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Lattice Dipole Gas and $(\nabla\phi)^4$ Models at Long Distances: Decay of Correlations and Scaling Limit

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Abstract. We prove that the scaling limit for a large class of weak $V(\nabla\phi)$ perturbations of the free massless lattice field ϕ is Gaussian with the covariance $c(V)(-\Delta)^{-1}$. The correlations as well as $c(V)$ are analytic in V. In particular the Mayer series for the dipole gas is convergent for small activity.

1. Introduction

The authors have been pursuing a program to gain a rigorous control of asymptotically free (AF) models of statistical mechanics and quantum field theory. This paper finishes such an analysis for infrared (IR) AF models, such as the dipole gas, $(\nabla\phi)^4$ model and related ones. We show that their correlations become those of a free massless field at long distances: the canonical scaling limit is shown to be the massless Gaussian Euclidean field with a definite field strength renormalization.

In a previous paper [1] the authors studied the renormalization group (RG) trajectory of the Hamiltonian in a general space of Hamiltonians. This analysis is now applied to the study of the correlations. The results of the present paper may also be interpreted as setting up rigorously the RG in a space of Gibbs states of certain critical (massless) theories and showing the convergence of its iterations to the state given by the massless Gaussian fixed point, in the sense of convergence of correlations. We, however, state our results only pragmatically, as a result about scaling limits and IR properties of the correlations.

When [1] was finished we obtained [2] where infrared behavior of the weakly coupled $(\nabla\phi)^4$ model was controlled by means of a phase-cell expansion. Both methods are similar as they are based on an analysis of contributions of different momenta on different scales of distances. In [2] different momentum scales are entangled in the expansion whereas we analyze the contribution of one momen-

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tum scale in a general inductive step. The price paid for greater conceptual clarity is that we have to consider iterations of much more general class than the starting $V(\nabla\phi)$ of interest. But in turn the results are quite model-independent (they apply e.g. to the dipole gas and to the $(\nabla\phi)^4$ at the same time).

One should also mention here a series of papers ([3] and references therein) which apply correlation inequalities to study the infrared behavior of massless models. This method gives results for any values of the coupling but is more model dependent and provides less understanding of the physics of the system. The future lies probably in applying the RG ideas together with the correlation inequalities (see e.g. an attempt in [4]).

Let us describe now the models that we consider, together with our results. The reader is recommended to have a look into [1] for more details and motivations. We state the results only for local potentials. Remark 2 below concerns the non-local ones fitting our scheme.

Let Λ be a periodic cube ($|\Lambda|=L^{Nd}$) in $\mathbb{Z}^d$ and ϕ_x, $x\in\Lambda$, the field with covariance $G_{0\Lambda}$, the inverse of

$$(G_{0\Lambda}^{-1})_{xy}=(-\Delta_\Lambda)_{xy}+L^{-Nd}\xi \tag{1}$$

(the infrared regulator ξ makes G_0 well defined), where Δ_Λ is the lattice periodic Laplacian. For each such Λ, let there be given a function $V_\Lambda(\chi)$ of the vector field $\chi_{\mu x}$, $\mu=1,\ldots,d$, $x\in\Lambda$, on Λ. Define the finite volume state

$$\langle - \rangle_{V_\Lambda}=\frac{1}{\mathcal{N}}\int - \exp[-V_\Lambda(\nabla\phi)]d\mu_{G_{0\Lambda}}(\phi), \tag{2}$$

where $d\mu_{G_{0\Lambda}}$ is the Gaussian measure with covariance $G_{0\Lambda}$ and $\mathcal{N}=\int\exp[-V_\Lambda]d\mu_{G_{0\Lambda}}$ is assumed to be non-zero. We shall also use the notation $\langle - \rangle_{\mathcal{H}_\Lambda}$, where the Hamiltonian $\mathcal{H}_\Lambda(\phi)=\frac{1}{2}(\phi, G_{0\Lambda}^{-1}\phi)+V_\Lambda(\nabla\phi)$. Denote the thermodynamic limit (TDL) $\Lambda\to\mathbb{Z}^d$, $\xi\to 0$ of $\langle - \rangle_{V_\Lambda}$ by $\langle - \rangle_V$ whenever it exists ($V=\{V_\Lambda\}$) (convergence here means the convergence of correlation functions). We define the scaling limit of $\langle - \rangle_V$ as follows. Let $x_1,\ldots,x_\mu\in\mathbb{R}^d$ be different points with $x_i\in L^{-N}\mathbb{Z}^d$ for some N. Define

$$G(x_1,\ldots,x_m)=\lim_{n\to\infty} L^{\frac{d-2}{2}nm}\left\langle \prod_{i=1}^m \phi_{L^n x_i}\right\rangle_V \tag{3}$$

whenever it exists. $\{G(x_1,\ldots,x_m)\}$ give the scaling limit of $\langle - \rangle_V$.

The Main Result. *Let $d\geqq 3$ and*

$$V_\Lambda(\chi)=\sum_{x\in\Lambda} v(\chi_x), \tag{4}$$

with $v(\chi)$ being a function invariant under rotations by multiples of $\frac{\pi}{2}$ of χ and under reflections in coordinate planes, vanishing together with the second derivatives at zero, even and analytic in χ for $|\mathrm{Im}\chi_\mu|<B$. Moreover, we assume that

(a) *for $|\chi_\mu|<B$, $|v(\chi)|<\eta$,*

(b) *for $|\mathrm{Im}\chi_\mu|<B$, $|\exp[-v(\chi)]|\leqq\exp[\kappa|\chi^2|]$ with $|\kappa|<O(1)<\frac{1}{2}$. Then for $B>B_0$ and $\eta<\eta_0$,*

(A) *the* TDL $\langle - \rangle_V \equiv \langle - \rangle_v$ *exists and the scaling limit is given by the massless gaussian field on* $\mathbb{R}^d$ *with the covariance* $c(v)^{-1}(-\Delta_c)^{-1}$, Δ_c *being the continuum Laplacian.*

(B) *In particular, the two-point function satisfies*

$$\langle \phi_x \phi_y \rangle_v = c(v)^{-1}(-\Delta)^{-1}_{xy} + O\left(\frac{1}{1+d(x,y)^{d-2+\varepsilon}}\right). \tag{5}$$

(C) $c(v_\lambda)$ *and* $\langle \prod \phi_{x_i} \rangle_{v_\lambda}$ *are analytic in* $\lambda \in \mathcal{R}$ *if* v_λ *is a family analytic in* λ *in some region* $\mathcal{R}$, *satisfying there the above requirements uniformly in* λ.

Remarks. 1. The functions

$$v(\nabla\phi) = \lambda \sum_\mu (\nabla_\mu \phi)^4 \qquad \text{(the anharmonic crystal)}, \tag{6}$$

and

$$v(\nabla\phi) = \lambda \sum_\mu (1 - \tfrac{1}{2}\varrho^2(\nabla_\mu\phi)^2 - \cos(\varrho\nabla_\mu\phi)) \qquad \begin{array}{l}\text{(the dipole gas in the}\\ \text{sine-Gordon representation)},\end{array} \tag{7}$$

satisfy our conditions for λ positive and small and for $|\lambda|$ small or $|\varrho|$ small respectively. In particular for the dipole gas the perturbation expansion in powers of λ (the Mayer expansion) converges for small $|\lambda|$. In fact let v be any invariant even function, vanishing together with the second derivative at zero, analytic in some strip around the reals with e^{-v} bounded by some Gaussian. Then $v(\lambda\phi)$ satisfies our conditions for λ small.

2. We may also consider non-local V's corresponding to the Boltzmann factors given by the formula (3.3) of [1], with the properties described in Sect. 4 therein, see also (2.14) below and what follows it. These V's constitute a class invariant under the RG. To be able to pass to the thermodynamic limit one has to take g^D_{AX} and $\tilde{V}_{AY}$ (being respectively the large field and the small field data) possessing infinite volume limits (note that they are functions of ϕ with finite support, X and Y respectively).

The organization of the rest of the paper is as follows:

In Sect. 2 we review the block spin formalism and the main results of [1] concerning the effective Hamiltonians.

Sections 3–5 are devoted to a careful study of the two-point function where the main ideas of our method are seen without unnecessary notational complications.

Finally, Sect. 6 shows how the argument may be applied to a general correlation: as an example we show that the scaling limit of the four-point function is Gaussian.

2. The Block-Spin Transformation

Let us consider a correlation function

$$\left\langle \prod_{i=1}^m \phi_{x_i} \right\rangle_{\mathcal{H}_\Lambda} \equiv \langle F \rangle_{\mathcal{H}_\Lambda}. \tag{1}$$

The idea of the RG is to compute (1) by successively integrating out fluctuations of short range. Explicitly, we introduce block spins ϕ_x^1, $x\in L^{-1}\Lambda\cap\mathbb{Z}^d$:

$$\phi_x^1 = L^{-\frac{d+2}{2}} \sum_{|y_\mu|<L/2} \phi_{Lx+y} \equiv (C\phi)_x, \tag{2}$$

and define $\mathcal{R}\mathcal{H}$, the renormalized Hamiltonian, (we drop the subscript Λ) by

$$\exp[-\mathcal{R}\mathcal{H}(\phi^1)] = \text{const}\int \exp[-\mathcal{H}(\phi)]\delta(\phi^1 - C\phi)D\phi. \tag{3}$$

For the correlation function (1), we get

$$\langle F\rangle_{\mathcal{H}} = \langle SF\rangle_{\mathcal{R}\mathcal{H}} = \ldots = \langle S^n F\rangle_{\mathcal{R}^n\mathcal{H}}, \tag{4}$$

$n\leqq N$, where

$$(SF)(\phi^1) = \int F(\phi)\exp[-\mathcal{H}(\phi)]\delta(\phi^1 - C\phi)D\phi \Big/ \int \exp[-\mathcal{H}(\phi)]\delta(\phi^1 - C\phi)D\phi. \tag{5}$$

Iterating N times we are finally left with the zero mode integral:

$$\langle F\rangle_{\mathcal{H}} = \int S^N F(\phi^N)d\mu_{L^{2N}\xi}(\phi^N). \tag{6}$$

In [1] we controlled $\mathcal{R}^n\mathcal{H}$, showing that (in the $\Lambda\nearrow\mathbb{Z}^d$ limit) it converges (in a sense specified below) to a Gaussian fixed point. The purpose of this paper is to control the iterations of S, given this information about $\mathcal{R}^n\mathcal{H}$.

Let us briefly recapitulate the main points and results of the analysis of [1]. Consider iterations of the form (1.4). It has been shown that one can introduce "scaling fields" ψ_z^1, $z\in L^{-1}\Lambda$, related in an approximately local manner to ϕ^1, and fluctuation fields Z_x, $x\in\Lambda\backslash L\mathbb{Z}^d$, so that $\mathcal{R}\mathcal{H}$ is given by the following integration over Z

$$\exp[-\mathcal{R}\mathcal{H}(\phi^1)]D\phi^1 = \text{const}\,d\mu_{G_1}(\phi^1)\int\exp[-V_\Lambda(L^{-d/2}\nabla\psi^1_{L^{-1}\cdot} + \nabla M^0 Z)]d\mu_1(Z), \tag{7}$$

with M_{xy}^0 an (approximately) local kernel, $x\in\Lambda$, $y\in\Lambda\backslash L\mathbb{Z}^d$, and G_1 being a new covariance for the unperturbed part,

$$G_1 = CG_0C^+. \tag{8}$$

Next one separates from the integral in (7) a "marginal" quadratic piece proportional to $(\phi^1|G_1^{-1}\phi^1)$ (except for the zero mode contribution) and absorbs it to $d\mu_{G_1}(\phi)$ turning the latter into $d\mu_{\bar{G}_1}(\phi)$. The whole process may be iterated giving

$$\begin{aligned}\exp[-\mathcal{R}^{n+1}\mathcal{H}(\phi^{n+1})]D\phi^{n+1} &= \text{const}\,d\mu_{C\bar{G}_nC^+}(\phi^{n+1})\\ &\quad\cdot\int\exp[-V^n(L^{-d/2}\nabla\psi^{n+1}_{L^{-1}\cdot} + \nabla M^n Z^n)]d\mu_{c_n^{-1}}(Z^n)\\ &= \text{const}\exp[-V^{n+1}(\nabla\psi^{n+1})]d\mu_{\bar{G}_{n+1}}(\phi^{n+1}),\end{aligned} \tag{9}$$

where $\bar{G}_{n+1}$ coincides with $c_{n+1}^{-1}C^{n+1}G_0(C^+)^{n+1}\equiv c_{n+1}^{-1}G_{n+1}$ on the subspace orthogonal to constants and with G_{n+1} on constants and

$$\begin{aligned}V^{n+1}(\nabla\psi^{n+1}) &= \tilde{V}^{n+1}(\nabla\psi^{n+1}) + \tfrac{1}{2}(\nabla\nabla\psi^{n+1}, K_{n+1}\nabla\nabla\psi^{n+1}),\\ \tilde{V}^{n+1}(0) &= \frac{\delta^2\tilde{V}^{n+1}(0)}{\delta\chi\delta\chi} = 0,\end{aligned} \tag{10}$$

($M^n = \mathcal{A}^n Q\Gamma_n^{1/2}$ in the notation of [1]).

