sum of operators on finite intervals, it has no absolutely continuous spectrum, so, as in Sect. 2, neither does H_0+V. □

4. Decoupling Local in Energy

As motivated in Sect. 1, we want to prove:

Theorem 4.1. *Let* $\|\alpha\| = \sup_{i=1,\ldots,\nu} |\alpha_i|$ *on* $\mathbb{Z}^\nu$. *Given* $\delta>0$, *there exists* $\varepsilon>0$ *and* C *so that for all intervals* $\Delta=(a,b)\subset\mathbb{R}$, *all potentials* $\{V(\alpha)\}_{\alpha\in\mathbb{Z}^\nu}$ *and all* $\alpha_0\in\mathbb{Z}^\nu$, $l>0$, *the following holds:*

Let $h=h_0+V$ *on* $l^2(\mathbb{Z}^\nu)$. *Suppose that there exists a bounded self-adjoint* A *on* $l^2(\mathbb{Z}^\nu)$ *obeying:*

(i) $Af=hf$ *if* f *vanishes on* $\{\beta|\ \|\beta-\alpha_0\|>l\}$,
(ii) $(a-\delta,b+\delta)\cap\sigma(A)=0$.

Then

$$\|E_\Delta(h)\delta_{\alpha_0}\|\leqq Ce^{-\varepsilon l}\ .$$

Remarks. 1. To understand the theorem, think of the case where $V(\beta)\geqq b+2\nu+\delta$ on $\{\beta|\ \|\beta-\alpha_0\|>l\}$ and let $A=h_0+W$, where W is an extension of V obeying $W(\beta)\geqq b+2\nu+\delta$ on all of $\mathbb{Z}^\nu$.

2. We take the norm we do on $\mathbb{Z}^\nu$ partly for notational convenience sine the "balls" are then cubes. The "right" norm is clearly the Euclidean norm for which the proof can be done also.

3. Section 7 contains an alternate approach to the decoupling expressed by Theorem 4.1.

Lemma 4.2. *For each* δ, *there exists* $\varepsilon>0$ *and* C_1 *so that if* $l=1,2,\ldots$ *is given and a self-adjoint* B *obeys:*

(i) $Bf=h_0f+\tilde{V}f$ *for some* $\tilde{V}$ *and all* f *vanishing on* $\{\alpha|\ \|\alpha\|>l\}$,
(ii) $\sigma(B)\cap(-\delta,\delta)=\emptyset$,

then

$$(\delta_\alpha,(B)^{-1}\delta_0)\leqq C_1e^{-2\varepsilon\|\alpha\|}\quad \textit{if}\ \|\alpha\|\leqq l\ .$$

Proof. This is a simple exercise in the Combes-Thomas [2] method. Define ϱ on $\mathbb{Z}^\nu$ by

$$\begin{aligned}\varrho(\alpha)&=0 &&\text{if } \|\alpha\|\geqq l\\ &=1-\|\alpha\| &&\text{if } \|\alpha\|\leqq l\ .\end{aligned}$$

Given any $g\in l^2(\mathbb{Z}^\nu)$ we decompose $g=f+q$, where f vanishes if $\|\alpha\|>l$, and q vanishes if $\|\alpha\|\leqq l$. If s vanishes for $\|\alpha\|\geqq l$, then

$$(s,Be^{\eta\varrho}q)=(Bs,e^{\eta\varrho}q)=0\ ,$$

since Bs is supported in $\{\alpha|\ \|\alpha\| \leqq l\}$. Thus

$$e^{-\eta\varrho} B e^{\eta\varrho} q = Bq \ .$$

Since $Bf = (h_0 + \tilde{V})f$ and for $|\eta| < 1$

$$e^{-\eta\varrho} h_0 e^{\eta\varrho} = h_0 + C_\eta$$

with $\|C_\eta\| \leqq d_1 |\eta|$ by an elementary calculation (depending on $|\varrho(\alpha) - \varrho(\beta)| \leqq C$, if $|\alpha - \beta| = 1$) we see that

