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Partial Spreads in Finite Projective Spaces and Partial Designs

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1. Introduction

A *partial t-spread* of a projective space **P** is a collection $\mathscr{S}$ of t-dimensional subspaces of **P** of the same order with the property that any point of **P** is contained in at most one element of $\mathscr{S}$. A partial t-spread $\mathscr{S}$ of **P** is said to be a *t-spread* if each point of **P** is contained in an element of $\mathscr{S}$; a partial t-spread which is not a spread will be called *strictly* partial. Partial t-spreads are frequently used for constructions of affine planes, nets, and Sperner spaces (see for instance Bruck and Bose [5], Barlotti and Cofman [2]).

The extension of nets to affine planes is related to the following problem:

When can a partial t-spread $\mathscr{S}$ of a projective space **P** *be embedded into a larger partial t-spread $\mathscr{S}'$ of* **P**?

A strictly partial t-spread $\mathscr{S}$ which cannot be embedded into any partial t-spread $\mathscr{S}'$ of the same projective space as a proper subset will be called a *maximal* strictly partial t-spread (or, shortly a msp t-spread). Mesner [8] and Bruen [6] have proved that if $|\mathscr{S}|$ denotes the cardinality of a msp 1-spread $\mathscr{S}$ of a three-dimensional projective space of finite order q, then

$$q+\sqrt{q}+1\leqq|\mathscr{S}|\leqq q^2-\sqrt{q}.$$

In this paper we shall investigate partial t-spreads in higher dimensional finite projective spaces. In Section 3 we shall generalize Mesner's result for msp t-spreads $\mathscr{S}$ in $(2t+1)$-dimensional projective spaces of order $q=p^a$ for arbitrary prime powers p^a and arbitrary integers t by showing that

$$|\mathscr{S}|\leqq q^{t+1}-\sqrt{q^t}.$$

Although a finite projective space $PG(d,q)$ of dimension $d\geqq 3$ and order q cannot contain t-spreads unless $d+1\equiv 0(t+1)$, any $PG(d,q)$ contains strictly partial t-spreads for $1\leqq t\leqq d$. In Section 4 partial t-spreads are considered in certain finite projective spaces containing no t-spreads. We give an upper bound for the cardinality of partial t-spreads in $PG(a(t+1),q)$ and a lower bound for the cardinality of msp t-spreads in $PG(a(t+1)-2,q)$ for arbitrary integers a, t and arbitrary prime powers q. Examples are provided to illustrate that the bounds obtained are the best possible. In particular it is shown that the upper and lower bounds obtained for msp 1-spreads in finite projective spaces of even dimension are the best possible.

In Section 5 special classes of partial spreads are described.

Finally, in Section 6 partial spreads are applied to the construction of partial designs by generalizing the well known method for the construction of affine planes and Sperner spaces from spreads. Among these partial designs nets and partial geometries can be found and characterized.

2. Definitions and Preliminary Results

Throughout the following investigations let us denote by $PG(d, q)$ the finite desargesian projective space of dimension d and order q. Under a subspace of $\mathbf{P}=PG(d,q)$ we shall understand a *linear* subspace of $\mathbf{P}$, i.e. a subspace of the same order as $\mathbf{P}$. For our considerations the following definitions and results are needed:

Result 2.1. $\mathbf{P}=PG(d,q)$ *contains a t-spread if and only if* $t+1$ *divides* $d+1$. *If* $\mathcal{S}$ *is a t-spread of* $PG(a(t+1)-1, q)$ *then*

$$|\mathcal{S}|=\sum_{i=0}^{a-1} q^{i(t+1)}.$$

For the proof of the first statement see Dembowski [7], p. 29; then the second statement follows easily.

For any two distinct elements V, V' of a partial t-spread $\mathcal{S}$ denote by $\langle V, V'\rangle$ the subspace of $\mathbf{P}$ generated by V and V'. We say that $\mathcal{S}$ *induces a partial spread* in $\langle V, V'\rangle$ if any element of $\mathcal{S}$ having a point in common with $\langle V, V'\rangle$ is contained in $\langle V, V'\rangle$. $\mathcal{S}$ is called *geometric* (Baer [1]) if for any two distinct elements V, V' of $\mathcal{S}$, it induces a partial t-spread in $\langle V, V'\rangle$.

Result 2.2 (see Segre [9]). $\mathbf{P}=PG(d,q)$ *contains a geometric t-spread if and only if* $t+1$ *divides* $d+1$.

For a geometric partial t-spread $\mathcal{S}$ of $\mathbf{P}=PG(d,q)$ let $\mathfrak{I}(\mathcal{S})$ be the following incidence structure: The points of $\mathfrak{I}(\mathcal{S})$ are the elements of $\mathcal{S}$, the blocks the subspaces $\langle V, V'\rangle$ for any two distinct elements V, V' of $\mathcal{S}$, and incidence in $\mathfrak{I}(\mathcal{S})$ is set-theoretic inclusion. Then the following holds:

Result 2.3 (see Segre [9]). *If* $\mathcal{S}$ *is a geometric t-spread of* $PG(a(t+1)-1, q)$, *then* $\mathfrak{I}(\mathcal{S})$ *is a projective space of order* q^{t+1} *and dimension* $a-1$.

For a generalization of this result see Theorem 5.1.

Furthermore, let us recall the definition of the quotient geometry of $\mathbf{P}$ modulo a subspace U (see Dembowski [7], p. 25). If $\mathbf{P}$ is a projective space of dimension d and order q and if U is a t-dimensional subspace of $\mathbf{P}$, then the *quotient geometry* $\mathbf{P}/U$ consists of all subspaces of $\mathbf{P}$ containing U. It can be shown that $\mathbf{P}/U$ is a projective space of dimension $d-t-1$ and order q.

Result 2.4 (Bose and Burton [4], Theorem 2). *Let $\mathfrak{B}$ be a set of points in $\mathbf{P} = PG(d, q)$ with the property that $\mathfrak{B}$ "blocks" the t-dimensional subspaces of $\mathbf{P}$ (i.e. any t-dimensional subspace of $\mathbf{P}$ contains at least one point of $\mathfrak{B}$). Then*

$$|\mathfrak{B}| \geqq q^{d-t} + q^{d-t-1} + \cdots + q + 1.$$

Equality holds if and only if $\mathfrak{B}$ is the point set of a $(d-t)$-dimensional subspace of $\mathbf{P}$.

Finally, for the investigations of partial designs in Section 6 we need some more definitions:

We call a finite incidence structure $\mathfrak{J}$ consisting of a point set $\mathscr{P}$ a block set $\mathscr{B}$, and incidence relation I a *tactical configuration* if each block of $\mathfrak{J}$ is incident with the same number k of points and each point of $\mathfrak{J}$ is incident with the same number r of blocks.

Let $\mathscr{P}$ be a finite set with $|\mathscr{P}| = v$. We denote by $\mathscr{P}(2)$ the set of all subsets $\mathscr{U}$ of $\mathscr{P}$ which contain exactly two elements. Consider a partition

$$\mathscr{A} = \{A_1, \ldots, A_m\}$$

of $\mathscr{P}(2)$. We call $\mathscr{A}$ an *association scheme* on $\mathscr{P}$ if the following condition is satisfied:

(2.1) *Given $\{P, Q\} \in A_n$, the number of $R \in \mathscr{P}$ for which $\{P, R\} \in A_i$ and $\{Q, R\} \in A_j$ depends only on n, i, j, and not on P and Q.*

We call the A_i $(i = 1, \ldots, m)$ the *association classes* of $\mathscr{A}$.