The following results were proven for the functions and kernels of (9) and (10), uniformly in the volume Λ. The initial V^0 is assumed to be given by (1.4) with v as described in Sect. 1. Given a (real) configuration $\chi^n = \nabla\psi^n$ which is uniquely determined by the n^{th} block spin $\phi^n \equiv C^n\phi$, we introduce a region of large fields, $D_n(\nabla\psi^n)$, as the smallest union of blocks of the lattice with spacing L^{N_0} (see [1], Sect. 3) satisfying

$$|\nabla_\mu \psi^n_{z'}| \leqq (n_0+n)^\nu \exp[\alpha d(z,z')] \tag{11}$$

for each $z \notin D_n(\nabla\psi^n)$. Here α is taken small and $\nu > \frac{1}{2}d^2+1$.

The following sets of (complex) vector fields χ^n on any $X \subset L^{-n}\Lambda$ (X, D unions of L^{N_0}-lattice blocks) were introduced in [1]:

$$\mathscr{K}_n(\chi) = \{\chi^n : |\chi^n_{\mu z}| < (n_0+n)^\nu,\ |\nabla_\nu \chi^n_{\mu z}| < C_1(n_0+n)^{\nu+d} \text{ if } z + L^{-n}e_\nu \in X\}, \tag{12}$$

and

$$B_n(D,X,a) = \bigcup_{D_n(\nabla\psi^n)\subset D} (\nabla\psi^n|_X + a\mathscr{K}_n(X)). \tag{13}$$

It was shown that $\exp[-\tilde{V}^n]$ is analytic on $B_n(L^{-n}\Lambda, L^{-n}\Lambda, 1)$ and has, for $\chi = \nabla\psi^n + \tilde{\chi}$ with $D_n(\nabla\psi^n) \subset D$, $\tilde{\chi} \in \mathscr{K}_n(L^{-n}\Lambda)$, a representation

$$\exp[-\tilde{V}^n(\chi)] = \sum_{\{X_j\}} \prod_j g^{nD}_{X_j}(\chi) \exp\left[-\sum_{Y \cap X_j = \emptyset} \tilde{V}^n_Y(\chi)\right] \tag{14}$$

with X_j disjoint, $\cup X_j \supset D$, X_j, Y being unions of L^{N_0}-blocks. The functions g^{nD}_X and $\tilde{V}^n_Y$ depend only on $\chi^n|_X$ or $\chi^n|_Y$ respectively and satisfy the following analyticity requirements and bounds inherited from our assumptions on v:

(1_n) g^{nD}_X is an even analytic function on $B_n(D,X,1)$. If X_j are disjoint and $D_1 \cap D = \bigcup_j D \cap X_j$, then for $\chi^n = \nabla\psi^n + \tilde{\chi}^n$ (on B_n),

$$\left|\prod_j g^{nD}_{X_j}(\chi^n)\right| \leqq \exp\left[\kappa \mathscr{D}_n(D_1, \nabla\psi^n) - 2\alpha \sum_j \mathscr{L}(X_j) + E \sum_j |D \cap X_j|\right]. \tag{15}$$

Here $|X|$ denotes the number of L^{N_0}-blocks in X. $\mathscr{L}(X)$ is the length of the shortest tree on the centers of the L^{N_0}-blocks building X and possibly other (continuum) points.

$$\mathscr{D}_n(K,\chi^n) = \left(\int_K dz + \int_{\partial K} d\sigma(z)\right)|\chi^n_z|^2. \tag{16}$$

(2_n) $\tilde{V}^n_Y$ is even, analytic on $2\mathscr{K}_n(Y)$ with

$$|\tilde{V}^n_Y| \leqq \delta^{n_0+n} \exp[-2\alpha \mathscr{L}(Y)], \quad 0 < \delta < 1. \tag{17}$$

$\tilde{V}^n_Y$ vanishes together with its second derivatives at $\chi = 0$.

(3_n)

$$\|K^n\|_{L^1(\square_1 \times \square_2)} \leqq C\delta^{n_0+n} \exp[-2\alpha d(\square_1, \square_2)] \tag{18}$$

for unit squares $\square_1, \square_2$.

(4) The infinite volume limit for c_n exists. Moreover, since

$$|c_{n+1} - c_n| \leqq C\delta^{n_0+n},$$

the infinite volume c_n tends to $c(v)$ when $n \to \infty$.

In the next sections we shall make inductive assumptions about $S^n F$, similar to the above ones, and shall iterate them much in the same way as we proceeded in [1].

3. The Two Point Function: A Representation for the Block Spin Correlations

Let us start by inserting the block spin decomposition

$$\phi_x = L^{-\frac{d-2}{2}} \psi^1_{L^{-1}x} + (MZ^0)_x \equiv \gamma \psi^1_{x^1} + z^0_x \qquad (x^n \equiv L^{-n} x) \tag{1}$$

to (2.1), getting

$$\langle F \rangle_{\mathscr{H}} \equiv \langle \phi_I \rangle_{\mathscr{H}} = \sum_{J \subset I} \gamma^{|J|} \langle \psi^1_J z^0_{I \setminus J} \rangle_{\mathscr{H}}, \tag{2}$$

where we use the notation

$$\psi^n_J = \prod_{j \in J} \psi^n_{x^n_j}. \tag{3}$$

Integrating out Z^0, we obtain from (2) [see also (2.5)]

$$SF = \sum_J \gamma^{|J|} \psi^1_J \langle z^0_{I \setminus J} \rangle_{Z^0}, \tag{4}$$

where, in general, we define

$$\langle f(Z^n) \rangle_{Z^n} = \int f(Z^n) \exp[-V^n(L^{-d/2} V \psi^n_{L^{-1}\cdot} + V z^n)] d\mu_{c_n^{-1}}(Z^n) / (f \equiv 1). \tag{5}$$

First let us consider the two point function. In this case (2) reads

$$G_{x_1 x_2} = \langle \phi_{x_1} \phi_{x_2} \rangle_{\mathscr{H}} = \langle \gamma^2 \psi^1_{x^1_1} \psi^1_{x^1_2} + \gamma \psi^1_{x^1_1} \langle z^0_{x_2} \rangle_{Z^0} + (1 \Leftrightarrow 2) + \langle z^0_{x_1} z^0_{x_2} \rangle_{Z^0} \rangle_{\mathscr{R}\mathscr{H}}. \tag{6}$$

Let us introduce the following notations

$$\langle z^k_{x^k_i} \rangle_{Z^k} = G^k_{k+1,i}, \tag{7}$$

$$\langle z^k_{x^k_1} z^k_{x^k_2} \rangle_{Z^k} = G^{kk}_{k+1,12}, \tag{8}$$

$$\langle z^k_{x^k_i} G^l_{k,j} \rangle_{Z^k} = G^{kl}_{k+1,ij} \qquad (k > l), \tag{9}$$

$$G^A_{n+1,B} = \langle G^A_{n,B} \rangle_{Z^n} \qquad (A = k, kl,\ B = i, ij,\ n < N - N_0), \tag{10}$$

and finally

$$G^A_{N,B} = \langle G^A_{N-N_0,B} \rangle_{\mathscr{R}^{N-N_0}\mathscr{H}}. \tag{11}$$

Iterating (6), we obtain

$$\begin{aligned} G_{x_1 x_2} = & \sum_{k=0}^{N-N_0-1} \gamma^{2k} G^{kk}_{N,12} + \sum_{l=0}^{N-N_0-2} \sum_{k=l+1}^{N-N_0-1} \gamma^{l+k} (G^{kl}_{N,12} + G^{kl}_{N,21}) \\ & + \gamma^{2(N-N_0)} \langle \psi^{N-N_0}_{x^{N-N_0}_1} \psi^{N-N_0}_{x^{N-N_0}_2} \rangle_{\mathscr{R}^{N-N_0}\mathscr{H}} \\ & + \sum_{k=0}^{N-N_0-1} \gamma^{N-N_0+k} \langle \psi^{N-N_0}_{x^{N-N_0}_1} G^k_{N-N_0,2} + (1 \Leftrightarrow 2) \rangle_{\mathscr{R}^{N-N_0}\mathscr{H}}. \end{aligned} \tag{12}$$

Thus, we only need to control the iterations of $\langle \text{—} \rangle_Z$ ($\equiv S$) on the various functions just introduced. Let G_n denote any of the objects $G^A_{n,B}$ $n \leqq N - N_0$. Note that G_n is a function of $\nabla\psi^n$ only (not of ψ^n) and can be extended naturally to vector fields χ^n. Let

$$G_n(\chi^n) = G_n(0) + \tilde{G}_n(\chi^n). \tag{13}$$

Of course, $\tilde{G}^k_{n,i} = G^k_{n,i}$, since $G^k_{n,i}$ is odd.

We shall assume (inductively in n, $k+1 \leqq n \leqq N - N_0$) that $\tilde{G}_n \exp[-\tilde{V}^n]$ is analytic on $\mathscr{B}(L^{-n}\Lambda, L^{-n}\Lambda, 1)$ and that for $\chi^n \in \mathscr{B}(D, L^{-n}\Lambda, 1)$,

$$\tilde{G}_n \exp[-\tilde{V}] = \sum_{\{X_j\},\{Y_\sigma\}} \prod_j \tilde{g}^{nD}_{X_j} \prod_\sigma \tilde{F}_{nY_\sigma} \exp\left[-\sum_{Y \cap X_j = \emptyset} \tilde{V}^n_Y\right], \tag{14}$$

where X_j, Y_σ are disjoint (built from L^{N_0}-blocks), $X_j \cap D$ is, for each j, a non-empty union of connected components (c.c.) of D, $\cup X_j \supset D$. Moreover, the set of points $x^n_J \subset \{x^n_1, x^n_2\}$ involved in G_n (i.e. $G^A_{n,J}$) satisfies $x^n_J \subset (\cup X_j) \cup (\cup Y_\sigma)$. For $D = \emptyset$, $\prod_j \tilde{g}^{nD}_{X_j}$ does not occur, for $X_j \cap x_J = \emptyset$, $\tilde{g}^{nD}_{X_j} = g^{nD}_{X_j}$. $\tilde{F}_{nY_\sigma}$ does not occur in (14), if $Y_\sigma \cap x_J = \emptyset$. Thus we see that for $G^k_{n,i}$ there is at most one $\tilde{F}$ in (14), whereas for $G^{kl}_{n,ij}$ there may be as many as two. $\tilde{g}^{nD}_{X_j}$ and $\tilde{F}_{nY_\sigma}$ implicity carry the indices k (or kl), i, (ij). Namely for $G^A_{n,B}$ we have $F^A_{n,BY}$, etc. Equation (14) is an analogue of (2.14) for the (unnormalized) block spin correlations. In analogy with (1_n) and (2_n) of Sect. 2, $\tilde{g}^{nD}_X$ and $\tilde{F}_{nY}$ possess the following properties, to be shown inductively.

(A_n) $\tilde{g}^{nD}_X$ are analytic on $\mathscr{B}_n(D, X, 1)$. They are even if $X \cap x^n_J$ is even. Otherwise they are odd. Equation (2.15) holds, if all or some of $g^{nD}_{X_j}$ are replaced by $\tilde{g}^{nD}_{X_j}$.

(B_n) $\tilde{F}_{nY}$ are analytic on $2\mathscr{K}_n(Y)$ and vanish at $\chi^n = 0$. $\tilde{F}^k_{n,iY}$ are odd and $\tilde{F}^{kl}_{n,ijY}$ are even. On $2\mathscr{K}_n(Y)$ they satisfy the bounds

$$|\tilde{F}^k_{n,iY}| \leqq L^{-\frac{d-\varepsilon}{2}(n-k)} \tilde{\delta}^{2n_0+k} \exp[-2\alpha \mathscr{L}(Y)], \tag{15}$$

and

$$|\tilde{F}^{kl}_{n,ijY}| \leqq L^{-\frac{d-\varepsilon}{2}(2n-k-l)} \tilde{\delta}^{n_0+l} \exp[-2\alpha \mathscr{L}(Y)], \tag{16}$$

where $\tilde{\delta} \equiv \delta^{1/3}$.

We will also trace the change of $G_n(0)$ with n.

(C_n)

$$|G^{kl}_{n+1,ij}(0) - G^{kl}_{n,ij}(0)| \leqq CL^{-\frac{d-\varepsilon}{2}(2n-k-l)} \tilde{\delta}^{n_0+l} \exp[-\alpha d(x^{n+1}_1, x^{n+1}_2)]. \tag{17}$$

In (15)–(17) $\varepsilon > 0$ may be chosen arbitrarily small if the parameters of our constructions (see the beginning of Sect. 4 of [1]) are chosen appropriately.