$$e^{-\eta\varrho} A e^{\eta\varrho} = A + \tilde{C}_\eta$$

with $\|\tilde{C}_\eta\| \leqq d_1 |\eta|$ for $|\eta| < 1$. Pick η_0 with $0 < \eta_0 < 1$ and $d_1 |\eta_0| \leqq \frac{1}{2}\delta$. Then $\|\tilde{C}_{\eta_0} A^{-1}\| \leqq \frac{1}{2}$, so by inverting the geometric series $e^{-\eta_0\varrho} A e^{\eta_0\varrho}$ is invertible and

$$\|e^{-\eta_0\varrho} A^{-1} e^{\eta_0\varrho}\| \leqq 2\delta^{-1} \ .$$

Thus

$$\begin{aligned} |(\delta_\alpha, A^{-1}\delta_0)| &= |(e^{\eta_0\varrho}, (e^{-\eta_0\varrho} A^{-1} e^{\eta_0\varrho}) e^{-\eta_0\varrho}\delta_0)| \\ &\leqq 2\delta^{-1} e^{\eta_0(\varrho(\alpha) - \varrho(0))} \ . \end{aligned}$$

But for $\|\alpha\| \leqq l$, $\varrho(\alpha) - \varrho(0) = -\|\alpha\|$. Take $C_1 = 2\delta^{-1}$ and $\varepsilon = \frac{1}{2}\eta_0$. □

Proof of Theorem 4.1. Without loss, we can suppose that $\alpha = 0$ by translation invariance. Let L be larger than l, and let $h_L = h_{0,L} + V$ on $l^2(\{\alpha|\ \|\alpha\| \leqq L\})$ with vanishing boundary conditions. If we prove that

$$\|E_{\bar{\Delta}}(h_L)\delta_0\| \leqq C e^{-\varepsilon l}$$

for all L, the result follows from the continuity of the functional calculus (see [12], Sect. VIII.7).

Let $E_0 \in \bar{\Delta}$ and suppose $h_L\phi = E_0\phi$ is an eigenfunction of h_L. Let

$$\begin{aligned} \tilde{\phi}(\alpha) &= 0 \qquad \|\alpha\| \geqq l \\ &= \phi(\alpha) \quad \|\alpha\| < l \ , \end{aligned}$$

and let $B = A - E_0$. Then $B\tilde{\phi}$ is supported on $\{\alpha|\ \|\alpha\| = l\}$ and $\|B\tilde{\phi}\|^2 \leqq C_2 \sum_{\|\alpha\| = l} |\phi(\alpha)|^2$ with C_2 a dimension dependent constant ($C_2 = \nu$, actually!). Thus

$$\begin{aligned} |\phi(0)| = |[B^{-1}(B\tilde{\phi})](0)| &\leqq \left| \sum_{|\alpha| = l} (\delta_0, B^{-1}\delta_\alpha)(B\tilde{\phi})(\alpha) \right| \\ &\leqq C_1 e^{-2\varepsilon l} \left| \sum_{|\alpha| = l} (B\tilde{\phi})(\alpha) \right| \leqq C_1 e^{-2\varepsilon l} [(2\nu)(2l)^{\nu-1}]^{1/2} \|B\tilde{\phi}\| \\ &\leqq C_1 C_2 e^{-2\varepsilon l} [(2\nu)(2l)^{\nu-1}]^{1/2} \left[\sum_{\|\alpha\| = l} |\phi(\alpha)|^2 \right]^{1/2} . \end{aligned}$$

So

$$\begin{aligned} \|E_{\bar{\Delta}}(h_L)\delta_0\|^2 &= \sum_{E \in \Delta} |\phi_E(0)|^2 \leqq C_1^2 C_2^2 e^{-4\varepsilon l} [(2\nu)(2l)^{\nu-1}]^4 \sum_{E;\, \|\alpha\| = l} |\phi_E(\alpha)|^2 \\ &= C_1^2 C_2^2 e^{-4\varepsilon l} [(2\nu)(2l)^{\nu-1}]^2 \ , \end{aligned}$$

where we have used $\sum_{\text{all } E} |\phi_E(\alpha)|^2 = (\delta_\alpha, \delta_\alpha) = 1$. (Here and above ϕ_E should have an extra index for possible degeneracy. We suppress this index.) Since ε is fixed, we have that

$$\sup_l e^{-2\varepsilon l}[(2\nu)(2l)^{\nu-1}]^2 \equiv C_3^2 < \infty \ ,$$

so that the theorem is proven with

$$C = C_1 C_2 C_3 \ . \quad \square$$

5. Distant Wells

Our main result in this section is the following (with $\|\alpha\| = \max_i |\alpha_i|$ as in the last section):