A *partial design* is a tactical configuration $\mathfrak{D} = (\mathscr{P}, \mathscr{B}, \mathrm{I})$ together with an association scheme $\mathscr{A} = \{A_1, \ldots, A_m\}$ on $\mathscr{P}$ such that:

(2.2) *Given $\{P, Q\} \in A_i$, the number $[P, Q]$ of blocks through P and Q depends only on $i \in \{1, \ldots, m\}$, and not on P, Q.*

An *n-fold parallelism* (n a positive integer) of an incidence structure $\mathfrak{J} = (\mathscr{P}, \mathscr{B}, \mathrm{I})$ is an equivalence relation $\|_n$ among the blocks, satisfying:

(2.3) *To any $P \in \mathscr{P}$ and $b \in \mathscr{B}$ there exist exactly n blocks $b' \in B$ such that $P \mathrm{I} b' \|_n b$.*

We call the equivalence classes of $\|_n$ the *parallel classes* of $\|_n$. An 1-fold parallelism will be called simply a *parallelism* $\|$.

A *net* is an incidence structure $\mathfrak{N} = (\mathscr{P}, \mathscr{B}, \mathrm{I})$ with $\mathscr{P} \neq \varnothing \neq \mathscr{B}$ such that

(i) *To every point (block) there exist at least two blocks (points) not incident with it.*

(ii) *$\mathfrak{N}$ admits a parallelism.*

(iii) *If b is a block not parallel to the block c, then b and c have exactly one point in common.*

A *partial geometry* (Bose [3]) is a tactical configuration $\mathfrak{G} = (\mathscr{P}, \mathscr{B}, \mathrm{I})$ satisfying the conditions:

(i) *Two distinct points are incident with at most one common block.*

(ii) *If P is a point not incident with the block b, there exist exactly l blocks b' which are incident with P and have a point in common with b (l a positive integer).*

A partial geometry with $l = 1$ is called a *4-gonal configuration.*

3. An Upper Bound for msp t-Spreads in $PG(2t+1, q)$

Throughout this chapter let $\mathscr{S}$ be a partial t-spread of $\mathbf{P}=PG(2t+1, q)$.

A t-spread of $\mathbf{P}$ contains $q^{t+1}+1$ elements; hence it is reasonable to call the number $q^{t+1}+1-|\mathscr{S}|$ the *deficiency* δ of $\mathscr{S}$.

Denote by $\mathfrak{A}$ the set of points in $\mathbf{P}$ which are not on elements of $\mathscr{S}$ and by $\mathfrak{B}$ the set of hyperplanes in $\mathbf{P}$ which contain no element of $\mathscr{S}$. Call the incidence structure $\mathscr{R}(\mathscr{S}):=(\mathfrak{A}, \mathfrak{B}, \in)$ the *residual geometry* of $\mathscr{S}$. By investigating this residual geometry $\mathscr{R}(\mathscr{S})$ we shall be able to give an upper bound for $|\mathscr{S}|$ (Theorem 3.1). First we shall prove several lemmas:

Lemma 3.1. $|\mathfrak{A}|=|\mathfrak{B}|=\delta(q^t+\dots+q+1)$.

Proof. From the definition of δ it follows that $\mathfrak{A}$ contains exactly $\delta(q^t+\dots+1)$ points.

$\mathbf{P}$ contains

$$h_1=q^{2t+1}+\dots+q+1$$

hyperplanes and any t-dimensional subspace of $\mathbf{P}$ is contained in $q^t+\dots+q+1$ hyperplanes of $\mathbf{P}$. Furthermore no two elements of $\mathscr{S}$ are contained in a common hyperplane. Hence the number of hyperplanes of $\mathbf{P}$ containing one element of $\mathscr{S}$ is equal to

$$h_2=|\mathscr{S}|(q^t+\dots+1)=(q^{t+1}+1-\delta)(q^t+\dots+1).$$

Therefore:

$$|\mathfrak{B}|=h_1-h_2=\delta(q^t+\dots+q+1).$$

Lemma 3.2. *Let U be a t-dimensional subspace of* $\mathbf{P}$. *Then U is skew to all elements of $\mathscr{S}$ (i.e. U is disjoint from all elements of $\mathscr{S}$) if and only if all the $q^t+\dots+q+1$ points of U are contained in $\mathfrak{A}$ or, equivalently all the $q^t+\dots+q+1$ hyperplanes through U are elements of* $\mathfrak{B}$.

Proof. Only the second equivalence has to be proved: If U is skew to all elements of $\mathscr{S}$, then all hyperplanes through U are contained in $\mathfrak{B}$.

Conversely assume that $V\in\mathscr{S}$ intersects U in a subspace of dimension $v\geqq 0$. It follows that

$$\dim\langle U, V\rangle \leqq 2t,$$

therefore there is a hyperplane $\mathbf{H}$ containing U and V. Since $\mathbf{H}$ contains V, $\mathbf{H}$ is not a member of $\mathfrak{B}$.

Lemma 3.3. *Let U be a t-dimensional subspace of* $\mathbf{P}$. *Then U is incident with the same number, say $\lambda=\lambda(U)$, of points of $\mathfrak{A}$ and hyperplanes of* $\mathfrak{B}$.

Proof. The statement is trivial for $U\in\mathscr{S}$; in this case $\lambda=0$.

If $U \notin \mathcal{S}$, let $V_1, \ldots, V_n$ be the elements of $\mathcal{S}$ which intersect U nontrivially. Denote by $v_i \geqq 0$ the dimension of $V_i \cap U$. Then $U - \mathfrak{A}$ contains exactly

$$\sum_{i=1}^{n} (q^{v_i} + \cdots + 1)$$

points. Since $\dim \langle U, V_i \rangle = 2t - v_i$, there exist exactly $q^{v_i} + \cdots + q + 1$ hyperplanes containing U and V_i. Furthermore V_i and V_j are not contained in a common hyperplane whenever $i \neq j$. Hence the number of hyperplanes containing U and not contained in $\mathfrak{B}$ is equal to

$$\sum_{i=1}^{n} (q^{v_i} + \cdots + 1).$$

Since the number of points incident with U is equal to the number of hyperplanes through U, Lemma 3.3 is proved.

Let $\mathbf{H}$ be a hyperplane of $\mathbf{P}$ containing exactly v elements of $\mathcal{S}$ where v is 0 or 1. Then $\mathbf{H}$ intersects the remaining $q^{t+1} + 1 - \delta - v$ elements of $\mathcal{S}$ in pairwise skew $(t-1)$-dimensional subspaces. Therefore the number h of points of $\mathbf{H}$ which are contained in an element of $\mathcal{S}$ is

$$\begin{aligned} h &= v(q^t + \cdots + 1) + (q^{t+1} + 1 - \delta - v)(q^{t-1} + \cdots + 1) \\ &= v\, q^t + q^{2t} + \cdots + q^{t+1} + q^{t-1} + \cdots + q + 1 - \delta(q^{t-1} + \cdots + 1). \end{aligned}$$

Hence we have

Lemma 3.4. *Let* $\mathbf{H}$ *be a hyperplane of* $\mathbf{P}$. *Then:*
If $\mathbf{H} \in \mathfrak{B} (\Leftrightarrow v = 0)$ *then* $\mathbf{H}$ *contains* $q^t + \delta(q^{t-1} + \cdots + 1)$ *points of* $\mathfrak{A}$;
if $\mathbf{H} \notin \mathfrak{B} (\Leftrightarrow v = 1)$ *then* $\mathbf{H}$ *contains* $\delta(q^{t-1} + \cdots + q + 1)$ *points of* $\mathfrak{A}$.