4. The Cluster Expansion

Here we shall show how, given (2.13) for one value of n, we may recover it for $n+1$. Since the initial steps that we take are analogous to those of [1], Sect. 3, we refer directly to this paper. Suppressing n and replacing $n+1$ by the prime, we have the following recursion:

$$G'(\chi')\exp[-\tilde{V}'(\chi')] = \int G(L^{-d/2}\chi'_{L^{-1}\cdot} + \nabla z)\exp[-V(L^{-d/2}\chi'_{L^{-1}\cdot} + \nabla z)]d\mu_{c^{-1}}(Z)$$
$$\cdot \exp[W'(0) + \tfrac{1}{2}\delta^2 W'(\chi')]. \tag{1}$$

Upon insertion of (3.13), this gives

$$\delta G(0) \equiv G'(0) - G(0) = \int \tilde{G}(\nabla z) \exp[-V(\nabla z)] d\mu_{c^{-1}}(Z) \exp[W'(0)], \tag{2}$$

and

$$\begin{aligned} \tilde{G}'(\chi') \exp[-\tilde{V}'(\chi')] = & \int \tilde{G}(\chi) \exp[-V(\chi)] d\mu_{c^{-1}}(Z) \exp[W'(0) + \tfrac{1}{2}\delta^2 W'(\chi')] \\ & - \int \tilde{G}(\nabla z) \exp[-V(\nabla z)] d\mu_{c^{-1}}(Z) \exp[W'(0)] \exp[-\tilde{V}'(\chi')], \end{aligned} \tag{3}$$

where

$$\chi = L^{-d/2} \chi'_{L^{-1}\cdot} + \nabla z. \tag{4}$$

Using (2.14) as an input, we obtain an analogoue of (3.15) and (3.16) of [1]:

$$\int \tilde{G}(\chi) \exp[-V(\chi)] d\mu_{c^{-1}}(Z) = \sum_{\bar{p}, \{X_j\}, \{Y_\sigma\}, \{Y_\alpha\}, \{Y_\beta\}} \int \tilde{\mathscr{F}}(X_j, Y_\sigma, Y_\alpha, Y_\beta; \chi) 1_{\bar{p}}(Z) d\mu_{c^{-1}}(Z), \tag{5}$$

where

$$\begin{aligned} \tilde{\mathscr{F}}(\ldots) = & \prod_j \tilde{g}^D_{X_j}(\chi) \prod_\sigma \tilde{F}_{Y_\sigma}(\chi) \prod_\alpha (\exp[-V_{Y_\alpha}(\chi)] - 1) \prod_\beta (\exp[-\tfrac{1}{2}\delta^2 V_{Y_\beta}(\chi)] - 1) \\ & \cdot \prod_{\Delta \not\subset X} \exp[-\tilde{V}_\Delta(\chi)] \prod_\Delta \exp[-\tfrac{1}{2}\delta^2 V_\Delta(\chi)]. \end{aligned} \tag{6}$$

Now (5) is decoupled as in [1], Sect. 3, leading to an analogue of (3.24) therein:

$$\int \tilde{G} \exp[-V] d\mu_{c^{-1}} = \prod_{\Delta \not\subset D'} \exp[-w'_\Delta] \sum_{\{\tilde{X}_\zeta\}} \prod_\zeta \tilde{\varrho}^{D'}_{\tilde{X}_\zeta}. \tag{7}$$

$\tilde{X}_\zeta$ are disjoint, $\cup \tilde{X}_\zeta$ has to contain $D' \cup \{x^n_J\}$. Equation (7) is an expression of the type of a polymer-gas unnormalized correlation function with polymer densities

$$\begin{aligned} \tilde{\varrho}^{D'}_{\tilde{X}}(\chi') = & \sum_{\bar{p}, \{X_j\}, \{Y_\sigma\}, \{Y_\alpha\}, \{Y_\beta\}, \{\bar{U}_\gamma\}} \int \prod_\gamma S(\bar{U}_\gamma) \tilde{\mathscr{F}}_{L\tilde{X}}(X_j, Y_\sigma, Y_\alpha, Y_\beta; \chi^s) \\ & \cdot 1_{\bar{p}}(Z_{L\tilde{X}}) d\mu_{c^{-1}}(Z_{L\tilde{X}}) \Big/ \prod_{\Delta \subset \tilde{X} \setminus D'} \exp[-w'_\Delta(\chi')]. \end{aligned} \tag{8}$$

In (8) $\tilde{\mathscr{F}}_{L\tilde{X}}$ is like $\mathscr{F}$ of (6) except that $X_j, Y_\sigma, Y_\alpha, Y_\beta \subset L\tilde{X}$ and Δ's in the products are taken from $L\tilde{X}$. The restrictions on the sums in (8) are as in (3.25) of [1], Y_σ playing the same role as Y_α and Y_β. The only additional restriction is that $(\cup X_j) \cup (\cup Y_\sigma)$ has to contain $x^n_J \cap L\tilde{X}$. Notice that if $x^{n+1}_J \cap \tilde{X} = \emptyset$, then $\tilde{\varrho}^{D'}_{\tilde{X}} = \varrho^{D'}_{\tilde{X}}$.

Now put

$$\begin{aligned} \bar{g}'^{D'}_{X'} = & \sum_{\{X_\zeta\}, \{Y_\xi\} \text{ in } X'} \prod_\zeta \tilde{\varrho}^{D'}_{\tilde{X}_\zeta} \prod_\xi (\exp[W'_{Y_\xi}(0) + \tfrac{1}{2}\delta^2 W'_{Y_\xi}] - 1) \\ & \cdot \exp\Big[-\sum_{\Delta \subset X' \setminus D'} w'_\Delta\Big] \exp\Big[\sum_{\Delta \subset X'} (w'_\Delta(0) + \tfrac{1}{2}\delta^2 w'_\Delta)\Big], \end{aligned} \tag{9}$$

as in (3.30) of [1]. Note again that if $x^{n+1}_J \cap X' = \emptyset$, then $\bar{g}'^{D'}_{X'} = g'^{D'}_{X'}$. For the odd case (one x^{n+1}_1 or x^{n+1}_2 in X'), $\bar{g}'^{D'}_{X'}$ will be the final $\tilde{g}'^{D'}_{X'}$ already. Define also $g'^{A}_{BX'}$ by (9) with the restriction $\cup X_\zeta \supset X' \cap D' \neq \emptyset$ replaced by $X' \cap D = \emptyset$ and x^{n+1}_1 or $x^{n+1}_2 \in X'$.

Equation (7) may be rewritten as

$$\begin{aligned}&\int \tilde{G}\exp[-V]d\mu_{c^{-1}}\exp[W'(0)+\tfrac{1}{2}\delta^2 W'] \\ &= \sum_{\{X'_j\},\{X'_\sigma\}} \prod_j \bar{g}'^{D'}_{X'_j} \prod_\sigma g'^{A_\sigma}_{B_\sigma X'_\sigma} \exp\Bigg[-\sum_{\substack{Y'\cap X'_j=\emptyset\\ Y'\cap X'=\emptyset}} \tilde{V}'_{Y'}\Bigg], \qquad (10)\end{aligned}$$

where X'_j, X'_σ are disjoint, $\cup X'_j \supset D'$ and x_J^{n+1} lies in $(\cup X'_j)\cup(X'_\sigma)$. Again there is no $\prod_j \bar{g}'^{D'}_{X'_j}$ if $D'=\emptyset$ and no $\prod_\sigma g'^{A_\sigma}_{B_\sigma X'_\sigma}$ if x_J^{n+1} lies inside $\cup X'_j$.

The last factor on the right-hand side of (10) is obtained by exponentiation of the polymer sum outisde $(\cup X_j)\cup(\cup X_\sigma)$, see Sect. 3 of [1].

In the next step of our expansion, we shall include $\exp\Bigg[-\sum_{\substack{Y'\cap X'_j=\emptyset\\ Y'\cap(\cup X'_\sigma)\neq\emptyset}} \tilde{V}'_{Y'}\Bigg]$ into this factor applying the Mayer expansion to the compensating one:

$$\begin{aligned}&\int \tilde{G}\exp[-V]d\mu_{c^{-1}}\exp[W'(0)+\tfrac{1}{2}\delta^2 W'] \\ &= \sum_{\{X'_j\},\{X'_\sigma\},\{Y'_\alpha\}} \prod_j \bar{g}'^{D'}_{X'_j} \prod_\sigma g'^{A_\sigma}_{B_\sigma X'_\sigma} \prod_\alpha (\exp[\tilde{V}'_{Y'_\alpha}]-1)\exp\Bigg[-\sum_{Y'\cap X'_j=\emptyset} \tilde{V}'_{Y'}\Bigg], \qquad (11)\end{aligned}$$

where $Y'_\alpha \cap X'_j=\emptyset$ and $Y'_\alpha\cap(\cup X'_\sigma)\neq\emptyset$. Introduce

$$F'^k_{iY'} = \sum_{X',\{Y'_\alpha\}} g'^k_{iX'} \prod_\alpha (\exp[\tilde{V}'_{Y'_\alpha}]-1), \qquad (12)$$

where $X'\cup(\cup Y'_\alpha)=Y'$, $X'\cap Y'_\alpha\neq\emptyset$, and

$$F'^{kl}_{ijY'} = \sum_{X',\{Y'_\alpha\}} g'^{kl}_{ijX'} \prod_\alpha (\exp[\tilde{V}'_{Y'_\alpha}]-1) + \sum_{X'_1,X'_2,\{Y'_\alpha\}} g'^k_{ijX'_1} g'^l_{jX'_2} \prod_\alpha (\exp[\tilde{V}'_{Y'_\alpha}]-1), \qquad (13)$$

where the restrictions on the first sum are as in (12) and in the second one we assume that $X'_1\cup X'_2\cup(\cup X'_\alpha)=Y'$, $X'_1\cap X'_2=\emptyset$, $(X'_1\cup X'_2)\cap Y'_\alpha\neq\emptyset$ and Y' is connected with respect to X'_1, X'_2 and Y'_α.

Note that $F'^k_{iY'}$ are odd and $F'^{kl}_{ijY'}$ are even. In fact $F'^k_{iY'}$ will be equal to the final $\tilde{F}'^k_{iY'}$. With this notation, (11) becomes

$$\int \tilde{G}\exp[-V]d\mu_{c^{-1}}\exp[W'(0)+\tfrac{1}{2}\delta^2 W'] = \sum_{\{X_j\},\{Y'_\sigma\}} \prod_j \bar{g}'^{D'}_{X_j} \prod_\sigma F'_{Y'_\sigma} \exp\Bigg[-\sum_{Y'\cap X_j=\emptyset} \tilde{V}'_{Y'}\Bigg], \qquad (14)$$

with the restrictions on the sums analogous to those of (10).

The last step in our expansion is to extract a constant term from G'. Substituting (14) and (2.14) to (2) and (3), we obtain

$$\delta G(0) = \sum_{Y'} F_{Y'}(0), \qquad (15)$$

and

$$\begin{aligned}\tilde{G}'\exp[-\tilde{V}'] &= \sum_{\{X'_j\},\{Y'_\sigma\}} \prod_j \bar{g}'^{D'}_{X'_j} \prod_\sigma F'_{Y'_\sigma} \exp\Bigg[-\sum_{Y'\cap X'_j=\emptyset} \tilde{V}'_{Y'}\Bigg] \\ &\quad -\sum_{\{X'_j\}} \sum_{Y'_1} \prod_j g'^{D'}_{X'_j} F'_{Y'_1}(0) \exp\Bigg[-\sum_{Y'\cap X'_j=\emptyset} \tilde{V}'_{Y'}\Bigg]. \qquad (16)\end{aligned}$$

Set

$$\tilde{F}'^{k}_{iY'} = F'^{k}_{iY'}, \tag{17}$$

$$\tilde{F}'^{kl}_{ijY'} = F'^{kl}_{ijY'} - F'^{kl}_{ijY'}(0), \tag{18}$$

$$\tilde{g}'^{D'}_{X'} = \bar{g}'^{D'}_{X'}, \tag{19}$$

if X' does not contain both x_1^{n+1} and x_2^{n+1}, and otherwise

$$\tilde{g}'^{D'}_{X'} = \bar{g}'^{D'}_{X'} - \sum_{\{X'_j\}, Y'_1, \{Y'_\alpha\}} \prod_j g'^{D'}_{X'_j} F'^{kl}_{ijY'_1}(0) \prod_\alpha (\exp[-\tilde{V}'_{Y'_\alpha}] - 1), \tag{20}$$

where X'_j are disjoint, $\cup X'_j \supset X' \cap D'$, $Y'_\alpha \cap X'_j = \emptyset$ and X' is connected with respect to X'_j, Y'_1, and Y'_α. Substitution of (17)–(20) to (16) gives

$$\tilde{G}' \exp[-\tilde{V}'] = \sum_{\{X'_j\}, \{Y'_\sigma\}} \prod_j \tilde{g}'^{D'}_{X'_j} \prod_\sigma \tilde{F}'_{Y'_\sigma} \exp\left[-\sum_{Y' \cap X'_j = \emptyset} \tilde{V}'_{Y'}\right], \tag{21}$$

which is (3.13) for $n+1$.

One may also show inductively that for $D_1 \supset D$ (compare (3.6) of [1])

$$\tilde{g}^{nD_1}_{X_1} = \sum_{\{X_j\}, \{Y_\sigma\}, \{Y_\alpha\}} \prod_j \tilde{g}^{nD}_{X_j} \prod_\sigma \tilde{F}_{nY_\sigma} \prod_\alpha (\exp[-\tilde{V}_{Y_\alpha}] - 1), \tag{22}$$

where X_j, Y_σ are disjoint, $(\cup X_j) \cap D = X_1 \cap D$, $Y_\alpha \subset X_1 \setminus \cup X_j$ and X_1 is connected with respect to X_j, Y_σ, Y_α, and c.c. of D_1. Again $\tilde{F}_{nY_\sigma}$ appears when $X_1 \setminus \cup X_j$ contains x_1^n or x_2^n. The proof (22) is deferred to the Appendix.