Theorem 5.1. *From $\mathbb{Z}^\nu$ choose a family, $\{C_n\}_{n=1}^\infty$ of disjoint hypercubes of side l_n. Define*

$$d_n = \min\{\|\alpha - \beta\| \ |\alpha \in C_n, \beta \in C_m \text{ some } m \neq n\} \ .$$

Suppose that for any $\varepsilon > 0$,

$$\sum_n l_n^{(\nu-1)} e^{-\varepsilon d_n} < \infty \ .$$

Let V be a function on $\mathbb{Z}^\nu$ obeying

$$V(\alpha) \geqq 0 \quad \text{if } \alpha \notin \bigcup_n C_n \ .$$

Then $\sigma_{\text{ac}}(h_0 + V) \cap (-\infty, 0) = \emptyset$.

Remarks. 1. If $\nu = 1$, $l_n^{\nu-1} = 1$, i.e. there is no restriction on the size of l_n, except that each is finite! Even if $\nu > 1$, if the d_n grow fast enough, the l_n can be much larger, e.g. $d_n = n^2$, $l_n = e^n$.

2. Think of the C_n as wells. This result says, if the wells are far enough apart, there can't be effective tunnelling out to infinity at negative energies.

Proof. Let S_n be the boundary of the cube of side $l_n + d_n$ with the same center as C_n. Let H_D be the operator obtained by removing all couplings between sets in S_n and the region A_n surrounded by S_n. Then H_D is a direct sum of the finite matrices and an operator on $l^2(\mathbb{Z}^\nu \backslash \cup A_n)$ which is positive so $\sigma_{\text{ac}}(H_D) \cap (-\infty, 0)$. Suppose that we prove for any finite $a < b < 0$,

$$(H - H_D) E_{(a,b)}(H) \equiv C_{(a,b)}$$

is trace class.

It follows by Pearson's theorem [13, Theorem XI.7] that

$$\operatorname*{s\text{-}lim}_{t \to \mp\infty} e^{itH_D}(H - H_D) E_{(a,b)}(H) e^{-itH} E_{\text{ac}}(H) = \tilde{\Omega}^{\pm}$$

exists. Since it intertwines e^{isH_D} with e^{isH}, it defines a map into $\operatorname{Ran} E_{\text{ac}}(H_D) E_{(-a,b)}(H_D) = \{0\}$. Since it is an isometry on $\operatorname{Ran}[E_{(a,b)}(H) E_{\text{ac}}(H)]$, this space must also be zero.

Thus we need only show that $C_{(a,b)}$ is trace class. $H-H_D$ can be written as a sum: $\sum C^{(n)}_{(a,b)}$ one for each set, S_n. S_n has $(2\nu)(l_n+d_n)^{\nu-1}$ points in it. Each can be surrounded by a cube of side d_n on which V is positive. Thus, for each of those points or a neighboring point, α, if $b<-\delta$:

$$\|E_{(a,b)}(H)\delta_\alpha\| \leqq C\exp(-\tfrac{1}{2}\varepsilon d_n)$$

by Theorem 4.1. Since the S_n contribution to $(H-H_D)$ can be written as a sum of $(4\nu)(l_n+d_n)^{\nu-1}$ rank one operators, the trace norm of that S_n contribution is bounded by

$$(4\nu C)(l_n+d_n)^{\nu-1}\exp(-\tfrac{1}{2}\varepsilon d_n) ,$$

so the sum of those terms is finite by the hypothesis. □

This result is of interest because there are examples where h has an interval inside $(-\infty, 0)$ in its spectrum. By using ideas in [3] (Sect. 3.5), based on the work of Klaus [7], one can easily prove:

Theorem 5.2. *Consider the operator h of Theorem 5.1. Suppose that $V(\alpha)=0$ if $\alpha\notin\bigcup_n C_n$. Let*

$$V_n(\alpha)=V(\alpha) \quad \alpha\in C_n$$
$$=0 \quad \alpha\notin C_n$$

and $h_n=h_0+V_n$. Then

$$\sigma_{\mathrm{ess}}(h)=\text{limit points of } \sigma(h_n) .$$

Example 1. In one dimension, one can construct well potentials of this type where $\sigma(h)=[-1,\infty)$ and where the spectrum is purely singular continuous [10]: this uses ideas derived from [11].

Example 2. Take a sequence of wells in two dimensions of constant size strung out in only one dimension. By varying the potential in the wells and using Theorem 5.2, one can arrange that $\sigma(h)=[-1,\infty)$. By Theorem 5.1, $\sigma_{\mathrm{ac}}(h)\subset[0,\infty)$. By using explicit states which move to infinity under the free evolution along a classical path orthogonal to the wells, it is easy to see that $[0,\infty)\subset\sigma_{\mathrm{ac}}(h)$ so $\sigma_{\mathrm{ac}}(h)=[0,\infty)$.

6. Random Potentials

In this final section, we will indicate how the idea of this paper can also be used to prove the absence of absolutely continuous spectrum in some one dimensional random Jacobi matrices. We will prove that

Theorem 6.1. *Let $\nu=1$. Let v be a random potential with $v(n)$* i.i.d.r.v. *with distribution $d\gamma$ which has an interval (c,d) in its support. Then h has no absolutely continuous spectrum.*

Remarks. 1. This result is certainly not new, although our proof requires less machinery than other proofs.

2. Basically, our proof shows that for any Hamiltonian if there are arbitrarily long intervals where v is within δ of a periodic potential for which (a, b) is in a gap, then $(a+\delta, b-\delta)$ is disjoint from the absolutely continuous spectrum of the original h (in accordance with our discussion in Sect. 1).

We need the following lemma, proven in [6]:

Lemma 6.2. *Fix $a>0$. Let $V_{a,l}$ be the potential.*

$$V_{a,l}(n)=a \quad n\equiv 0, \bmod l$$
$$=0 \quad n\not\equiv 0, \bmod l\ .$$

Then for each l sufficiently large there is $\varepsilon_l>0$ so that for $j=1,2,\dots,l-1$, the intervals

$$\left(2\cos\left(\frac{\pi j}{l}\right), 2\cos\left(\frac{\pi j}{l}\right)+\varepsilon_l\right)$$

are disjoint from the spectrum of $h+V_{a,l}$.

Remarks. 1. The energies $\cos\left(\frac{\pi j}{l}\right)$ are the points gaps can open. The point [6] is that large l is weak coupling and one can use a kind of perturbation argument.

2. In fact the ε_l go to zero exponentially [6].

Proof of Theorem 6.1. By the argument in Sect. 5 and Theorem 4.1, it is sufficient to find for any real α, an ε, a sequence $I_{\pm 1},\dots,I_{\pm n},\dots$, of intervals whose length diverges with $I_{\pm n}\to\pm\infty$ as $n\to\infty$ and a sequence of potentials $\{w_n\}_{n=-\infty}^{\infty}$ so that

(a) $(\alpha-\varepsilon, \alpha+\varepsilon)\cap\sigma(h_0+w_n)=0$,
(b) $w_n=v$ on I_n.

Pick $x, y\in(c, d)$ the interval in $\sup\gamma$ with $x<y$. If $\alpha<x-2$, pick $\delta>0$ and note that since $\gamma(x, x+\delta)>0$, there are arbitrarily long intervals where $v(n)>x$, and so we can take $w_n>x$ and thus $\sigma(h+w_n)\subset[x-2, \infty)$. Similarly for $\alpha>x+2$.