Dually it follows

Lemma 3.5. *Let* P *be a point of* $\mathbf{P}$. *Then:*
If $P \in \mathfrak{A}$ *then* P *is contained in* $q^t + \delta(q^{t-1} + \cdots + 1)$ *hyperplanes of* $\mathfrak{B}$;
if $P \notin \mathfrak{A}$ *then* P *is contained in* $\delta(q^{t-1} + \cdots + 1)$ *hyperplanes of* $\mathfrak{B}$.

Now we are ready to state the following theorem:

Theorem 3.1. *Let* $\mathcal{S}$ *be a* msp *t-spread of* $\mathbf{P} = PG(2t+1, q)$. *Then:*

$$|\mathcal{S}| \leqq q^{t+1} - \sqrt{q^t}.$$

Furthermore: If $t > 1$ *then*

$$|\mathcal{S}| < q^{t+1} - \sqrt{q^t}.$$

Proof. Let U be a t-dimensional subspace of $\mathbf{P}$ which contains exactly λ points of $\mathfrak{A}$ and is (by Lemma 3.3) contained in exactly λ hyperplanes of $\mathfrak{B}$. According to Lemma 3.1 there are

$$\delta(q^t + \cdots + 1) - \lambda$$

points of $\mathfrak{A}$ not in U. From these points

$$q^t+\delta(q^{t-1}+\cdots+1)-\lambda$$

lie on each of the λ hyperplanes of $\mathfrak{B}$ through U in view of Lemma 3.4. Let x be the average number of hyperplanes in $\mathfrak{B}$ containing U and a point $P\in\mathfrak{A}-U$. Then:

$$[\delta(q^t+\cdots+1)-\lambda]\,x=\lambda[q^t+\delta(q^{t-1}+\cdots+1)-\lambda].$$

This together with

$$x\leqq q^{t-1}+\cdots+1$$

yields the inequality:

$$(q^{t-1}+\cdots+1)[\delta(q^t+\cdots+1)-\lambda]\geqq\lambda[q^t+\delta(q^{t-1}+\cdots+1)-\lambda],$$

which reduces to

$$(3.1)\quad [\lambda-\delta(q^{t-1}+\cdots+1)][\lambda-(q^t+\cdots+1)]\geqq 0.$$

Since $\mathscr{S}$ is maximal, U contains fewer than $q^t+\cdots+q+1$ points of $\mathfrak{A}$; otherwise $\{U\}\cup\mathscr{S}$ would be a partial t-spread containing $\mathscr{S}$ as a proper subset contradicting our assumption on $\mathscr{S}$. That is

$$\lambda-(q^t+\cdots+1)<0$$

which implies together with (3.1)

$$(3.2)\quad \lambda\leqq\delta(q^{t-1}+\cdots+1).$$

Let λ_{ij} be the number of points of $\mathfrak{A}$ which are in the intersection of the i-th and j-th hyperplane of $\mathfrak{B}$, and let $\bar{\lambda}$ be the average number of the λ_{ij} over the

$$\delta(q^t+\cdots+1)[\delta(q^t+\cdots+1)-1]$$

ordered pairs (i,j) for $i\neq j$. There are $\delta(q^t+\cdots+1)$ points of $\mathfrak{A}$ altogether, each of which is on

$$q^t+\delta(q^{t-1}+\cdots+1)$$

hyperplanes of $\mathfrak{B}$ (Lemma 3.5). Counting the incidences in two ways we obtain:

$$\bar{\lambda}\,\delta(q^t+\cdots+1)[\delta(q^t+\cdots+1)-1]$$
$$=\delta(q^t+\cdots+1)[q^t+\delta(q^{t-1}+\cdots+1)][q^t+\delta(q^{t-1}+\cdots+1)-1].$$

$\mathscr{S}$ is not a spread, hence $\delta\neq 0$, therefore:

$$(3.3)\quad \bar{\lambda}[\delta(q^t+\cdots+1)-1]=[q^t+\delta(q^{t-1}+\cdots+1)][q^t+\delta(q^{t-1}+\cdots+1)-1].$$

The next step is to provide an upper bound for $\bar{\lambda}$.

Let $\mathbf{M}:=\mathbf{H}_i\cap\mathbf{H}_j$, where $\mathbf{H}_i$ and $\mathbf{H}_j$ are the i-th and j-th hyperplanes in $\mathfrak{B}$ for $i\neq j$. $\mathbf{M}$ contains no element of $\mathscr{S}$. By (3.2) any t-dimensional subspace U contained in $\mathbf{M}$ is incident with

$$\lambda(U)\leqq\delta(q^{t-1}+\cdots+1)$$

points of $\mathfrak{A}$. Counting the incident pairs (P, U) where $P \in \mathfrak{A} \cap \mathbf{M}$ and U is a t-dimensional subspace of $\mathbf{M}$ we get:

$$R\,\overline{\lambda(U)} = \bar{\lambda}\, S$$

where $\overline{\lambda(U)}$ is the average number of the $\lambda(U)$'s, R is the number of the t-dimensional subspaces of $\mathbf{M}$, and S the number of the t-dimensional subspaces of $\mathbf{M}$ through a point $Q \in \mathfrak{A} \cap \mathbf{M}$. It is easy to verify that

$$RS^{-1} = T := (q^{2t-1} + \cdots + 1)(q^t + \cdots + 1)^{-1},$$

hence

(3.4) $\bar{\lambda} \leqq T\delta(q^{t-1} + \cdots + 1)$.

Also we remark that:

(3.5) *From* $t \geqq 1$ *it follows that* $T \geqq 1$; *if* $t > 1$ *then* $T > 1$.

The statements (3.3) and (3.4) imply:

$$T\delta(q^{t-1} + \cdots + 1)[\delta(q^t + \cdots + 1) - 1]$$
$$\geqq [q^t + \delta(q^{t-1} + \cdots + 1)][q^t + \delta(q^{t-1} + \cdots + 1) - 1].$$

By solving this quadratic inequality for δ and disregarding negative solutions, we obtain

(3.6) $\delta \geqq 1 + \dfrac{(T-1) + \sqrt{(T-1)^2 + 4(T-1)\, q^t + 4\, q^{3t}}}{2\, q^t}$.

(3.5) and (3.6) imply:

$$\delta \geqq \sqrt{q^t} + 1,$$

moreover for $t > 1$:

$$\delta > \sqrt{q^t} + 1.$$

This finishes the proof of Theorem 3.1.

Remark. It should be mentioned that this proof is a generalization of the proof of Theorem 1 in Mesner [8].

4. Partial *t*-Spreads in Finite Projective Spaces Containing no *t*-Spreads

According to Result 2.1 a finite projective space $\mathbf{P}$ of dimension d contains a t-spread if and only if

$$d \equiv -1 \bmod (t+1).$$

In this section we consider partial t-spreads in $\mathbf{P} = PG(d, q)$ with

$$d \not\equiv -1 \bmod (t+1).$$

In particular we obtain the best possible upper bound for partial t-spreads in $PG(d, q)$ where $d \equiv 0 \bmod (t+1)$ and the best possible lower bound for maximal partial t-spreads in the case $d \equiv -2 \bmod (t+1)$. A combination of these results shows that our upper and lower bounds for maximal partial 1-spreads in finite projective spaces of even dimension is the best possible.