5. The Estimates

The essential feature of the RG transformation which allows inductive proof of $(\mathrm{A_n})$–$(\mathrm{C_n})$ is the scaling of fields (by $L^{-(d-2)/2}$ and of distances (by L^{-1}). These scalings give rise to contractive properties of the RG.

We assume $(\mathrm{A_n})$ and $(\mathrm{B_n})$, $k+1 \leqq n < N - N_0$, and start with $\tilde{\varrho}^{D'}_{\tilde{X}}$ as given by (4.8). We may follows word by word the analysis of Sect. 5 of [1]. Namely, $\tilde{g}^D_X$ have the same bounds as g^D_X and the bounds on $\tilde{F}^A_{BY}$ (although weaker than those for $\exp[-\tilde{V}_Y] - 1$ are sufficient to produce (5.42), (5.48), and (5.49) of [1]. This settles the $D' \cap \tilde{X} \neq \emptyset$ case. Consider the $D' \cap \tilde{X} = \emptyset$ one (we put $\tilde{\varrho}^{D'}_{\tilde{X}} = \tilde{\varrho}_{\tilde{X}}$ then). For $\bar{p} \neq \emptyset$ terms of (4.8), we obtain immediately the bound

$$\exp[-O((n_0+n)^2)] \exp[-8\alpha \mathscr{L}(X)] G^{-|\tilde{X}|} \tag{1}$$

due to small probability of large Z, see Sect. 5 of [1]. Take now $\bar{p} = 0$. Call $\tilde{X}$ small if $|\tilde{X}| \leqq 2^d$ and $\mathscr{L}(\tilde{X})$ is minimal for given $|\tilde{X}|$. $\tilde{X}$ will be called big if it is not small. For big $\tilde{X}$ there is enough contractive strength coming from the resclaing of the distances to extract the bound

$$|(\tilde{\varrho}_{\tilde{X}})_{\bar{p}=0\ \text{terms}}| \leqq \begin{cases} L^{-\frac{d-\varepsilon}{2}(n+1-k)} \tilde{\delta}^{2n_0+k} \exp[-8\alpha \mathscr{L}(\tilde{X})] G^{-|\tilde{X}|} & \text{for the odd case}, \\ L^{-\frac{d-\varepsilon}{2}(2n+2-k-l)} \tilde{\delta}^{n_0+l} \exp[-8\alpha \mathscr{L}(\tilde{X})] G^{-|\tilde{X}|} & \text{for the even case}. \end{cases} \tag{2}$$

For $\tilde{X}$ small, the only dangerous term in (4.8) is the one with no Y_α, Y_β, no s-derivatives and a single Y_σ with $|\tilde{X}|=|Y_\sigma|$ and $\mathscr{L}(\tilde{X})=\mathscr{L}(Y_\sigma)$ (there is only one such Y_σ containing x_1^n, x_2^n or both for a given $\tilde{X}$). This term is, up to a contribution suppressed by $O((n_0+n)^{v+d}\delta^{n_0+n})$,

$$\int \tilde{F}^{A_\sigma}_{B_\sigma Y_\sigma}(\chi^0)1_0(Z_{L\tilde{X}})d\mu_{c^{-1}}(Z_{L\tilde{X}}). \tag{3}$$

Let us consider the odd case first. To use more efficiently the contraction coming from the rescaling of the fields, write

$$\tilde{F}^k_{iY}(\chi^0)=\frac{d}{dt}\bigg|_{t=0}\tilde{F}^k_{iY}(t\chi^0)+\tilde{\tilde{F}}^k_{iY}(\chi^0). \tag{4}$$

The first term is linear in χ^0. Notice that the function $t\to\tilde{\tilde{F}}^k_{iY}(t\chi^0)$ has the Taylor series at zero starting with t^3 and for $\chi'\in 2\mathscr{K}_{n+1}(\tilde{X})$ and $Z_{L\tilde{X}}$ in the support of 1_0, it is analytic for $|t|<\frac{3}{4}L^{d/2}$, say, and bounded there by twice the right-hand side of (3.15). Hence, at $t=1$, $|\tilde{\tilde{F}}^k_{iY}(\chi^0)|\leqq 2(\frac{3}{4}L^{d/2})^{-3}\cdot$right-hand side of (3.15) by the maximum principle. The first term on the right-hand side of (4) contributes to (3),

$$L^{-d/2}\frac{d}{dt}\bigg|_{t=0}\tilde{F}^k_{iY}(t\chi'_{L^{-1}})\int 1_\sigma(Z_{L\tilde{X}})d\mu_{c^{-1}}(Z_{L\tilde{X}}), \tag{5}$$

which is bounded by

$$L^{-d/2}(1+(n_0+n)^{-1})^v L^{-\frac{d-\varepsilon}{2}(n-k)}\tilde{\delta}^{2n_0+k}\exp[-2\alpha\mathscr{L}(\tilde{X})]. \tag{6}$$

The contribution of the second one is bounded by

$$2(\tfrac{3}{4}L^{d/2})^{-3}L^{-\frac{d-\varepsilon}{2}(n-k)}\tilde{\delta}^{2n_0+k}\exp[-2\alpha\mathscr{L}(\tilde{X})], \tag{7}$$

both for $\chi'\in 2\mathscr{K}_{n+1}(\tilde{X})$. Combining (6) and (7), we conclude that in the odd case

$$|(3)|\leqq L^{-\frac{d}{2}+\frac{\varepsilon}{8}}L^{-\frac{d-\varepsilon}{2}(n-k)}\tilde{\delta}^{2n_0+k}\exp[-2\alpha\mathscr{L}(\tilde{X})] \tag{8}$$

on $2\mathscr{K}_{n+1}(\tilde{X})$ for L and n_0 big.

In the even case we proceed cimilarly writing

$$\tilde{F}^{kl}_{ijY}(\chi^0)=\frac{d^2}{dt^2}\bigg|_{t=1}\tilde{F}^{kl}_{ijY}(t\chi^0)+\tilde{\tilde{F}}^{kl}_{ij}(\chi^0). \tag{9}$$

The first term contributes a term quadratic in χ' bounded on $2\mathscr{K}_{n+1}(\tilde{X})$ by

$$L^{-d}(1+(n_0+n)^{-1})^{2v}L^{-\frac{d-\varepsilon}{2}(2n-k-l)}\delta^{n_0+l}\exp[-2\alpha\mathscr{L}(\tilde{X})], \tag{10}$$

and a constant term bounded by, say,

$$\tfrac{1}{2}L^{-d}L^{-\frac{d-\varepsilon}{2}(2n-k-l)}\tilde{\delta}^{n_0+l}\exp[-2\alpha\mathscr{L}(\tilde{X})] \tag{11}$$

(we recall that $\nabla MZ_{L\tilde{X}}$ is small on the support of 1_0). $\tilde{\tilde{F}}^{kl}_{ijY}$ contributes to (3) a term bounded by

$$2(\tfrac{3}{4}L^{d/2})^{-4}L^{-\frac{d-\varepsilon}{2}(2n-k-l)}\tilde{\delta}^{n_0+l}\exp[-2\alpha\mathscr{L}(X)]. \tag{12}$$

Altogether we obtain in the even case:

$$|(3)-(3)|_{\chi'=0}|\leqq L^{-d+\varepsilon/4}L^{-\frac{d-\varepsilon}{2}(2n-k-l)}\tilde{\delta}^{n_0+l}\exp[-2\alpha\mathscr{L}(X)], \tag{13}$$

and $(3)|_{\chi'=0}$ also satisfies this bound.

The contributions to $\tilde{\varrho}_{\tilde{X}}$, for $\tilde{X}$ small, other than (3) always gain some small factors and we may absorb them into (8) and (13) by increasing ε.

Summarizing, for $\chi'\in 2\mathscr{K}_{n+1}(\tilde{X})$,

$$|\tilde{\varrho}_{\tilde{X}}|\leqq\begin{cases}L^{-\frac{d-\varepsilon}{2}(n+1-k)}\tilde{\delta}^{2n_0+k}\exp]-8\alpha\mathscr{L}(\tilde{X})]G^{-|\tilde{X}|} & \text{for } \tilde{X} \text{ big},\\ L^{-\frac{\varepsilon}{4}-\frac{d-\varepsilon}{2}(n+1-k)}\tilde{\delta}^{2n_0+k}\exp[-2\alpha\mathscr{L}(\tilde{X})] & \text{for } \tilde{X} \text{ small}\end{cases} \tag{14}$$

in the odd case and

$$|\tilde{\varrho}_{\tilde{X}}-\tilde{\varrho}_{\tilde{X}}|_{\chi'=0}|\leqq\begin{cases}L^{-\frac{d-\varepsilon}{2}(2n+2-k-l)}\tilde{\delta}^{n_0+l}\exp[-8\alpha\mathscr{L}(\tilde{X})]G^{-|\tilde{X}|} & \text{for } \tilde{X} \text{ big},\\ \mathrm{L}^{-\frac{\varepsilon}{2}-\frac{d-\varepsilon}{2}(2n+2-k-l)}\tilde{\delta}^{n_0+l}\exp[-2\alpha\mathscr{L}(\tilde{X})] & \text{for } \tilde{X} \text{ small}\end{cases} \tag{15}$$

in the even case. $\tilde{\varrho}_{\tilde{X}}|_{\chi'=0}$ also satisfies (15).

Having bounded $\tilde{\varrho}^{D'}_{\tilde{X}}$, $\varrho^{D'}_{\tilde{X}}$ and their products, we proceed as in Sect. 5 of [1] to obtain the bound of the type (2.15) for $\bar{g}'^{D'}_{X'}$'s with $D'\cap X'\neq\emptyset$ and their products among themselves and with $g'^{D'}_{X'}$'s except that the constant E is increased. g'^{A}_{BX} and $\tilde{F}'^{A}_{BY}$ are bounded immediately with the use of (14), (15) and their definitions (4.9), (4.12), (4.13), (4.17), and (4.18). As a result we obtain (3.15) and (3.16) with n replaced by $n+1$ and

$$|F'^{kl}_{ijY}(0)|\leqq L^{-\frac{d-\varepsilon}{2}(2n+2-k-l)}\tilde{\delta}^{n_0+l}\exp[-2\alpha\mathscr{L}(Y)]. \tag{16}$$

Now, using (4.19), (4.20), and (16) we obtain (2.15) for $n+1$ with some or all $g'^{D'}_{X_j}$ replaced by $\tilde{g}'^{D'}_{X_j}$ and E by a big constant. Finally, the constant is brought down to E by the use of (4.22) as in Sect. 5 of [1]. This ends the proof of (A_{n+1}) and (B_{n+1}), given (A_n) and (B_n). (D_{n+1}) follows from (4.15) and (16).

To show that (A_n)–(C_n) hold for all n, $k+1\leqq n\leqq N-N_0$, we have to start the induction. For the first step [see (3.7)–(3.9)] the procedure is exactly the same as for the next ones, except that for (3.7) we need to decouple the M kernels in the $z_{x_i^n}=(MZ)_{x_i^n}$ as we did for the ∇M kernels (see (3.17) in [1]). We only have to check, that sufficiently small factors arise in (4.8). For $\tilde{G}^k_{k+1,i}$ one may always extract an $O((n_0+k)^{\nu+d}\delta^{n_0+k})$ factor, since $\partial_s\int z^s 1_0(Z)d\mu_{c^{-1}}(Z)=0$. For $\tilde{G}^{kl}_{k+1,ij}$, $k>l$, $\tilde{F}^l_{k,iY}$ provide the necessary small contributions (to control the combinatorics we use one $\tilde{\delta}^{n_0}$ factor). Moreover (still for $k>l$)

$$|G^{kl}_{k+1,ij}(0)|\leqq CL^{-\frac{d-\varepsilon}{2}(k-l)}\tilde{\delta}^{n_0+l}\exp[-\alpha d(x_1^{k+1},x_2^{k+1})]. \tag{17}$$

Finally consider $G^{kk}_{k+1,12}$. $F^{kk}_{k+1,12Y}$ has extra $O((n_0+k)^{\nu}\delta^{n_0+k})$ factor in all other terms except the one given by (here $Y=\Delta_1\cup\Delta_2\cup\overline{x_1^{k+1}}\cup\overline{x_2^{k+1}}$, Δ_i blocks)

$$\begin{aligned}F^{kk,0}_{k+1,12Y}&\equiv\sum_{u\in\Delta_1,v\in\Delta_2}\int Z_uZ_v1_0(Z)d\mu_{c^{-1}}(Z)(M)_{x_1^ku}(M)_{x_2^kv}\\&=\sum_{u,v}(c_k^{-1}+O(e^{-\varepsilon n^2}))(M)_{x_1^ku}(M)_{x_2^kv}\delta_{uv}\,.\end{aligned}\tag{18}$$

Thus $\tilde{F}^{kk}_{k+1,12Y}$ satisfies our claims and

$$\begin{aligned}G^{kk}_{k+1,12}(0)=\sum_Y F^{kk}_{k+1,12Y}(0)&=c_k^{-1}\sum_u(M)_{x_1^ku}(M)_{x_2^ku}\\&\quad+O(\tilde{\delta}^{n_0+k}\exp[-\alpha d(x_1^{k+1},x_2^{k+1})])\,.\end{aligned}\tag{19}$$

Since

$$\sum_u(M)_{xu}(M)_{yu}=\mathscr{T}_{kxy}\tag{20}$$

($\mathscr{T}_k$ is the free covariance of z^k), we obtain

$$|G^{kl}_{k+1,ij}(0)-\delta_{kl}c_k^{-1}\mathscr{T}_{kx_1^kx_2^k}|\leqq CL^{-\frac{d-\varepsilon}{2}(k-l)}\tilde{\delta}^{n_0+l}\exp[-\alpha d(x_1^{k+1},x_2^{k+1})]\,.\tag{21}$$

In order to control G_{xy}, as given by (3.12), we still have to estimate the expectations $\langle - \rangle_{\mathscr{R}^{N-N_0}\mathscr{H}}$ appearing there and in (3.11). Notice that

$$\langle - \rangle_{\mathscr{R}^{N-N_0}\mathscr{H}}=\frac{1}{\mathscr{N}}\int - \exp[-V_\Delta^{N-N_0}(\nabla\psi^{N-N_0})d\mu_{\bar{G}_{N-N_0}}(\phi^{N-N_0})]\,.\tag{22}$$

Both in the numerator and in the denominator we consider separately ϕ^{N-N_0} such that $D_{N-N_0}(\nabla\psi^{N-N_0})=\Delta$ (large fields) and $D_{N-N_0}(\nabla\psi^{N-N_0})=\emptyset$ (small fields).