For any energy, α, in $[x-2, x+2]$ write $\alpha=z+2\cos\left(\frac{\pi j}{l}\right)$ with $z\in(c, d)$. Use the lemma to be sure that α is in a gap for a potential with values $z-\varepsilon$ and y with y only at sites which are a multiple of l and $\varepsilon<\frac{\varepsilon_l}{n}$ and so that $z-\varepsilon\in(c, d)$. Suppose that the distance from α to $\mathbb{R}\backslash$gaps is $d>0$. There will be arbitrarily long intervals with $|v(n)-y|<\frac{d}{2}$ if n is divisible by 1 and $|v(n)-z+E|<\frac{d}{2}$ otherwise, so the theorem is proven. □

7. Long Range Free Hamiltonians

We want to show that the results of Sect. 6 extends to certain situations where h_0 is not the usual free Hamiltonian but only an operator with some decay off-diagonal. We will consider bounded operators h_0 on $l^2(\mathbb{Z})$ so that for some $l=1,2,\dots$ we have that

$$[x,h_0],\ldots,[x,\ldots[x,h_0]]\ldots] \; (l \textit{ times}) \textit{ are bounded} , \tag{2}$$

where $(xu)(n)=nu(n)$.

Lemma 7.1. *Suppose that*

$$|(h_0)_{ij}| \leqq C(|i-j|^{-l-1-\varepsilon}) \tag{3}$$

for some $\varepsilon>0$. *Then* (2) *holds.*

Proof. The discrete version of Holmgren's estimate says that

$$\|a\| \leqq \max\left[\sup_i \sum_j |a_{ij}|, \sup_j \sum_i |a_{ij}|\right] .$$

Since $[x,[x,[\ldots,h_0]\ldots]_{ij}$ (m times) $=(i-j)^m(h_0)_{ij}$ we see that (3) implies (2). □

Lemma 7.2. *Suppose that* (2) *holds. Let* $h=h_0+v$ *with* v *diagonal. Then*

$$\|[x,[x,\ldots,e^{ith}]\ldots](p \textit{ times})\| \leqq C(1+|t|)^P$$

for $p \leqq l$.

Proof. We know that $[x,e^{ith}]=i\int_0^t e^{isth}[x,h]e^{i(t-s)h}ds$ from which we obtain the results inductively. □

Corollary 7.3. *Suppose that* (2) *holds and that* $f\in C_0^\infty(\mathbb{R})$. *Then*

$$|f(h)_{ij}| \leqq C(|i-j|+1)^{-l} .$$

Proof. By Fourier transforms

$$f(h)=(2\pi)^{-1/2}\int \hat{f}(s)e^{ish}ds ,$$

so that Lemma 7.2 implies that $[x,[x,\ldots,f(h)]\ldots]$ is bounded, from which the result follows. □

Now define

$$\Lambda_L(j)=\{i\in\mathbb{Z}|\ |i-j|\leqq L\} ,$$

$$I_\delta(E_0)\equiv\{E\in\mathbb{R}|\ |E-E_0|\leqq\delta\} .$$

We suppose that $h=h_0+v$; $h_1=h_0+v_1$.

Theorem 7.4. *Fix* ε, δ, E_0, M, *and* $f\in C_0^\infty$ *supported in* $I_{\delta-\varepsilon}(E_0)$, h_0 *and* v. *Suppose that* h_0 *obeys* (2) *with* $l\geqq 1$ *and* $|v|\leqq M$. *Then there is a* C *so that if there is a* v_1 *with*

$$|v_1|\leqq M ,$$

$$|v-v_1|\leqq\varepsilon/2 \quad on \quad \Lambda_L(j) ,$$

$$\sigma(h_1)\cap I_\delta(E_0)=\phi ,$$

then, for $i_1, i_2\in\Lambda_{L/2}(j)$:

$$|f(h)_{j_1j_2}|\leqq CL^{-(2l-1)} .$$

Proof. Define v_2 by

$$v_2(k)=v(k) \quad k\in\Lambda_L(j)$$

$$=v_1(k) \quad k\notin\Lambda_L(j) .$$

Then $|v_2 - v_1| \leqq \varepsilon/2$ on all of $\mathbb{Z}$ so $\sigma(h_2) \cap I_{\delta-\varepsilon}(E_0) = \emptyset$ and thus, $f(h_2) = 0$. Thus by DuHamel's formula for $e^{ith} - e^{ith_2}$:

$$f(h) = (2\pi)^{-1/2} i \int \hat{f}(t) \left(\int_0^t e^{i(t-s)h}(v - v_2) e^{ish_2} ds \right) dt \ .$$