Theorem 4.1. *Let $\mathscr{S}$ be a partial t-spread in $\mathbf{P} = PG(a(t+1), q)$, where a is a positive integer. Then:*

$$|\mathscr{S}| \leqq \sum_{i=1}^{a-1} q^{i(t+1)+1} + 1.$$

Proof. Since the number v of points of $\mathbf{P}$ is equal to

$$v = q^{at+a} + q^{at+a-1} + \cdots + 1,$$

we have

$$\text{(4.1)} \quad |\mathscr{S}| \leqq (q^{at+a} + \cdots + 1)(q^t + \cdots + 1)^{-1}$$
$$= q^{(a-1)(t+1)+1} + q^{(a-2)(t+1)+1} + \cdots + q^{t+2} + q + (q^t + \cdots + 1)^{-1}.$$

Hence:

$$|\mathscr{S}| \leqq q^{(a-1)(t+1)+1} + \cdots + q^{t+2} + q.$$

Let us call the number

$$\delta := \delta(\mathscr{S}) := q^{(a-1)(t+1)+1} + \cdots + q^{t+2} + q - |\mathscr{S}|$$

the *deficiency* of $\mathscr{S}$. Denote by $\mathfrak{A}$ the set of those points in $\mathbf{P}$ which are not incident with an element of $\mathscr{S}$. (Note that δ and $\mathfrak{A}$ are defined similarly as above in Section 3.) By (4.1) we have

$$\text{(4.2)} \quad |\mathfrak{A}| = \delta(q^t + \cdots + 1) + 1.$$

Suppose that the statement of Theorem 4.1 is false. Then $\mathscr{S}$ has deficiency δ where

$$\text{(4.3)} \quad 0 \leqq \delta \leqq q - 2.$$

Suppose that this is the case. We shall show first that there is a hyperplane $\mathbf{H}$ of $\mathbf{P}$ containing at most $\delta(q^{t-1} + \cdots + 1)$ points of $\mathfrak{A}$.

For this denote by x the average number of points of $\mathfrak{A}$ in a hyperplane of $\mathbf{P}$. Using (4.2) we get:

$$(q^{at+a} + \cdots + 1)\, x = [\delta(q^t + \cdots + 1) + 1](q^{at+a-1} + \cdots + 1).$$

It follows that

$$x = \delta(q^{t-1} + \cdots + 1) + \frac{\delta q^t (q^{(a-1)(t+1)} + \cdots + 1) + q^{at+a-1} + \cdots + 1}{q^{at+a} + \cdots + 1}.$$

(4.3) implies

$$\frac{\delta q^t (q^{(a-1)(t+1)} + \cdots + 1) + q^{at+a-1} + \cdots + 1}{q^{at+a} + \cdots + 1} < 1,$$

hence

$$x<\delta(q^{t-1}+\cdots+1)+1.$$

Therefore there must be a hyperplane **H** of **P** which contains exactly m points of $\mathfrak{A}$, where

(4.4) $\quad 0\leqq m\leqq\delta(q^{t-1}+\cdots+1).$

Now consider a hyperplane **H** containing m points of $\mathfrak{A}$ where $m\leqq\delta(q^{t-1}+\cdots+1)$. Let $s_{\mathbf{H}}$ be the number of elements of $\mathscr{S}$ whose points are all contained in **H**. Then

$$\begin{aligned}q^{at+a-1}+\cdots+q+1&=s_{\mathbf{H}}(q^t+\cdots+1)+(|\mathscr{S}|-s_{\mathbf{H}})(q^{t-1}\cdots+1)+m\\&=s_{\mathbf{H}}\,q^t+|\mathscr{S}|\,(q^{t-1}+\cdots+1)+m.\end{aligned}$$

Since

$$|\mathscr{S}|=q^{(a-1)(t+1)+1}+\cdots+q^{t+2}+q-\delta,$$

it follows that

$$\begin{aligned}&q^{at+a-1}+\cdots+1\\&=s_{\mathbf{H}}\,q^t+(q^{(a-1)(t+1)+1}+\cdots+q^{t+2}+q-\delta)(q^{t-1}+\cdots+1)+m.\end{aligned}$$

From this we deduce that

$$q^{(a-1)(t+1)}+q^{(a-2)(t+1)}+\cdots+q^{t+1}+1=s_{\mathbf{H}}\,q^t-\delta(q^{t-1}+\cdots+1)+m.$$

Therefore:

$$s_{\mathbf{H}}=\frac{q^{(a-1)(t+1)}+\cdots+q^{t+1}+1+\delta(q^{t-1}+\cdots+1)-m}{q^t},$$

hence

(4.5) $\quad q^t\mid\delta(q^{t-1}+\cdots+1)+1-m.$

By (4.4) we know that

$$\delta(q^{t-1}+\cdots+1)+1-m>0$$

and so

(4.6) $\quad q^t\leqq\delta(q^{t-1}+\cdots+1)+1-m.$

According to (4.2) and (4.4) we know that $m\geqq 0$ and $\delta\leqq q-2$; therefore

$$q^t\leqq\delta(q^{t-1}+\cdots+1)+1-m\leqq q^t-(q^{t-1}+\cdots+1)<q^t.$$

This contradiction proves Theorem 4.1.

The next statement shows that Theorem 4.1 is the best possible. We even prove a little more:

Theorem 4.2. *Let* **P** *be the* $PG(a(t+1)+r, q)$, *where* a *and* r *are positive integers with* $0 \leqq r \leqq t-1$. *Then* **P** *contains a maximal partial* t-*spread* $\mathcal{S}$ *with*

$$|\mathcal{S}| = \sum_{i=1}^{a-1} q^{i(t+1)+r+1} + 1.$$

Proof. The proof is given by using induction on a.

For $a=1$ the assertion is trivial since clearly $PG(t+1+r, q)$ contains a t-dimensional subspace U. If $\mathcal{S} := \{U\}$, then $\mathcal{S}$ is maximal since $r \leqq t-1$ and so $t+1+r \leqq 2t$.

Consider now the case $a>1$ and assume that the assertion of Theorem 4.2 is true for $a-1$.

Let **P** be embedded as a subspace in

$$\Sigma = PG(2[(a-1)(t+1)+r]+1, q).$$

Σ contains an $((a-1)(t+1)+r)$-spread $\mathcal{S}'$. One can choose $\mathcal{S}'$ such that **P** contains an element, U_0, of $\mathcal{S}'$. By Result 2.1 we know that

$$(4.7) \quad |\mathcal{S}' - \{U_0\}| = q^{(a-1)(t+1)+r+1}$$

Next we show that any subspace $U_i \in \mathcal{S}' - \{U_0\}$ intersects **P** in a t-dimensional subspace. Namely:

(a)
$$\begin{aligned} \dim(U_i \cap \mathbf{P}) &= \dim U_i + \dim \mathbf{P} - \dim \langle U_i, \mathbf{P}\rangle \\ &\geqq \dim U_i + \dim \mathbf{P} - \dim \Sigma \\ &= (a-1)(t+1)+r+a(t+1)+r-2[(a-1)(t+1)+r]-1 \\ &= t; \end{aligned}$$

(b) on the other hand suppose that $\dim(U_i \cap \mathbf{P}) \geqq t+1$. Then it follows:

$$\begin{aligned} \dim[(U_i \cap \mathbf{P}) \cap U_0] &= \dim(U_i \cap \mathbf{P}) + \dim U_0 - \dim \langle (U_i \cap \mathbf{P}), U_0 \rangle \\ &\geqq \dim(U_i \cap \mathbf{P}) + \dim U_0 - \dim \mathbf{P} \\ &\geqq t+1+(a-1)(t+1)+r-a(t+1)-r \\ &= 0. \end{aligned}$$