For large fields the integrands are easily bounded (with use of (A_{N-N_0})) by

$$\text{const}\exp[O(\kappa)\int_\Delta dz(\nabla\psi_z^{N-N_0})^2]\Big(1+\sum_{x\in\Delta}(\phi^{N-N_0})^2\Big).\tag{23}$$

The latter is integrable with respect to $d\mu_{\bar{G}_{N-N_0}}$, since

$$(\phi^{N-N_0}|\bar{G}^{-1}_{N-N_0}\phi^{N-N_0})=c_{N-N_0}\int_\Delta(\nabla\psi^{N-N_0})^2+L^{2(N-N_0)}\xi L^{-N_0d}\Big(\sum_{x\in\Delta}\phi_x^{N-N_0}\Big)^2\tag{24}$$

(take $\xi>L^{-2N}$). Moreover, using (7) of Appendix 3 in [1], we may extract from its integral an $\exp[-O((n_0+N-N_0)^{2\nu-d^2})]$ factor ($2\nu-d^2>1$!).

For small field integral, we use the small field bounds of (B_{N-N_0}). The constant contribution to $G^{kl}_{N-N_0ij}$, bounded with the use of (21) and $(\mathrm{D_n})$ goes through the expectation $\langle - \rangle_{\mathscr{R}^{N-N_0}\mathscr{H}}$. The results are

$$\begin{aligned}|G^{kl}_{Nij}-c_k^{-1}\delta_{kl}\mathscr{T}_{kx_1^kx_2^k}|&\leqq C\sum_{n=k}^{N-N_0}L^{-\frac{d-\varepsilon}{2}(2n-k-l)}\tilde{\delta}^{n_0+l}\exp[-\alpha d(x_1^{n+1},x_2^{n+1})]\\&\leqq CL^{-\frac{d-\varepsilon}{2}(k-l)}\tilde{\delta}^{n_0+l}[1+d(x_1^k,x_2^k)]^{-d+\varepsilon}\,,\end{aligned}\tag{25}$$

$$|\langle\psi^{N-N_0}_{x_1^{N-N_0}}\psi^{N-N_0}_{x_2^{N-N_0}}\rangle_{\mathscr{R}^{N-N_0}\mathscr{H}}|\leqq C\,,\tag{26}$$

$$|\langle\psi^{N-N_0}_{x_1^{N-N_0}}G^k_{N-N_0,i}\rangle_{\mathscr{R}^{N-N_0}\mathscr{H}}|\leqq CL^{-\frac{d-\varepsilon}{2}(N-k)}\tilde{\delta}^{2n_0+k}\,.\tag{27}$$

Substituting (25)–(27) to (3.12), we obtain

$$\left|G_{xy}-\sum_{k=0}^{N-N_0-1}\gamma^{2k}\mathcal{T}_{kx_1^kx_2^k}c_k^{-1}\right|\leqq C\sum_{l=0}^{N-N_0-1}\sum_{k=l}^{N-N_0-1}\gamma^{l+k}L^{-\frac{d-\varepsilon}{2}(k-l)}$$
$$\cdot\tilde{\delta}^{n_0+l}(1+d(x_1^k,x_2^k))^{-d+\varepsilon}+C\gamma^{2N}$$
$$\leqq C\sum_{k=0}^{N-N_0-1}\gamma^{2k}\tilde{\delta}^{n_0+k}(1+d(x_1^k,x_2^k))^{-d+\varepsilon}+C\gamma^{2N}$$
$$\leqq C\tilde{\delta}^{n_0}(1+d(x_1,x_2))^{-d+2-\varepsilon}+C\gamma^{2N}. \tag{28}$$

Now

$$\sum_{k=0}^{N-N_0-1}\gamma^{2k}\mathcal{T}_{kx_1^kx_2^k}c_k^{-1}=c_{N-N_0}^{-1}\sum_{k=0}^{N-N_0-1}\gamma^{2k}\mathcal{T}_{kx_1^kx_2^k}$$
$$-\sum_{k=0}^{N-N_0-1}\gamma^{2k}\mathcal{T}_{kx_1^kx_2^k}(c_{N-N_0}^{-1}-c_k^{-1}). \tag{29}$$

The first term on the right-hand side of (29) differs from $c_{N-N_0}^{-1}$ times the free two-point function $G^0_{x_1,x_2}$ by $c_{N-N_0}^{-1}\gamma^{2(N-N_0)}\langle\psi^{N-N_0}_{x_1^{N-N_0}}\psi^{N-N_0}_{x_2^{N-N_0}}\rangle_{G_{N-N_0}}$, which is smaller than $C\gamma^{2N}$, compare (26). The second one is bounded, with the use of (2.19) and $|\mathcal{T}_{kxy}|\leqq C\exp[-\alpha|x-y|]$ (see [1]) by

$$C\sum_{k=0}^{N-N_0-1}\gamma^{2k}\delta^{n_0+k}\exp[-\alpha d(x_1^k,x_2^k)]\leqq C\delta^{n_0}(1+d(x_1^k,x_2^k))^{-d+2-\varepsilon}. \tag{30}$$

Summarizing,

$$|G_{x_1x_2}-c_{N-N_0}^{-1}G^0_{x_1x_2}|\leqq C\delta^{n_0}(1+d(x_1,x_2))^{-d+2-\varepsilon}+C\gamma^{2N}. \tag{31}$$

As far as the thermodynamic limit is concerned, it is straightforward to prove by induction that $\tilde{g}_X^{nD}$ and $\tilde{F}^A_{nBY}$, as well as g_X^{nD} and $\tilde{V}_Y^n$, K^n, and c_n, converge (for $n+1$ the volume dependence enters only through $\tilde{g}_X^{nD}$, F^A_{nBY}, g_X^{nD}, $\tilde{V}_Y^n$, $\delta^2V_Y^n$, c_n and the kernels M^n; all our estimates are uniform in volume). As a consequence, also G_N^{kl} has the limit when $N\to\infty$ ($G^{kl}_{N-N_0}(0)$ does since $\delta G_n(0)$ converge and fulfill ($\mathrm{C_n}$); the contribution of $\tilde{G}^{kl}_{N-N_0}$ to G_N^{kl} goes down with N by virtue of ($\mathrm{B_n}$)). As a consequence of (3.12), $G_{x_1x_2}$ has the thermodynamic limit. Since $c_{N-N_0}\xrightarrow[N\to\infty]{}c(v)$, (31) becomes for the infinite volume quantities

$$|G_{x_1x_2}-c(v)^{-1}G^0_{x_1x_2}|\leqq C\delta^{n_0}(1+d(x_1,x_2))^{-d+2-\varepsilon}. \tag{32}$$

This gives (1.5).

The analyticity of the infinite volume limit in v also follows via a straightforward inductive argument.

6. The General Correlations

It is now rather straightforward to generalize the above analysis to a general correlation function. In this section we will explain first the idea for the general case and then carry out the analysis in more detail for the 4-point function.

Thus consider iterating (3.2) and (3.4) for a general $\langle \psi_I \rangle_{\mathscr{H}}$:

$$\langle \psi_I \rangle_{\mathscr{H}} = \sum_{J_1 \subset I} \gamma^{|J_1|} \langle \psi^1_{J_1} \langle z^0_{I \setminus J_1} \rangle_{Z^0} \rangle_{\mathscr{R}\mathscr{H}}$$
$$= \sum_{J_2 \subset I} \sum_{J_1 \subset I \setminus J_2} \gamma^{2|J_2|} \langle \psi^2_{J_2} \gamma^{|J_1|} \langle z^1_{J_1} \langle z^0_{I \setminus (J_1 \cup J_2)} \rangle_{Z^0} \rangle_{Z^1} \rangle_{\mathscr{R}^2\mathscr{H}}$$
$$= \sum_{p=1}^{|I|} \sum_{N-N_0 > n_1 > n_2 > \ldots > n_p} \sum_{\substack{\{I_j\}_1^p \\ \text{part. of } I}} \langle G(\{n_j\}, \{I_j\}) \rangle_{\mathscr{R}^{N-N_0}\mathscr{H}}$$
$$+ \sum_{\emptyset \neq J \subset I} \gamma^{(N-N_0)|J|} \left\langle \psi^{N-N_0}_J \sum_{p=1}^{|I \setminus J|} \sum_{n_1 > \ldots > n_p} \sum_{\{I_j\}} G(\{n_j\}, \{I_j\}) \right\rangle_{\mathscr{R}^{N-N_0}\mathscr{H}}, \tag{1}$$

where

$$G(\{n_j\}, \{I_j\}) = S^{N-N_0-n_1-1} \langle \gamma^{n_1|J_1|} z^{n_1}_{J_1} S^{n_1-n_2-1} \langle \gamma^{n_2|J_2|} z^{n_2}_{J_2} S^{n_2-n_3-1} \ldots$$
$$\ldots S^{n_{p-1}-n_p-1} \langle \gamma^{n_p|J_p|} z^{n_p}_{J_p} \rangle_{Z^{n_p}} \rangle_{Z^{n_{p-1}}} \ldots \rangle_{Z^{n_1}}. \tag{2}$$

Thus, to start with, let us analyze

$$\langle z^k_I \rangle_{Z^k} \equiv \left\langle \prod_{i=I} z^k_{x^k_i} \right\rangle_{Z^k} \tag{3}$$

(we will often suppress the index k below). Expanding, as in the case of the two point function, and gathering clusters around D' and the x^{k+1}_i's, we obtain the analogue of (4.14):

$$\langle z_I \rangle_Z \exp[-\tilde{V}'] = \sum_{\{X'_j\}} \sum_{\{Y'_\sigma\}} \prod \bar{g}'^{D'}_{X'_j} \prod F'_{Y_\sigma} \exp\left[-\sum_{Y' \cap X'_j = \emptyset} \tilde{V}'_{Y'}\right], \tag{4}$$

where the X'_j, Y'_σ are disjoint, $X' \equiv \cup X'_j \supset D'$, $X'_j \cap D' = \cup \text{c.c.}\, D' \neq \emptyset$ and $x'_I \equiv \cup x^{k+1}_i \subset X' \cup (\cup Y'_\sigma)$. Again, if $X'_j \cap x'_I = \emptyset$, $\bar{g}'^{D'}_{X'_j} = g'^{D'}_{X'_j}$, and there is no j-product if $D' = \emptyset$ and no σ-one if $x'_I \subset X'$, and of course $x'_I \cap Y_\sigma \neq \emptyset$. In order to control the iterations of (4) (after applying S to it or to z^{k+1}_J times it) we need to take out the constant parts of F'_{Y_σ}'s which would not contract in the iteration. We repeat the analysis (4.15)–(4.21) for the expansion (4). For this purpose, denote explicitly in (4) the dependence of $\bar{g}$ and F on $I : \bar{g}^{D,I}_X$, F^I_Y (we drop also the primes now). Consider, for some fixed $i \in I$ (we often identify below i and x^k_i)

$$\sum_{j \neq i} \left(\sum_{Y \supset i \cup j} F^{\{ij\}}_Y(0) \right) \sum_{\{X_l\}, \{Y_\sigma\}} \prod_l \bar{g}^{D, I \setminus \{ij\}}_{X_l} \prod_\sigma F^{I \setminus \{ij\}}_{Y_\sigma} \exp\left[-\sum_{Y \cap X = \emptyset} \tilde{V}_Y\right], \tag{5}$$

which is a term we will subtract from (4); it equals

$$\sum_{j \neq i} \langle z_i z_j \rangle_Z(0) \langle z_{I \setminus \{ij\}} \rangle_Z \exp[-\tilde{V}]. \tag{6}$$

Expand now $\exp[-\sum \tilde{V}_Y]$ in (5), gather disjoint clusters and resum; (5) becomes

$$\sum_{\{X_l\}\{Y_\sigma\}} \prod_l \bar{g}^{D,I,i}_l \prod_\sigma F^{I,i}_{Y_\sigma} \exp\left[-\sum_{Y \cap X_l = \emptyset} \tilde{V}_Y\right], \tag{7}$$

with

$$\bar{g}^{D,I,i}_X = \sum_{j : \{ij\} \subset X} \sum_{\{ij\} \subset Y \subset X} \sum_{\substack{\{X_l\}, \{Y_\sigma\} \\ \{Y_\alpha\}}} F^{\{ij\}}_Y(0) \prod \bar{g}^{D, I \setminus \{ij\}}_{X_l} \prod F^{I \setminus \{ij\}}_{Y_\sigma} \prod_\alpha (\exp[-\tilde{V}_{Y_\alpha}] - 1), \tag{8}$$

$$F^{I,i}_{\tilde{Y}} = \sum_{j : \{ij\} \subset \tilde{Y}} \sum_{\{ij\} \subset Y \subset \tilde{Y}} \sum_{\{Y_\sigma\} \text{disj.}} F^{\{ij\}}_Y(0) \prod F^{I \setminus \{ij\}}_{Y_\sigma}, \tag{9}$$

where $\{X_l\}$, $\{Y_\sigma\}$ in (8) are disjoint, $Y_\alpha \cap X_l = \emptyset$ and X is connected with respect to Y, X_l, Y_σ and Y_α and $\tilde{Y}$ in (9) with respect to Y, Y_σ. Also note that in (7) only one $\bar{g}$ or F is not $\bar{g}^{D,I}_{X_l}$ or $F^I_{Y_\sigma}$, namely the one for which $i \in X_l$ or Y_σ.