Since $v - v_2$ is supported outside $\Lambda_L(j)$ we see by Corollary 7.3 that

$$\begin{aligned} |f(h)_{j_1 j_2}| &\leqq C \int |\hat{f}(t)| \left(\sum_{k \notin \Lambda_L(j)} \int_0^t (t-s)^l |j_1 - k|^{-l} s^l |j_2 - k|^{-l} ds \right) dt \\ &\leqq C L^{-2l+1} \ . \end{aligned}$$

Theorem 7.5. *Let h_0 obey (2) with $l = 3$. Suppose v is bounded. Suppose there exists $\delta > \varepsilon > 0$, E_0, $\{L_k\}_{k=0}^{\infty}$, $\{j_k\}_{k=-\infty}^{\infty}$, and $\{v_k\}_{k=-\infty}^{\infty}$ so that*

(i) $\pm j_k \to \infty$ *as* $k \to \pm\infty$,
(ii) $L_k \to \infty$,
(iii) $|v - v_k| \leqq \varepsilon/2$ *on* $\Lambda_{L_{|k|}}(j_k)$,
(iv) $\sigma(h_0 + v_k) \cap I_\delta(E_0) = \emptyset$ *all* k.

Then $\sigma_{\mathrm{ac}}(h_0 + v) \cap I_{\delta-\varepsilon}(E_0) = \emptyset$.

Remarks. 1. It suffices that

$$|(h_0)_{ij}| \leqq C|i - j|^{-4-\varepsilon} \ .$$

2. This theorem provides an alternate proof of (c) stated in the introduction.

Proof. By passing to a subset, we can suppose that

$$\sum L_k^{-1} < \infty \ .$$

Break $\mathbb{Z}$ into regions R_k with $R_k = (j_k, j_{k+1})$. Pick a function f supported in $I_{\delta-\varepsilon}(E_0)$. Let

$$\begin{aligned} A_{i_1 i_2} &= f(H)_{i_1 i_2} \quad & i_1, i_2 \text{ in the same } R_k \\ &= 0 & i_1, i_2 \text{ in different } R_k \ . \end{aligned}$$

A is a direct sum of finite matrices, so if we show that $A - f(H)$ is trace class, then $f(H)$ has empty absolutely continuous spectrum. Since f was arbitrary (except for smoothness and support), H has no absolutely continuous spectrum in $I_{\delta-\varepsilon}(E_0)$.

If C is any matrix with

$$\sum_{i,j} |C_{ij}| < \infty \ ,$$

then C is trace class as a sum of rank one operators. Thus it suffices to show that

$$\sum_{i_1, i_2 \text{ in different } R_k} |f(H)_{i_1 i_2}| < \infty \ .$$

The sum is bounded by

$$2 \sum_k \sum_{i_1 < j_k < i_2} |f(H)_{i_1 i_2}| \ .$$

We break the sum over i_1, i_2 into the regions where both are smaller than $L_k/2$ and those where $|i_1 - i_2| \geqq L_k/2$. The points where $i_1, i_2 < L_k/2$ are controlled by Theorem 7.4 so their sum is bounded by

$$C(L_k/2)^2 L_k^{-2l+1} .$$

Since $l \geqq 3$, this is $O(L^{-3})$.

For the points where $|i_1 - i_2| \geqq L_k/2$, we use Corollary 7.3 and bound by

$$\int_{\substack{|x-y| \geqq L/2 \\ x<0 \\ y>0}} \frac{dx\,dy}{|x-y|^l} ,$$

which is $O(L_k^{-(l-2)})$.

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