This is a contradiction since otherwise U_i and U_0 would have a point in common. Let

$$\mathcal{S}_1 := \{V_i \mid V_i = U_i \cap P,\ U_i \in \mathcal{S}' - \{U_0\}\}.$$

U_0 has dimension $(a-1)(t+1)+r$ and hence U_0 contains – by induction – a maximal partial t-spread $\mathcal{S}_2$ with

$$(4.8) \quad |\mathcal{S}_2| = q^{(a-2)(t+1)+r+1} + \dots + q^{t+2+r} + 1.$$

Finally put

$$\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2.$$

By using (4.7) and (4.8) it follows that there are

$$\sum_{i=1}^{a-1} q^{i(t+1)+r+1}+1$$

elements in $\mathscr{S}$. Any point of $\mathbf{P}$ outside U_0 is contained in an element of $\mathscr{S}_1$ and $\mathscr{S}_2$ is a maximal partial t-spread of U_0; this implies that $\mathscr{S}$ is a maximal partial t-spread of $\mathbf{P}$. Since $\mathscr{S}$ has the required cardinality, Theorem 4.2 is proved.

Our next aim is to give lower bounds for maximal partial t-spreads in certain special cases:

Theorem 4.3. *Let $\mathscr{S}$ be a maximal partial t-spread of $\mathbf{P}=PG(b(t+1)-2,q)$ where b is a positive integer. Then:*

$$|\mathscr{S}|\geqq \sum_{i=0}^{b-2} q^{i(t+1)}.$$

Proof. Since $\mathscr{S}$ is maximal, the set $\mathscr{P}(\mathscr{S})$ consisting of all points of $\mathbf{P}$ which are incident with an element of $\mathscr{S}$, blocks the t-dimensional subspaces of $\mathbf{P}$. According to Result 2.4 we have:

$$|\mathscr{P}(\mathscr{S})|\geqq q^{b(t+1)-2-t}+q^{b(t+1)-3-t}+\cdots+q+1$$
$$=q^{(b-1)(t+1)-1}+\cdots+q+1.$$

Therefore it is

$$|\mathscr{S}|\geqq q^{(b-2)(t+1)}+q^{(b-3)(t+1)}+\cdots+q^{t+1}+1.$$

Theorem 4.4. *Let $\mathbf{P}$ be the $PG(a(t+1)+r,q)$ where a and r are positive integers with $0\leqq r\leqq t-1$. Then $\mathbf{P}$ contains a maximal partial t-spread $\mathscr{S}$ with*

$$|\mathscr{S}|=\sum_{i=0}^{a-1} q^{i(t+1)}.$$

Proof. Let U be a subspace of dimension $a(t+1)-1$ of $\mathbf{P}$. By Result 2.1 the space U contains a t-spread $\mathscr{S}$ with

$$|\mathscr{S}|=\sum_{i=0}^{a-1} q^{i(t+1)}.$$

$\mathscr{S}$ is a maximal partial t-spread of $\mathbf{P}$ since any t-dimensional subspace of $\mathbf{P}$ intersects U.

Remark. The assertion of Theorem 4.4 for $r=t-1$ and $a=b-1$ shows that the result of Theorem 4.3 is the best possible.

Some of the partial t-spreads constructed in Theorem 4.4 can be characterized as follows:

Theorem 4.5. *Let $\mathscr{S}$ be a maximal partial t-spread of $\mathbf{P}=PG(b(t+1)-2,q)$ with*

$$|\mathscr{S}|=\sum_{i=0}^{b-2} q^{i(t+1)}.$$

Then $\mathcal{S}$ is a t-spread of an appropriate $((b-1)(t+1)-1)$-dimensional subspace U of **P**.

Proof. If $|\mathcal{S}| = q^{(b-2)(t+1)} + \cdots + q^{t+1} + 1$, then

$$|\mathcal{P}(\mathcal{S})| = q^{(b-1)(t+1)-1} + q^{(b-1)(t+1)-2} + \cdots + q + 1.$$

Furthermore since $\mathcal{S}$ is maximal, $\mathcal{P}(\mathcal{S})$ blocks the t-dimensional subspaces of **P**. From Result 2.4 it follows that $\mathcal{P}(\mathcal{S})$ is the point set of an $((b-1)(t+1)-1)$-dimensional subspace U of **P**. Hence $\mathcal{S}$ is a t-spread of U.

By applying Theorems 4.1–4.5 the following properties of maximal partial 1-spreads in finite projective spaces of even dimension can be obtained:

Theorem 4.6. *Let $\mathcal{S}$ be a maximal partial 1-spread of* $\mathbf{P} = PG(2a, q)$. *Then:*

(a) $q^{2a-2} + q^{2a-4} + \cdots + q^2 + 1 \leqq |\mathcal{S}| \leqq q^{2a-1} + q^{2a-3} + \cdots + q^3 + 1$.

(b) *For any positive integer a there are examples $\mathcal{S}_1$ and $\mathcal{S}_2$ of maximal partial 1-spreads in* **P** *such that*

$$|\mathcal{S}_1| = q^{2a-2} + q^{2a-4} + \cdots + q^2 + 1,$$
$$|\mathcal{S}_2| = q^{2a-1} + q^{2a-3} + \cdots + q^3 + 1.$$

(c) *A maximal partial 1-spread of* **P** *has cardinality*

$$q^{2a-2} + q^{2a-4} + \cdots + q^2 + 1$$

if and only if it is an 1-spread of a hyperplane of **P**.

5. On *n*-Uniform and *n*-Geometric Partial Spreads

There is an enormous variety of nonisomorphic partial t-spreads in projective spaces and it seems an extremely difficult problem to establish a satisfactory classification of these structures. In this section we shall describe two families of partial t-spreads satisfying certain additional combinatorial conditions. The first class represents a generalization of the geometric t-spreads introduced by Baer [1].

Definition. A geometric partial t-spread $\mathcal{S}$ of **P** is called *n-geometric* if for any two different elements V, V' of $\mathcal{S}$ the following condition is satisfied:

$\langle V, V' \rangle$ contains exactly $n+1$ elements of $\mathcal{S}$.

The second class is defined as follows:

Definition. Let $\mathcal{S}$ be a partial t-spread of **P**. Then $\mathcal{S}$ is called *n-uniform* if any $(t+1)$-dimensional subspace of **P** containing an element of $\mathcal{S}$ intersects exactly n $(n \geqq 0)$ additional elements of $\mathcal{S}$ in a (necessarily unique) point.

The following theorems provide us with some insight into the nature of the partial spreads defined above.

It is possible to show that geometric maximal partial t-spreads are n-geometric. We even prove a little more:

Theorem 5.1. *Let $\mathcal{S}$ be a geometric maximal partial t-spread of $\mathbf{P}=PG(d,q)$ with $d \geqq 2t+1$. Then the incidence structure $\mathfrak{I}(\mathcal{S})$ defined in Section 2 consists of the points and lines of a projective space of dimension at least one.*

Proof. It is clear that any two distinct points of $\mathfrak{I}(\mathcal{S})$ are joined by a unique block of $\mathfrak{I}(\mathcal{S})$. Therefore the blocks of $\mathfrak{I}(\mathcal{S})$ are also called *lines*.