The idea now is as follows. Subtracting (5) from (4) amounts to subtracting a "gaussian" contribution: $F^I_{\tilde{Y}} - F^{I,i}_{\tilde{Y}}$ for $i \in \tilde{Y}$ will be $0(\delta^{n_0+k})$; (5) will be the main contribution to (4). We now apply S to the difference, which again is of the same form (see (10) below). We repeat this until there are $x^{k+p}_m \neq x^{k+p}_n$ in a common small Y. Then we need to subtract $F^I_Y(0)$, as is evident from Sect. 5; the F_Y with Y small will not contract in our scheme (in fact the estimates would blow up). If other such pairs exist, we need to subtract them too. Thus, in more detail, fix i and write

$$\langle z_I \rangle_Z \exp[-\tilde{V}] = \sum_{j \neq i} \langle z_i z_j \rangle_Z(0) \langle z_{I \setminus \{ij\}} \rangle_Z \exp[-\tilde{V}]$$

$$+ \sum_{\{X_l\},\{Y_\sigma\}} \prod_l \tilde{g}^{D,I,i}_{X_l} \prod_\sigma \tilde{F}^{I,i}_{Y_\sigma} \exp\Bigg[-\sum_{Y \cap X_l = \emptyset} \tilde{V}_Y\Bigg], \tag{10}$$

with

$$\tilde{g}^{D,I,i}_X = \bar{g}^{D,I}_X - \bar{g}^{D,I,i}_X, \tag{11}$$

$$\tilde{F}^{I,i}_Y = F^I_Y - F^{I,i}_Y. \tag{12}$$

Let us consider the estimation of the $\tilde{g}$ and the $\tilde{F}$. Consider F^I_Y first. Since the main contribution for it is given by

$$F^I_{Y,0} = \sum_{\{\bar{U}_y\}} \int \prod S(\bar{U}_y) z_{I \cap LY} 1_0(Z_{LY}) d\mu_{c^{-1}}(Z_{LY}), \tag{13}$$

we get easily the bound

$$|F^I_Y| \leqq C_{|I \cap Y|} \exp[-2\alpha \mathscr{L}(Y)], \tag{14}$$

where $C_{|I \cap Y|}$ depends on the number of points, $|I \cap Y|$, in $I \cap Y$. Similarily $\bar{g}^{D,I}_X$ satisfies the bound (2.15) with a multiplicative constant $C_{|I \cap X|}$ as in (14). For $\tilde{F}^{I,i}_Y$ we claim that the bound

$$|\tilde{F}^{I,i}_Y| \leqq C_{|I \cap Y|} \tilde{\delta}^{n_0+k} \exp[-2\alpha \mathscr{L}(Y)] \tag{15}$$

holds. To prove this, note, that it suffices to consider (9) with F^J_Y replaced by $F^J_{Y,0}$ (given by (13)) as well as F^I_Y replaced by $F^I_{Y,0}$, the error being bounded by (15). Moreover, we may omit 1_0 in (13), again with the error bounded by (15). This reduces F^I_Y to $\bar{F}^I_{Y,0}$, given by

$$\bar{F}^I_{Y,0} = \sum_{\{\langle ij \rangle\}} \sum_{\{Y_{ij}\} \supset \{ij\}} \prod \bar{F}^{\{ij\}}_{Y_{ij},0}, \tag{16}$$

with Y connected with respect to Y_{ij}, $\{\langle ij \rangle\}$ running through the pairings of I. Equation (16) follows from (13) since we have a gaussian integral left as $1_0 \to 1$. But (15) for F's replaced by $\bar{F}$'s is straightforward.

Consider finally $\tilde{g}^{D,I,i}_X$. By (8), (11), (14) and the corresponding bound for $\bar{g}^{D,I,i}_X$, $\tilde{g}^{D,I,i}_X$ also satisfies (2.15) with some multiplicative constant $C_{|X \cap I|}$, which we now take the same for $\bar{g}$, F, $\tilde{F}$ and $\tilde{g}$. For being able to iterate the bounds for $\tilde{g}$, we need the "cocycle" property, analogous to (4.22) for $\tilde{g}$. Namely, for $D' \supset D$

$$\tilde{g}^{D',I,i}_X = \sum_{\{X_j\},\{Y_\sigma\},\{Y_\alpha\}} \prod_j \tilde{g}^{D,I,i}_{X_j} \prod_\sigma \tilde{F}^{I,i}_{Y_\sigma} \prod_\alpha (\exp[-\tilde{V}_{Y_\alpha}] - 1) \tag{17}$$

with X_j, Y_σ disjoint, $Y_\alpha \cap X_j = \emptyset$ and X connected with respect to X_j, Y_σ, Y and c.c. of D'. This follows from the corresponding one for $\bar{g}$: (proved as for $|I|=2$, see Appendix)

$$\bar{g}_{X_j}^{D',I} = \sum_{\{X_{jk}\},\{Y_{\sigma j}\},\{Y_{\alpha j}\}} \prod \bar{g}_{X_{jk}}^{D,I} \prod F_{Y_{\sigma j}}^{I} \prod (\exp[-\tilde{V}_{\alpha j}]-1) \tag{18}$$

which, when inserted to (8), yields (17) for $\bar{g}_X^{D',I,i}$, and then by (11) for $\tilde{g}_X^{D',I,i}$. Equation (10) is the starting point for the iteration. The main (only) contribution in the scaling limit will be given by the first term. We shall assume, inductively in $|I|$, that we can cope with it.

Thus we wish to address the iteration of the second term of (10), namely, the application of S to it or to z_J times it as we are advised to do by (1) and (2). Consider first case. Denote

$$He^{-\tilde{V}} = \sum \prod_l \tilde{g}_{X_l}^{D,I,i} \prod_\sigma \tilde{F}_{Y_\sigma}^{I,i} \exp[-\sum \tilde{V}_Y]. \tag{19}$$

The expansion described in Sect. 4 may be applied to (19), giving

$$H'e^{-\tilde{V}'} = S(H)\exp[-\tilde{V}'] = \sum \prod \bar{g}_{X_l}^{\prime D',I,i} \prod F_{Y_\sigma}^{\prime I,i} \exp[-\sum \tilde{V}_Y']. \tag{20}$$

We claim the $\bar{g}'$, F' satisfy the bounds

$$(\alpha) \qquad \bar{g}_X' \text{ the same as } \tfrac{1}{2}G^{-L^{-N_0}\mathscr{L}(X)}\tilde{g}_X, \quad \text{with} \quad n \to n+1, \tag{21}$$

$$(\beta) \ |F_Y^{\prime I,i}| \leqq C_{|I\cap Y|} G^{-L^{-N_0}\mathscr{L}(Y)} \exp[-2\alpha\mathscr{L}(Y)] \begin{Bmatrix} \tilde{\delta}^{n_0+k}, & i \in Y \\ 1, & i \notin Y \end{Bmatrix} \quad \text{for } Y \text{ big}, \tag{22}$$

$$(\gamma) \qquad |F_Y^{\prime I,i}(0)| \leqq (1+\varepsilon) C_{|I\cap Y|} \exp[-2\alpha\mathscr{L}(Y)] \begin{Bmatrix} \tilde{\delta}^{n_0+k} \\ 1 \end{Bmatrix} \quad \text{for } Y \text{ small}, \tag{23}$$

$$(\delta) \ |F_Y^{\prime I,i} - F_Y^{\prime I,i}(0)| \leqq C_{|I\cap Y|} L^{-(d-\varepsilon)} \exp[-2\alpha\mathscr{L}(Y)] \begin{Bmatrix} \tilde{\delta}^{n_0+k} \\ 1 \end{Bmatrix} \quad \begin{matrix} \text{for } Y \text{ small}, \\ I\cap Y \text{ even}, \end{matrix} \tag{24}$$

$$(\varepsilon) \qquad |F_Y^{\prime I,i}| \leqq C_{|I\cap Y|} L^{-\frac{1}{2}(d-\varepsilon)} \exp[-2\alpha\mathscr{L}(Y)] \begin{Bmatrix} \tilde{\delta}^{n_0+k} \\ 1 \end{Bmatrix} \quad \begin{matrix} \text{for } Y \text{ small}, \\ I\cap Y \text{ odd}. \end{matrix} \tag{25}$$

In fact, (21)–(25) will hold provided we choose the constants (C_n) properly (see below). Consider e.g. (β).

The leading term, i.e. no R, $(e^{-\tilde{V}}-1)$ etc. factors in the cluster integral, is

$$\sum_{\{Y_\sigma\}} \sum_{\{\bar{U}_\gamma\}} \int \prod S(\bar{U}_\gamma) \prod \tilde{F}_{Y_\sigma}^{I,i} 1_0(Z_{LY}) d\mu(Z_{LY}). \tag{26}$$

The G factor (recall from [1]; G may be chosen big) arises as before from the contraction of space. The only difference with our previous analysis is the constants $C_{|I\cap Y|}$. Let us choose (C_n) so rapidly increasing in n that

$$\sum_{\{I_\sigma\} \text{ part. of } \tilde{I}} \prod C_{|I_\sigma|} \leqq (1+\varepsilon) C_{|\tilde{I}|} \quad \text{for all} \quad \tilde{I} \subseteqq I. \tag{27}$$

This is possible of course. Now the bound (22) and similarly all the others follow by an analysis similar to that for the two point function. (We may extract the $\frac{1}{2}G^{-\frac{1}{N_0}\mathscr{L}(X)}$ in (α) from the redefinition of κ and contraction as in (β): see [1]).

The terms (γ) in (20) are of course not satisfactory since they expand: we need to subtract them. Let for some j there be several k in a common small Y_j containing them.

Denote $I_j = I \cap Y_j$ and write

$$\sum \prod_l \bar{g}_{X_l}^{\prime D', I, i} \prod_\sigma F_{Y_\sigma}^{\prime I, i} \exp[-\sum \tilde{V}_Y'] = F_{Y_j}^{\prime I_j, i}(0) \sum \prod \bar{g}_{X_l}^{\prime D, I \backslash I_j, i}$$
$$\cdot \prod_\sigma F_{Y_\sigma}^{\prime I \backslash I_j, i} \exp[-\sum \tilde{V}_Y'] + \sum \prod_l \tilde{g}_{X_l}^{\prime D, I, i, j} \prod \tilde{F}_{Y_\sigma}^{\prime I, i, j} \exp[-\sum \tilde{V}_Y'], \tag{28}$$

where of course $\bar{g}'^{D, I \backslash I_j, i}$ are obtained from applying S to (19) with I replaced by $I \backslash I_j$, and if $i \notin I \backslash I_j$, the i-index is superfluous. Again (28) is derived by first writing the first term on the right-hand side as

$$\sum \prod \bar{g}_{X_l}^{\prime D', I, i, j} \prod F_{Y_\sigma}^{\prime I, i, j} \exp[-\sum \tilde{V}_Y'], \tag{29}$$

with e.g.

$$F_Y^{\prime I, i, j} = \sum F_{Y_j}^{\prime I_j, i}(0) \prod_\sigma \tilde{F}_{Y_\sigma}^{\prime I \backslash I_j, i} \tag{30}$$

(Y_σ disjoint, Y connected with respect to Y_j, Y_σ). There is again only one term in the products in (29) different from the $\bar{g}'$, F' on the left-hand side in (28), namely, the one with X_l (or Y_σ) containing Y_j (if $Y_j \cap X = \emptyset$, then $\bar{g}_X^{\prime D, I, i, j} = \bar{g}_X^{\prime D, I \backslash I_j, i} = \bar{g}_X^{\prime D, I, i}$; similarly for F's). This is why we may define the new $\tilde{g}$, $\tilde{F}$ on the right-hand side of (28) in the second term. The point of (28) is that

$$\tilde{F}_{Y_j}^{\prime I, i, j} = F_{Y_j}^{\prime I, i} - F_{Y_j}^{\prime I_j, i}(0) = F_{Y_j}^{\prime I_j, i} - F_{Y_j}^{\prime I_j, i}(0), \tag{31}$$

which by (24) has now contracted.