Next we show that the span of any three non-collinear points of $\mathfrak{I}(\mathcal{S})$ is a projective plane. Let $\mathbf{U}$ be a subspace of dimension $3t+2$ of $\mathbf{P}$ which is generated by elements of $\mathcal{S}$. Let $\langle V_1, V_2\rangle$ and $\langle V_3, V_4\rangle$ be two different lines of $\mathfrak{I}(\mathcal{S})$ which are contained in $\mathbf{U}$, where $V_i \in \mathcal{S}$ $(i=1,\dots,4)$. Hence:

$$\dim(\langle V_1, V_2\rangle \cap \langle V_3, V_4\rangle) \geqq t.$$

Since $\mathcal{S}$ is maximal, $\langle V_1, V_2\rangle \cap \langle V_3, V_4\rangle$ contains a point Q which is incident with an element $V_0 \in \mathcal{S}$. Since $\mathcal{S}$ is geometric:

$$V_0 \subseteq \langle V_1, V_2\rangle \cap \langle V_3, V_4\rangle.$$

Assume that $\dim(\langle V_1, V_2\rangle \cap \langle V_3, V_4\rangle) \geqq t+1$. The space $\langle V_1, V_2\rangle$ contains an element $V \in \mathcal{S}$ different from V_0. Since $\dim\langle V_1, V_2\rangle = 2t+1$, V would intersect $\langle V_1, V_2\rangle \cap \langle V_3, V_4\rangle$ and hence would be contained in $\langle V_1, V_2\rangle \cap \langle V_3, V_4\rangle$. Therefore:

$$\langle V, V_0\rangle = \langle V_1, V_2\rangle = \langle V_1, V_2\rangle \cap \langle V_3, V_4\rangle$$

and so

$$\langle V_1, V_2\rangle = \langle V_3, V_4\rangle.$$

This is a contradiction. Therefore any two lines of $\mathfrak{I}(\mathcal{S})$ contained in U intersect in an unique point of $\mathfrak{I}(\mathcal{S})$.

Since $\mathcal{S}$ is maximal, $\mathcal{S}$ induces a maximal partial t-spread in any $\langle V_1, V_2\rangle$. It can be checked easily that a maximal partial t-spread of $PG(2t+1, q)$ contains more than two elements.

Hence $\mathfrak{I}(\mathcal{S})$ is a projective space of dimension at least one.

Since in a finite projective space any line is incident with the same number, say $k+1$, of points, it follows as a corollary that under the assumptions of Theorem 5.1 $\mathcal{S}$ is n-geometric with $n=k$.

In our next statement we characterize the 1-uniform partial t-spreads.

Theorem 5.2. *Let $\mathcal{S}$ be a partial t-spread of $\mathbf{P}=PG(d,q)$. Then $\mathcal{S}$ is 1-uniform if and only if the following conditions hold:*

(i) *Any three elements of $\mathcal{S}$ generate a $(3t+2)$-dimensional subspace of $\mathbf{P}$, and*

(ii) *for any $V_0 \in \mathcal{S}$ the set $\{\langle V_0, V\rangle \mid V \in \mathcal{S} - \{V_0\}\}$ is a t-spread of the quotient geometry $\mathbf{P}/V_0$.*

Proof. First let us assume that $\mathcal{S}$ is a partial 1-uniform t-spread of $\mathbf{P}$.

Suppose that an element $V_0 \in \mathcal{S}$ has a point Q in common with $\langle V_1, V_2 \rangle$ where V_1 and V_2 are two different elements of $\mathcal{S}$; $V_1 \neq V_0 \neq V_2$. Then there is a line g of $\langle V_1, V_2 \rangle$ containing Q and intersecting both V_1 and V_2. Hence there is a $(t+1)$-dimensional subspace U containing V_1 and intersecting V_0 and V_2. Therefore $\mathcal{S}$ cannot be 1-uniform. Hence $\langle V_0, V_1, V_2 \rangle$ has dimension $3t+2$.

The proof of (ii) follows from the fact that any $(t+1)$-dimensional subspace U through $V_0 \in \mathcal{S}$ intersects exactly one element $V_i \neq V_0$ of $\mathcal{S}$; hence U is contained in exactly one subspace $\langle V_i, V_0 \rangle$ of $\mathbf{P}$.

On the other hand let $\mathcal{S}$ be a partial t-spread of $\mathbf{P}$ with properties (i) and (ii). Let V_0 be an element of $\mathcal{S}$. Assume that a $(t+1)$-dimensional subspace U containing V_0 intersects two different elements V_1 and V_2 of $\mathcal{S}$. Then

$$\dim \langle V_0, V_1, V_2 \rangle < 3t+2.$$

This contradiction to (i) shows that $n \leqq 1$.

But according to (ii) any $(t+1)$-dimensional subspace U containing V_0 intersects an additional element of $\mathcal{S}$. Thus $n \geqq 1$, hence $n=1$ and $\mathcal{S}$ is 1-uniform.

Corollary. *If $\mathbf{P}=PG(d,q)$ contains a 1-uniform partial t-spread then*

$$d \equiv -1 \bmod (t+1).$$

Proof. In view of (ii) the projective space $\mathbf{P}/V_0$ contains a t-spread. Hence by Result 2.1:

$$\dim \mathbf{P}/V_0 \equiv -1 \bmod (t+1).$$

Since $\dim \mathbf{P}/V_0 = d-(t+1)$, it follows that

$$\dim \mathbf{P} \equiv -1 \bmod (t+1).$$

It is easy to verify the following propositions:

Proposition 5.1. *Any t-spread $\mathcal{S}$ of $\mathbf{P}=PG(d,q)$ is n-uniform with $n=q^{t+1}$.*

Namely: Let U be a $(t+1)$-dimensional subspace of $\mathbf{P}$ containing $V \in \mathcal{S}$. Since $\mathcal{S}$ is a spread, any of the q^{t+1} points of $U-V$ is contained in exactly one element of $\mathcal{S}-\{V\}$.

Proposition 5.2. *Let $\mathcal{S}$ be a n-geometric partial t-spread of $\mathbf{P}$ with the property that any $(t+1)$-dimensional subspace of $\mathbf{P}$ containing an element $V \in \mathcal{S}$ intersects at least one element $V' \in \mathcal{S}-\{V\}$. Then $\mathcal{S}$ is n-uniform.*

Finally we prove

Theorem 5.3. *Let $\mathcal{S}$ be a partial t-spread in $\mathbf{P}=PG(d,q)$ with $d \geqq 3t+2$. If $\mathcal{S}$ is maximal, geometric, and n-uniform then $\mathcal{S}$ is a t-spread.*

Proof. Step 1. For all $V \in \mathcal{S}$ the set

$$\mathcal{S}_V := \{\langle V, V_i \rangle \mid V_i \in \mathcal{S} - \{V\}\}$$

is a t-spread of $\mathbf{P}/V$.

Namely: Since $\mathcal{S}$ is uniform, any $(t+1)$-dimensional subspace U of $\mathbf{P}$ containing V intersects an element $V_i \in \mathcal{S}-\{V\}$. Thus any point of $\mathbf{P}/V$ is contained in at least one t-dimensional subspace $\langle V, V_i\rangle$ of $\mathbf{P}/V$. Since $\mathscr{S}$ is geometric, any point of $\mathbf{P}/V$ is contained in at most one subspace $\langle V, V_i\rangle$. Hence $\mathscr{S}_V$ is a t-spread of $\mathbf{P}/V$.

It follows that $d=a(t+1)-1$ (see corollary to Theorem 5.2). One can even deduce that

$$\dim \mathfrak{J}(\mathscr{S})=a-1.$$

By the assumptions of Theorem 5.3 we know that $a \geqq 3$.