Equation (28) will now be repeated to the second term on its right-hand side until all (γ)-type F's are subtracted (the first terms on the right-hand side are treated inductively). There might be many such subtractions and they contribute new terms to the $\tilde{g}_X'$, $\tilde{F}_Y'$. These may be bounded using the G-factors in (α) and (β): the more contributions to $\tilde{g}_X'$ or $\tilde{F}_Y'$, the bigger X or Y has to be. The reader may easily convince himself that after all the subtractions we have obtained

$$H' \exp[-\tilde{V}'] = \sum \prod \tilde{g}_{X_j}^{\prime D', I} \prod \tilde{F}_Y^{\prime I} \exp[-\sum \tilde{V}_Y'], \tag{32}$$

where we suppress the i, j, etc., such that $\tilde{g}_X^{\prime D', I}$ satisfies (2.15) with the constant $C_{|I \cap X|}$, and denoting by $\mathcal{N}(J)$ the biggest number of disconnected blocks in $\overline{x_J^{k+1}}$,

$$|\tilde{F}_Y^{\prime I}| \leq C_{|I \cap Y|} L^{-\frac{d-\varepsilon}{2} \mathcal{N}(I \cap Y)} \exp[-2\alpha \mathcal{L}(Y)] \begin{Bmatrix} \bar{\delta}^{n_0 + k} \\ 1 \end{Bmatrix},$$
$$\leq C_{|I \cap Y|} L^{-(d-\varepsilon)} \exp[-2\alpha \mathcal{L}(Y)] \begin{Bmatrix} \bar{\delta}^{n_0 + k} \\ 1 \end{Bmatrix} \quad \text{for } Y \text{ small, } I \cap Y \text{ even}. \tag{33}$$

The iteration may proceed now. For $S(z_J H)$ we may derive an analogous expansion as for H, now with $\tilde{g}_X^{D, I \cup J}$, $\tilde{F}_Y^{D, I \cup J}$. Similar bounds follow for them (with suitable (C_n)) and then applications of S are controlled as before. There are a finite number of steps when z_J are added and after a finite number of steps all x_i^{k+p} are in the same block, whence the iteration is as that of the two-point function. To see how this process may be carried through in detail to prove the triviality of the scaling limit, let us restrict ourselves to the four-point function. To extend this analysis to a general correlation is just a matter of bookkeeping.

Thus take $|I|=4$ in (1) and consider the first term; the second one will turn out to vanish as $N\to\infty$. Take the $p=1$ term first. This is given by

$$\sum_{n=0}^{N-N_0-1} \gamma^{4n}\langle S^{N-N_0-n-1}\langle z^n_{x^n_1}\dots z^n_{x^n_4}\rangle_{Z^n}\rangle_{\mathscr{R}^{N-N_0}\mathscr{H}}. \tag{34}$$

By (10) we may write

$$\langle z^n_{x^n_1}\dots z^n_{x^n_4}\rangle_{Z^n}\exp[-\tilde V^{n+1}]=\sum_{j=2}^{4}\langle z^n_{x^n_1}z^n_{x^n_j}\rangle_{Z^n}(0)\langle z^n_{x^n_k}z^n_{x^n_l}\rangle_{Z^n}e^{-\tilde V^{n+1}}+H_{n+1}e^{-\tilde V^{n+1}}, \tag{35}$$

$$H_{n+1}e^{-\tilde V^{n+1}}=\sum\prod\tilde g^{nD}_{X_j}\prod\tilde F^n_{Y_\sigma}\exp[-\tilde V^{n+1}_Y], \tag{36}$$

where we suppress I, i.

We already know how to control the first term from the analysis of the two-point function [see (5.25), (5.19)]:

$$\langle S^{N-N_0-n-1}\langle z^n_{x^n_k}z^n_{x^n_l}\rangle_{Z^n}\rangle_{\mathscr{R}^{N-N_0}\mathscr{H}}=c_n^{-1}\mathscr{T}_{nx^n_kx^n_l}+O(\tilde\delta^{n_0+n}(1+d(x^n_k,x^n_l))^{-d+\varepsilon}), \tag{37}$$

$$|\langle z^n_{x^n_1}z^n_{x^n_j}\rangle_{Z^n}(0)-c_n^{-1}\mathscr{T}_{nx^n_1x^n_j}|\leqq C\tilde\delta^{n_0+n}\exp[-\alpha d(x^{n+1}_1,x^{n+1}_2)]. \tag{38}$$

Since

$$\sum_{n=0}^{\infty}\gamma^{2n}\tilde\delta^{n_0+n}\exp[-2\alpha d(x^n_1,x^n_j)]\leqq C\tilde\delta^{n_0}(1+d(x_1,x_j))^{-d+2-\varepsilon}, \tag{39}$$

$$\sum_{n=0}^{\infty}\gamma^{2n}\tilde\delta^{n_0+n}(1+d(x^n_k,x^n_l))^{-d+\varepsilon}\leqq C\tilde\delta^{n_0}(1+d(x_k,x_l))^{-d+2-\varepsilon}, \tag{40}$$

we get from the first term in (35) a contribution to (34),

$$\sum_{\text{pairings}}\left[\sum_{n=0}^{N-N_0-1}\gamma^{4n}c_n^{-2}\mathscr{T}_{nx^n_ix^n_j}\mathscr{T}_{nx^n_kx^n_l}+O(\tilde\delta^{n_0}(1+d(x_i,x_j))^{-d+2}(1+d(x_k,x_l))^{-d+2-\varepsilon})\right]. \tag{41}$$

H_{n+1} will be studied as explained above. Denote $x\approx y$ if x and y lie in the same small Y. Let m_1 be the first m such that for some i,j, $x^{m_1-1}_i\approx x^{m_1-1}_j$. We may assume that the present i coincides with the original one. We write

$$\begin{aligned}H_{m_1}\exp[-\tilde V^{m_1}]&=S^{m_1-n-1}(H^{n+1})\exp[-\tilde V^{m_1}]=F^{ij}_{m_1Y}(0)\sum\prod\tilde g^{m_1,D,kl}_{X_l}\prod F^{kl}_{m_1Y_\sigma}\\&\quad\cdot\exp[-\sum\tilde V^{m_1}_Y]+\tilde H_{m_1}e^{-\tilde V^{m_1}}\equiv(F^{ij}_{m_1Y}(0)G_{m_1kl}+\tilde H_{m_1})\exp[-\tilde V^{m_1}].\end{aligned} \tag{42}$$

The first term is again of the two point type whereas to $\tilde H^{m_1}$ we apply $S^{m_2-m_1}$, where m_2 is for the next pair: $x^{m_2-1}_{i'}\approx x^{m_2-1}_{j'}$; note that this might correspond to two new x_i's or one new collapsing in the next step to the small set where $x^{m_2}_i$ and $x^{m_2}_j$ lie. Thus

$$S^{m_2-m_1}H_{m_1}=\sum_{p=0}^{m_2-m_1-1}\sum_{Y_p}F^{ij}_{m_1+pY_p}(0)S^{m_2-m_1-p}G_{m_1+p,kl}+S(\tilde H_{m_2-1}). \tag{43}$$

We estimate the two-point terms

$$|F^{ij}_{m_1+pY_p}(0)| \leqq C\tilde{\delta}^{n_0+n} L^{(-d+\varepsilon)(m_1+p-n)}, \tag{44}$$

$$\begin{aligned}|\langle S^{N-N_0-m_1-p-1} G_{m_1+p,kl}\rangle_{\mathscr{R}^{N-N_0}\mathscr{H}}| &\leqq CL^{-(d-\varepsilon)(m_1+p-n)}(1+d(x_k^{m_1+p}, x_l^{m_1+p}))^{-d+\varepsilon} \\ &\leqq C(1+d(x_k^n, x_l^n))^{-d+\varepsilon},\end{aligned} \tag{45}$$

which imply that the first piece in (43) contributes to (34) (if $n > m_1$ the analysis is similar)

$$\begin{aligned}&\sum_{n=0}^{m_2} C\gamma^{4n}\tilde{\delta}^{n_0+n} \sum_{l=\max(n,m_1)}^{m_2-1} L^{(-d+\varepsilon)(l-n)}(1+d(x_k^n, x_l^n))^{-d+\varepsilon} \\ &\leqq C\tilde{\delta}^{n_0}(1+d(x_i, x_j))^{-d+2-\varepsilon}(1+d(x_k, x_l))^{-d+2-\varepsilon},\end{aligned} \tag{46}$$

and we are left with $S(\tilde{H}_{m_2-1}) \equiv \bar{H}_{m_2}$. Let us consider the (more complicated) case where $(i'j') \cap (ij) = \emptyset$, i.e. a totally new pair of points collapses to a common small Y. (The other case where first three and then four points collapse is left to the reader.) We write ($Y_1 \supset \{ij\}$, $Y_2 \supset \{i'j'\}$ small)

$$\begin{aligned}\bar{H}_{m_2}\exp[-\tilde{V}^{m_2}] &\equiv \sum \prod \bar{g}^{m_2 D, I}_{X_l} \prod F^I_{m_2 Y_\sigma} \exp[-\sum \tilde{V}_Y] \\ &= F^{ij}_{m_2 Y_1}(0) \sum \prod \bar{g}^{m_2 D, i'j'}_{X_l} \prod F^{i'j'}_{m_2 Y_\sigma} \exp[-\sum V_Y] + \bar{H}^{ij}_{m_2}\exp[-\tilde{V}^{m_2}]\end{aligned} \tag{47}$$

with, subtracting once more,

$$\begin{aligned}\bar{H}^{ij}_{m_2} e^{-\tilde{V}^{m_2}} &\equiv \sum \prod \bar{g}^{m_2 D, I, ij}_{X_l} F^{I,ij}_{m_2 Y_\sigma} \exp[-\tilde{V}_Y] \\ &= F^{I,ij}_{m_2 Y_2}(0) \sum \prod \bar{g}^{m_2 D, ij}_{X_l} \prod F^{ij}_{m_2 Y_\sigma} \exp[-\sum \tilde{V}_Y] \\ &\quad + \sum \prod \bar{g}^{m_2 D, I}_{X_l} \prod \tilde{F}^I_{m_2 Y_\sigma} \exp[-\sum \tilde{V}_Y].\end{aligned} \tag{48}$$

(In (47) $\bar{g}$, F are not those of (4); we are suppressing the indices of all the previous subtractions.) In (48) we note that $F^{I,ij}_{m_2 Y_2} = F^{i'j'}_{m_2 Y_2}$ and of course $F^{I,ij}_{m_2 Y_\sigma} = F^I_{m_2 Y_\sigma} = F^{i'j'}_{m_2 Y_\sigma}$, if $Y_\sigma \cap \{ij\} = \emptyset$. Thus the last term in (48) indeed has in $\tilde{F}^I_{m_2 Y_1}$ and $\tilde{F}^I_{m_2 Y_2}$ a subtraction at zero. Equations (47) and (48) may be expressed as

$$\bar{H}_{m_2} = F^{ij}_{m_2 Y_1}(0) G_{m_2, i'j'} + F^{i'j'}_{m_2 Y_2}(0) G_{m_2, ij} + \tilde{H}_{m_2}. \tag{49}$$

Let m_3 now be the first m such that all x_i^{m-1} are in the same small Y. We have

$$S^{m_3-m_2}\bar{H}_{m_2} = \sum_{q=m_2}^{m_3-1} \sum_{Y_{1q}, Y_{2q}} (F^{ij}_{qY_{1q}}(0) S^{q-m_2} G_{qi'j'} + F^{i'j'}_{qY_{2q}} S^{q-m_2} G_{qij}) + S(\tilde{H}_{m_3-1}). \tag{50}$$

Again the two-point function pieces give a contribution that can be absorbed to the $O(-)$ term in (41), whereas $\bar{H}_{m_3} \equiv S(\tilde{H}_{m_3-1})$ is given by

$$\bar{H}_{m_3} = \sum_{\{X_j\}, Y} \prod \bar{g}^{m_3 D}_{X_j} \prod F_{m_3 Y} \exp[-\sum \tilde{V}^{m_3}_Y], \tag{51}$$

with

$$|F_{m_3 Y}| \leqq C\tilde{\delta}^{n_0+n} L^{-2(m_3-n)(d-\varepsilon)} \exp[-2\alpha \mathscr{L}(Y)]. \tag{52}$$

Iteration of $S^k \bar{H}_{m_3}$ is now as in case of the two point function, yielding

$$|\langle S^{N-N_0-m_3-1}\bar{H}_{m_3}\rangle_{\mathscr{R}^{N-N_0}\mathscr{H}}| \leqq C\tilde{\delta}^{n_0+n} L^{-2(m_3-n)(d-\varepsilon)}. \tag{53}$$

Summing this over n with γ^{4n} produces the $O(-)$ term in (41). Equation (41) is thus our bound for (34).