Step 2. Now we are able to complete the proof of the Theorem using induction on a.

Let a be equal to 3. Then $\mathfrak{J}(\mathscr{S})$ is a projective plane. Since $\mathscr{S}_V$ is a t-spread of $\mathbf{P}/V=PG(2t+1,q)$ for any $V\in\mathscr{S}$, it follows that through V there are $q^{t+1}+1$ different subspaces $\langle V, V_i\rangle$. Hence $\mathfrak{J}(\mathscr{S})$ is a projective plane of order q^{t+1}. Therefore:

$$|\mathscr{S}|=q^{2(t+1)}+q^{t+1}+1.$$

Since this is the cardinality of a t-spread in $PG(3t+2,q)$, $\mathscr{S}$ is a t-spread.

Now let us assume that the assertion of the Theorem is true for $a-1\geqq 3$ and let $\mathscr{S}$ be a maximal, geometric, uniform partial t-spread of $\mathbf{P}=PG(a(t+1)-1,q)$. Then:

$$\dim \mathfrak{J}(\mathscr{S})=a-1.$$

Let $\mathscr{S}'\subset\mathscr{S}$ be the point set of a hyperplane $\mathfrak{J}(\mathscr{S}')$ of $\mathfrak{J}(\mathscr{S})$. Then – by induction – any line $\langle V_1, V_2\rangle$ of $\mathfrak{J}(\mathscr{S})$ contained in $\mathfrak{J}(\mathscr{S}')$ contains $q^{t+1}+1$ points. Therefore any line $\langle V, V'\rangle$ of $\mathfrak{J}(\mathscr{S})$ contains $q^{t+1}+1$ points since $\mathfrak{J}(\mathscr{S})$ is a projective space.

Furthermore we know by induction that

$$|\mathscr{S}'|=q^{(a-2)(t+1)}+q^{(a-3)(t+1)}+\cdots+q^{t+1}+1.$$

Let V be an element of $\mathscr{S}$ not contained in $\mathscr{S}'$. Then through V there are $|\mathscr{S}'|$ lines each of which containing $q^{t+1}+1$ points. It follows that

$$|\mathscr{S}|=q^{(a-1)(t+1)}+q^{(a-2)(t+1)}+\cdots+q^{t+1}+1.$$

Since this is the number of elements of a t-spread in $PG(a(t+1)-1,q)$, $\mathscr{S}$ is a t-spread.

Thus the proof of Theorem 5.3 is finished.

The following question remains open: Are there maximal geometric partial t-spreads in $PG(a(t+1)-1,q)$ where $a\geqq 3$ which are not spreads?

6. Partial Designs Constructed from Partial Spreads

Let $\mathscr{S}$ be a partial t-spread of $\mathbf{P}=PG(d,q)$ which is embedded as a subspace in $\Sigma=PG(D,q)$ where $D>d$. Let r be an integer such that $r\geqq t$. Then define the incidence structure $\mathbf{J}_r=\mathbf{J}(\mathscr{S},D,d,r)$ as follows:

The points of $\mathbf{J}_r$ are the points of Σ not in $\mathbf{P}$;

the blocks of $\mathbf{J}_r$ are the r-dimensional subspaces of Σ intersecting $\mathbf{P}$ exactly in an element of $\mathscr{S}$;

incidence is induced by the incidence of Σ.

The aim of this section is the study of the incidence structures $\mathbf{J}_r$.

Clearly, if $\mathbf{J}_r$ is nontrivial (i.e. $\mathbf{J}_r$ contains blocks), then $r \geqq t+1$. Furthermore, since a block of $\mathbf{J}_r$ intersects $\mathbf{P}$ in a t-dimensional subspace, we have that $r \leqq D-d+t$. Hence

(6.1) $t+1 \leqq r \leqq D-d+t$.

Theorem 6.1. $\mathbf{J}_r$ *is a partial design admitting an n-fold parallelism.*

Proof. (1) $\mathbf{J}_r$ *is a tactical configuration.*
The number of points incident with a block is equal to

$$q^r+\cdots+1-(q^t+\cdots+1)=q^{t+1}(q^{r-t-1}+\cdots+1).$$

Let V be an element of $\mathscr{S}$. Consider the quotient geometry $\Pi=\Sigma/\mathbf{P}$. Let P be a point of $\mathbf{J}_r$. Then it is easy to see that the number n of blocks of $\mathbf{J}_r$ containing P and intersecting $\mathbf{P}$ in V is equal to the number n of the $[r-(t+1)]$-dimensional subspaces in $\Pi=PG(D-(t+1),q)$ which contain the point $\langle V,P\rangle$ of Π and do not intersect the $[d-(t+1)]$-dimensional subspace $\mathbf{P}$ of Π. Hence the number of the blocks in $\mathbf{J}_r$ through a point P of $\mathbf{J}_r$ is $|\mathscr{S}|\cdot n$, which is independent of the choice of P.

Obviously, in view of (6.1) we have:

(6.2) $n \geqq 1$ and if $r>t+1$, then even $n>1$.

(2) The next step is to define the association classes of $\mathbf{J}_r$. For two arbitrary distinct points X, Y of $\mathbf{J}_r$ denote by XY the line of Σ through X and Y. In general, there are three association classes A_1, A_2, and A_3 in $\mathbf{J}_r$ consisting of the following 2-sets of points:

A_1 consists of the 2-sets $\{X, Y\}$ of points of $\mathbf{J}_r$ for which the line XY of Σ intersects $\mathbf{P}$ in a point of $\mathscr{P}(\mathscr{S})$;

the class A_2 contains exactly those 2-sets $\{X, Y\}$ for which the line XY intersects $\mathbf{P}$ in a point outside $\mathscr{P}(\mathscr{S})$;

finally the elements of A_3 are the 2-sets $\{X, Y\}$ such that the line XY has no point in $\mathbf{P}$.

The properties of a projective space imply that

$$\mathscr{A}=\{A_1, A_2, A_3\}$$

is an association scheme on the points of $\mathbf{J}_r$. Furthermore, given $\{X, Y\} \in A_i$, the number of blocks in $\mathbf{J}_r$ through X and Y depends only on i and not on the choice of X and Y. Hence $\mathbf{J}_r$ is a partial design.

(3) We define: Two blocks b, c of $\mathbf{J}_r$ are called n-fold parallel, $b \|_n c$, if b and c intersect $\mathbf{P}$ in the same element of $\mathscr{S}$.

This is obviously an equivalence relation among the blocks and to any point-block pair (P, b) there are exactly n blocks b_i which contain P and are n-fold parallel to b. (The integer n is the same as in (1).)

Theorem 6.2. *If* $\mathbf{J}_r$ *is the partial design defined as above, then*

(a) $\mathbf{J}_r$ *admits a parallelism if and only if* $r=t+1$;

(b) $\mathbf{J}_r$ *is a net with at least two parallel classes if and only if* $D=2t+2$ *and* $|\mathscr{S}| \geqq 2$;

(c) $\mathbf{J}_r$ *is a net with one parallel class if and only if* $r=t+1$ *and* $|\mathcal{S}|=1$.

Proof. (a) The number n is equal to 1 if and only if $r=t+1$.