Estimation of the other terms in (1) proceeds similarly. Consider e.g. $p=2$, $|I_1|=|I_2|=2$, given by

$$\sum_{N-N_0>n_1>n_2} \sum_{\substack{\text{ordered}\\ \text{pairings}}} \gamma^{2(n_1+n_2)} \langle S^{N-N_0-n_1-1} \langle z^{n_1}_{x_i^{n_1}} z^{n_1}_{x_j^{n_1}} S^{n_1-n_2-1} \langle z^{n_2}_{x_k^{n_2}} z^{n_2}_{x_l^{n_2}} \rangle_{Z^{n_2}} \rangle_{Z^{n_1}} \rangle_{\mathscr{R}^{N-N_0}\mathscr{H}} \,. \tag{54}$$

Here $S^{n_1-n_2-1}\langle z^{n_2}_{x_k^{n_2}} z^{n_2}_{x_l^{n_2}}\rangle_{Z^{n_2}} \equiv G_{n_1,kl}$ we have already computed:

$$G_{n_1,kl} = G_{n_1,kl}(0) + \tilde{G}_{n_1,kl}\,, \tag{55}$$

with

$$|G_{n_1,kl}(0) - c_{n_2}^{-1} \mathscr{T}_{n_2 x_k^{n_2} x_l^{n_2}}| \leqq C\tilde{\delta}^{n_0+n_2}(1+d(x_k^{n_2}, x_l^{n_2}))^{-d+\varepsilon}\,, \tag{56}$$

and

$$\tilde{G}_{n_1,kl} = \sum \prod \tilde{g}^{nD,kl}_{X_j} \prod \tilde{F}^{kl}_{n_1 Y_\sigma} \exp[-\sum V_{\tilde{Y}}]\,, \tag{57}$$

with

$$|\tilde{F}^{kl}_{n_1 Y}| \leqq C\tilde{\delta}^{n_0+n_2} L^{-\frac{d-\varepsilon}{2}(n_1-n_2)|Y\cap\{k,l\}|} \exp[-2\alpha\mathscr{L}(Y)]\,, \tag{58}$$

and the usual bounds for $\tilde{g}$.

Thus

$$\langle z^{n_1}_{x_i^{n_1}} z^{n_1}_{x_j^{n_1}} G_{n_1,kl}\rangle_{Z^{n_1}} = G_{n_1,kl}(0) \langle z^{n_1}_{x_i^{n_1}} z^{n_1}_{x_j^{n_1}}\rangle_{Z^{n_1}} + \langle z^{n_1}_{x_i^{n_1}} z^{n_1}_{x_j^{n_1}} \tilde{G}_{n_1,kl}\rangle_{Z^{n_1}}\,. \tag{59}$$

The first term contributes, by (56) and Sect. 5,

$$\begin{aligned} G_{n_1,kl}(0)\langle z^{n_1}_{x_i^{n_1}} z^{n_1}_{x_j^{n_1}}\rangle &= c_{n_1}^{-1} c_{n_2}^{-1} \mathscr{T}_{n_1 x_i^{n_1} x_j^{n_1}} \mathscr{T}_{n_2 x_k^{n_2} x_l^{n_2}} \\ &\quad + \mathscr{O}(\tilde{\delta}^{n_0+n_2}(1+d(x_i^{n_1}, x_j^{n_1}))^{-d+\varepsilon}(1+d(x_k^{n_2}, x_l^{n_2}))^{-d+\varepsilon})\,, \end{aligned} \tag{60}$$

and upon summation over n_1 and n_2 in (54)

$$\begin{aligned} \sum_{\text{pairings}} \Bigg[\Bigg(\sum_{n_1 \neq n_2} (c_{n_1}^{-1} \mathscr{T}_{n_1 x_i^{n_1} x_j^{n_1}} c_{n_2}^{-1} \mathscr{T}_{n_2 x_k^{n_2} x_l^{n_2}}) \Bigg) \\ + \mathscr{O}(\tilde{\delta}^{n_0}(1+d(x_i, x_j))^{-d+2}(1+d(x_k, x_l))^{-d+2-\varepsilon} + \mathscr{O}(\{ij\} \Leftrightarrow \{kl\})) \Bigg]\,, \end{aligned} \tag{61}$$

whereas, defining

$$H_{n_1+1} = L^{(d-\varepsilon)(n_1-n_2)} \langle z^{n_1}_{x_i^{n_1}} z^{n_1}_{x_j^{n_1}} \tilde{G}_{n_1,kl}\rangle\,, \tag{62}$$

it has the expansion (36) with analogous bounds. Thus

$$\begin{aligned} |\langle S^{N-N_0-n_1-1} H_{n_1+1}\rangle_{\mathscr{R}^{N-N_0}\mathscr{H}}| \leqq C\tilde{\delta}^{n_0+n_2} \sum_{\text{pairings}} (1+d(x_i^{n_1}, x_j^{n_1}))^{-d+\varepsilon} \\ \cdot (1+d(x_k^{n_2}, x_l^{n_2}))^{-d+\varepsilon}\,, \end{aligned} \tag{63}$$

and combining (63) with (62) and (59) these terms in (54) can again be absorbed to the $\mathscr{O}(-)$ in (61). Similar analysis is now carried out by inspection to the other p, $\{I_j\}$ combinations in (1). We get

$$\langle\phi_{x_1}\dots\phi_{x_4}\rangle_{\mathscr{H}}=\sum_{\text{pairings}}\left[\left(\sum_{n=0}^{N-N_0-1}c_n^{-1}\mathscr{T}_{nx_i^nx_j^n}\right)\left(\sum_{n=0}^{N-N_0-1}c_n^{-1}\mathscr{T}_{nx_k^nx_l^n}\right)\right.$$
$$\left.+\mathcal{O}(\tilde{\delta}^{n_0}(1+d(x_i,x_j))^{-d+2}(1+d(x_k,x_l))^{-d+2-\varepsilon})+\mathcal{O}(\{ij\}\Leftrightarrow\{kl\})\right]$$
$$+\mathcal{O}(\gamma^N),\tag{64}$$

where the $\mathcal{O}(\gamma^N)$ comes from the last contribution in (1). Proceeding as in (5.29) and taking $N\to\infty$, we obtain

$$\langle\phi_{x_1}\dots\phi_{x_4}\rangle=c(v)^{-2}\sum_{\text{pairings}}[G_{0ij}G_{0kl}$$
$$+\mathcal{O}(\tilde{\delta}^{n_0}(1+d(x_i,x_j))^{-d+2}(1+d(x_k,x_l))^{-d+2-\varepsilon})+\mathcal{O}(\{ij\}\Leftrightarrow\{kl\})].\tag{65}$$

In the scaling limit (1.3) the $\mathcal{O}(-)$'s drop away and

$$G(x_1\dots x_4)=c(v)^{-2}\sum_{\text{pairings}}(-\Delta_c)^{-1}_{x_ix_j}(-\Delta_c)^{-1}_{x_kx_l},\tag{66}$$

which was the claim.

Appendix

Here we prove that (4.22) for n implies the same relation for $n+1$. The repetition of the arguments of Appendix 2 of [1] gives

$$\bar{g}'^{D'_1}_{X'}=\sum_{\{X_l\},\{X_\sigma\}}\prod_l\bar{g}'^{D'}_{X_l}\prod_\sigma g'^{A_\sigma}_{B_\sigma X_\sigma}\sum_{\{Y_\alpha\}}\prod_\alpha(\exp[-\tilde{V}'_{Y_\alpha}]-1),\tag{1}$$

where X_l, X_σ are disjoint, $Y_\alpha\subset X'\backslash(\cup X_l)\cup(\cup X_\sigma)$, $\cup X_l\supset X'\cap D'$ and X' is connected with respect to X_l, X_σ, Y_α and connected components of D'_1. To proceed further, we need relations inverse to (4.12) and (4.13). These are

$$g'^k_{iX}=\sum_{Y,\{Y_\alpha\}\text{ in }X}F'^k_{iY}\prod_\alpha(\exp[-\tilde{V}'_{Y_\alpha}]-1),\tag{2}$$

and

$$g'^{kl}_{ijX}=\sum_{Y,\{Y_\alpha\}\text{ in }X}F'^{kl}_{ijY}\prod_\alpha(\exp[-\tilde{V}'_{Y_\alpha}]-1)+\sum_{\substack{Y_1,Y_2,\{Y_\alpha\}\text{ in }X\\ Y_1\cap Y_2=\emptyset}}F'^k_{iY_1}F'^l_{jY_2}\prod_\alpha(\exp[-\tilde{V}'_{Y_\alpha}]-1),\tag{3}$$

with X connected with respect to Y (Y_1, Y_2) and Y_α in both (2) and (3). It is easy to see that (4.12) and (4.13) as well as (2) and (3) establish one-to-one relations between (g'^A_{BX}) and (F'^A_{BY}). In order to show that one is the inverse of the other it is then sufficient to prove that substitution of (4.12) and (4.13) to (2) and (3) yields initial (g'^A_{BX}). But, with X connected with respect to X_1, Y_α and Y_β,

$$\sum_{\substack{X_1,\{Y_\alpha\},\{Y_\beta\}\text{ in }X\\ Y_\alpha\cap X_1\neq\emptyset}}g'^k_{iX_1}\prod_\alpha(\exp[\tilde{V}'_{Y_\alpha}]-1)\prod_\beta(\exp[-\tilde{V}'_{Y_\beta}]-1)$$
$$=\sum_{\substack{X_1,(Y_1,\dots,Y_r)\text{ in }X\\ X\text{ conn. with respect to }X_1\text{ and }Y_\sigma}}\frac{(-1)^l}{l!}g'^k_{iX_1}\prod_{\sigma=1}^r(1-U(X_1,Y_\sigma))\tilde{V}_{Y_\sigma}=g'^k_{iX}\tag{4}$$

($U(X, Y)=1$ if $X \cap Y \neq \emptyset$ and vanishes otherwise). Similarly, with X connected with respect to X_1, (X_2), Y_α and Y_β

$$\sum_{\substack{X_1,\{Y_\alpha\},\{Y_\beta\} \text{ in } X \\ Y_\alpha \cap X_1 \neq \emptyset}} g'^{kl}_{ijX_1} \prod_\alpha (\exp[\tilde{V}_{Y_\alpha}]-1) \prod_\beta (\exp[-\tilde{V}_{Y_\beta}]-1)$$

$$+ \sum_{\substack{X_1, X_2, \{Y_\alpha\},\{Y_\beta\} \text{ in } X \\ X_1 \cap X_2 = \emptyset,\, Y_\alpha \cap (X_1 \cup X_2) \neq \emptyset}} g'^k_{iX_1} g'^l_{jX_2} \prod_\alpha (\exp[\tilde{V}_{Y_\alpha}]-1) \prod_\beta (\exp[-\tilde{V}_{Y_\beta}]-1)$$

$$= \sum_{\substack{X_1, (Y_1, \dots, Y_r) \text{ in } X \\ X \text{ conn. with respect to } X_1 \text{ and } Y_\sigma}} \frac{(-1)^l}{l!} g'^k_{ijX_1} \prod_{\sigma=1}^r (1 - U(X_1, Y_\sigma)) \tilde{V}_{Y_\sigma}$$

$$+ \sum_{\substack{X_1, X_2, (Y_1, \dots, Y_r) \text{ in } X \\ X \text{ conn. with respect to } X_{1,2} \text{ and } Y_\sigma \\ X_1 \cap X_2 = \emptyset}} \frac{(-1)^l}{l!} g'^k_{iX_1} g'^l_{jX_2} \prod_{\sigma=1}^r (1 - U(X_1 \cup X_2, Y_\sigma)) \tilde{V}_{Y_\sigma} = g'^{kl}_{ijX} . \tag{5}$$

Insertion of (2) and (3) to (1) yields

$$\tilde{g}'^{D'_1}_{X'} = \sum_{\{X_l\}} \prod_l \bar{g}'^{D'}_{X_l} \sum_{\{Y_\sigma\},\{Y_\alpha\}} \prod_\sigma F'^{A_\sigma}_{B_\sigma Y_\sigma} (\exp[-\tilde{V}'_{Y_\alpha}]-1), \tag{6}$$

with X_l disjoint, $\cup X_l \supset X' \cap D'$, Y_σ, $Y_\alpha \subset X'_l \backslash \cup X$ and X' connected with respect to X_l, Y_σ, Y_α and connected components of D'_1. This proves (3.39) for $n+1$ except for the case when $x_1^{n+1}, x_2^{n+1} \in X'$. In the latter case, using (6), (4.18), (4.20), and (3.6) of [1] we obtain

$$\tilde{g}^{D'_1}_{X'} = \sum_{\{X_l\},\{Y_\sigma\},\{Y_\alpha\}} \prod_j \bar{g}'^{D'}_{X_l} \prod_\sigma F'^{A_\sigma}_{B_\sigma Y_\sigma} \prod_\alpha (\exp[-\tilde{V}'_{Y_\alpha}]-1)$$
$$- \sum_{\{X_l\},\{Y_\alpha\}, Y} g'^{D'}_{X_l} \prod_\alpha (\exp[-\tilde{V}'_{Y_\alpha}]-1) F'^{kl}_{uvY}(0),$$

where in the first sum the restrictions are as in (6) and in the second one X_l are disjoint, $Y_\alpha \subset X' \backslash \cup X_l$ and X' is connected with respect to X_l, Y_α, Y and connected components of D'_1. The part of the second sum with $Y \cap X_l = \emptyset$ cancels the constant term in F'^{kl}_{uvY}, see (4.18), whereas the one with $Y \cap X_l \neq \emptyset$ provides the correction for the $\bar{g}'^{D'}_X$ with $x_1^{n+1}, x_2^{n+1} \in X$, appearing in the first sum, necessary to convert it into $\tilde{g}'^{D'}_X$, see (4.20). This completes the proof of (4.22) for $n+1$.

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