(b) First let us suppose that $\mathbf{J}_r$ is a net with at least two parallel classes. $\mathbf{J}_r$ admits a parallelism, hence – by (a) – $r=t+1$. Since the net $\mathbf{J}_r$ admits at least two parallel classes, we have $|\mathcal{S}|\geqq 2$ and therefore $d\geqq 2t+1$. Now let V_1 and V_2 be two distinct elements of $\mathcal{S}$. It follows that $D<2t+3$, since otherwise there would exist two $(t+1)$-dimensional subspaces U_i of $\Sigma-\mathbf{P}$ with $V_i\subseteq U_i$ $(i=1,2)$ such that $U_1\cap U_2=\varnothing$. This is a contradiction to the definition of a net since the nonparallel blocks U_1, U_2 of $\mathbf{J}_r$ would have no point in common. Therefore it follows:

$$2t+2\geqq D\geqq d+1\geqq 2t+2,$$

hence

$$D=2t+2.$$

On the other hand suppose that $D=2t+2$ and $|\mathcal{S}|\geqq 2$. From $|\mathcal{S}|\geqq 2$ we can deduce that $d\geqq 2t+1$, hence

$$D=d+1.$$

Then from (6.1) we obtain that $r=t+1$. Therefore $\mathbf{J}_r$ admits – by (a) – a parallelism with at least two parallel classes since $|\mathcal{S}|\geqq 2$.

It remains to show that any two nonparallel blocks U_1, U_2 of $\mathbf{J}_r$ have exactly one point in common.

U_1 and U_2 have at least one point in common: Since $D=2t+2$, any two $(t+1)$-dimensional subspaces of Σ must intersect. U_1 and U_2 cannot have a point of $\mathbf{P}$ in common since U_1 and U_2 are nonparallel.

Assume that U_1 and U_2 have a line g of Σ in common. Then the line g would intersect the hyperplane $\mathbf{P}$ of Σ in a point of $U_1\cap\mathbf{P}$ and $U_2\cap\mathbf{P}$. But this is a contradiction to the fact that U_1 and U_2 are nonparallel.

(c) $\mathbf{J}_r$ has one parallel class if and only if $|\mathcal{S}|=1$. Together with (a) this shows (c).

Next we characterize the partial geometries among the $\mathbf{J}(\mathcal{S}, D, d, r)$.

Theorem 6.3. *Let* $\mathbf{J}_r$ *be the partial design defined above. Then:*

(a) *Any two distinct points of* $\mathbf{J}_r$ *are incident with at most one common block if and only if* $r=t+1$;

(b) $\mathbf{J}_r$ *is a partial geometry with* $l\geqq 1$ *if and only if* $D=d+1$ *and* $\mathcal{S}$ *is l-uniform with* $l\geqq 1$;

(c) $\mathbf{J}_r$ *is a partial geometry with* $l=0$ *if and only if* $r=t+1$ *and* $|\mathcal{S}|=1$.

Proof. (a) The number n is equal to 1 if and only if $r=t+1$.

(b) First let us suppose that $\mathbf{J}_r$ is a partial geometry with $l\geqq 1$. Then any two points of $\mathbf{J}_r$ are incident with at most one common block; hence by (a) it follows that $r=t+1$. Let V be an element of $\mathcal{S}$. If we assume that $D\geqq d+2$ then there would exist a $(t+2)$-dimensional subspace W of Σ intersecting $\mathbf{P}$ exactly in V. Then W would contain a non-incident point-block pair (P, U) of $\mathbf{J}_r$. From the construction of W it follows that no block of $\mathbf{J}_r$ containing P intersects U. This is a contradiction to the fact that $\mathbf{J}_r$ is a partial geometry with $l\geqq 1$.

It remains to show that $\mathscr{S}$ is l-uniform. Let V be an element of $\mathscr{S}$, U a $(t+1)$-dimensional subspace of $\mathbf{P}$ containing V, and M a block of $\mathbf{J}_r$ intersecting $\mathbf{P}$ in V. Furthermore let P be a point of $\mathbf{J}_r$ contained in the subspace $\langle U, M\rangle$ but not contained in M. From the definition of a partial geometry we know that there are exactly l blocks N_i of $\mathbf{J}_r$ containing P and intersecting M in a (necessarily unique) point Q_i resp. The line PQ_i intersects the hyperplane $\mathbf{P}$ of Σ in a point T_i of U $(i=1, \ldots, l)$. Each of these points is contained in an element $V_i \in \mathscr{S}-\{V\}$ and no two distinct points T_i, T_j are contained in the same element $V_i \in \mathscr{S}-\{V\}$, since an element $V' \in \mathscr{S}-\{V\}$ can intersect U in at most one point. Hence U intersects at least l elements of $\mathscr{S}-\{V\}$. But U cannot intersect any other element V' of $\mathscr{S}-\{V\}$ because then there would be one more block b' containing P and intersecting b. Hence $\mathscr{S}$ is l-uniform.

On the other hand suppose that $D=d+1$ and $\mathscr{S}$ is l-uniform with $l \geqq 1$.

From $D=d+1$ it follows that $r=t+1$. Hence by (a) any two points of $\mathbf{J}_r$ are incident with at most one common block. Let (P, M) be a non-incident point-block pair of $\mathbf{J}_r$. Since $\mathbf{P}$ is a hyperplane of Σ, it follows that $\langle P, M\rangle$ intersects $\mathbf{P}$ in a $(t+1)$-dimensional subspace U containing $V := M \cap \mathbf{P} \in \mathscr{S}$. Since $\mathscr{S}$ is l-uniform, U intersects exactly l elements $V_i \in \mathscr{S}-\{V\}$ $(i=1, \ldots, l)$. It follows that $\langle P, V_i\rangle$ are l pairwise distinct blocks of $\mathbf{J}_r$ containing P and intersecting M. Conversely any block N of $\mathbf{J}_r$ being incident with P and intersecting M intersects $\mathbf{P}$ in an element V_i $(i=1, \ldots, l)$. Therefore there are exactly l blocks of $\mathbf{J}_r$ containing P and intersecting M. Hence $\mathbf{J}_r$ is a partial geometry with $l \geqq 1$.

(c) The number l is equal to 0 if and only if $|\mathscr{S}|=1$. Then the assertion follows by using (a).

Remark. If $\mathscr{S}$ is a 1-uniform partial t-spread of $PG(d, q)$ then

$$\mathbf{Q} := \mathbf{J}_{t+1} := \mathbf{J}(\mathscr{S}, d+1, d, t+1)$$

is a 4-gonal configuration. The 4-gonal configurations constructed by Thas ([10], "*Construction 1*") provide examples for the structures $\mathbf{Q}$.

If the partial t-spreads $\mathscr{S}$ in the definition of $\mathbf{J}(\mathscr{S}, d+1, d, t+1)$ can be embedded into t-spreads of $\mathbf{P}$, then the corresponding partial designs can be completed to $2-(v, k, 1)$ designs. By applying our Theorem 3.1 on partial t-spreads we can state the following sufficient condition for such a completion:

Theorem 6.4. *Let* $\mathbf{J}$ *be* $\mathbf{J}(\mathscr{S}, 2t+2, 2t+1, t+1)$ *and suppose through any point of* $\mathbf{J}$ *there are*

$$R > q^{t+1} - \sqrt{q^t}$$

blocks of $\mathbf{J}$. *Then* $\mathbf{J}$ *can be completed by adjoining additional blocks to a*

$$\mathbf{J}' := \mathbf{J}(\mathscr{S}', 2t+2, 2t+1, t+1),$$

which is a $2-(q^{2(t+1)}, q^{t+1}, 1)$ *design. Actually* $\mathbf{J}'$ *is a translation plane.*

Proof. Theorem 3.1.

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