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THE ORDER ASPECT OF THE FUZZY REAL LINE

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It is the purpose of this paper to put the "fuzzy real line" in a setting which proves to be advantageous to a more fundamental study of that space.
Actually there are three different fuzzy real lines to be found in the literature, mainly defined by U. Höhle in [1], [2] and by B. Hutton in [4], [5], and a fourth one shall be added in this work.
The main result of this paper is the fact that three of the four spaces are homeomorphic to fuzzy topological spaces the underlying sets of which are, in each case, the probability measures on $\mathbb{R}$, and the fuzzy (resp. quasi fuzzy and translation-closed fuzzy) topologies of which are determined by the left and right sections of a canonical fuzzy extension of the strict order relation on $\mathbb{R}$. From this it will follow very fundamentally that it is the order of $\mathbb{R}$, and not the topology, which determines the fuzzy real line.

1. Preliminaries

The unit interval is denoted I, $I_0 :=]0,1]$ and $I_1 := [0,1[$. If X is a set and $A \subset X$ then the characteristic function of A is denoted 1_A. If f is any real valued function on $\mathbb{R}$ then by f(x+) and f(x-) we denote right and left limit of f in x.

Given a set X, by a quasi fuzzy topology Δ on X [9] we understand a collection of fuzzy sets on X, closed for finite infima and arbitrary suprema. If moreover Δ contains all constant fuzzy sets we call it a fuzzy topology [9] and if it is stable for translations, i.e. for all $\mu \in \Delta$ and $\alpha \in I$: $(\mu+\alpha) \wedge 1 \in \Delta$ and $(\mu-\alpha) \vee 0 \in \Delta$, then we call it a translation closed fuzzy topology [1]. In the terminology of U. Höhle [1], [3], this is a probabilistic topology for the case

$L = [0,1]$ (see [1] for notations).
If TOP denotes the category of topological spaces and FTS that of fuzzy topological spaces then by $\omega : \text{TOP} \to \text{FTS}$, $\iota : \text{FTS} \to \text{TOP}$ and $\iota_\alpha : \text{FTS} \to \text{TOP}$ we denote, the functors introduced in [9] and [11]. The topological space $(X, \iota(\Delta))$ is called the topological modification of Δ.
If X is a separable metrizable space, then we denote $\mathcal{B}(X)$ the Borel σ-algebra on X and $\mathcal{T}(X)$ the topology on X. The collection of all probability measures on $\mathcal{B}(X)$ is denoted $M(X)$ and the collection of all degenerate probability measures is denoted $\mathcal{D}(X)$. Recall that $P \in M(X)$ is called degenerate if there exists $x \in X$ such that $P(B) = 1$ if and only if $x \in B$. Throughout this work such a measure is denoted P_x.
If $Y \in \mathcal{B}(X)$ then we denote $M^Y(X)$ the collection of all $P \in M(X)$ for which $P(Y) = 1$. If $f : X \to Y$ is a measurable map then it has a natural extension

$$\hat{f} : M(X) \to M(Y) \tag{1.1}$$

defined by $\hat{f}(P)(B) = P(f^{-1}(B))$ for all $B \in \mathcal{B}(Y)$ [18].
$H \subset M(X)$ is called tight if for all $\varepsilon \in I_0$ there exists $K \subset X$, compact, such that $P(K) > 1 - \varepsilon$ for all $P \in H$ [19].

We recall some basic notions from [12]. If $\{0,1\}$ is equipped with its natural, i.e. discrete, Borel σ-algebra then for any $A \in \mathcal{B}(X)$ the map $1_A : X \to \{0,1\}$ is measurable and thus has an extension

$$\hat{1}_A : M(X) \to M(\{0,1\}) : P \to (1-P(A))\, P_0 + P(A)\, P_1 \tag{1.2}$$

by (1.1). Since any probability measure on $\{0,1\}$ is determined by its value on f.i. $\{1\}$, the mapping

$$\theta : M(\{0,1\}) \to I : (1-\alpha)P_0 + \alpha P_1 \to \alpha \tag{1.3}$$

is a natural isomorphism between $M(\{0,1\})$ and I. Then, upon identifying these two sets, (1.2) becomes,

$$\delta_A : M(X) \to I : P \to P(A) \tag{1.4}$$

We now extend the topology of X to the fuzzy topology $\Delta(X)$ on $M(X)$, generated by

$$\Sigma(X) := \{\delta_G | G \in \mathcal{T}(X)\} \tag{1.5}$$

The space $(M(X),\Delta(X))$ is denoted simply $M(X)$. Since the extension of a continuous map $f : X \to Y$ to $\hat{f} : M(X) \to M(Y)$ is again continuous, we obtain a functorial relationship. Let SMS denote the subcategory of TOP consisting of all separable metrizable spaces.

THEOREM 1.1. The correspondence Ext_{top} : SMS $\to$ FTS defined by

$$\text{Ext}_{\text{top}}(X) := M(X) \quad \text{on objects}$$

$$\text{Ext}_{\text{top}}(f) := \hat{f} \quad \text{on morphisms}$$

is a covariant functor.

In [12] it was shown that X is canonically embedded in $M(X)$ by $\varphi : X \to M(X) : x \to P_x$. Regardless of structures on X and on $M(X)$, φ shall always denote this map.

2. Three fuzzy real lines

In the literature there are three spaces which are referred to as "fuzzy real line".

The coarsest space is that of B. Hutton [4]. Let R denote all non increasing real-valued functions on $\mathbb{R}$ with infimum equal to 0 and supremum equal to 1. Let $\sim$ be the equivalence relation on R defined as $\lambda \sim \mu$ if and only if $\lambda(x-) = \mu(x-)$ and $\lambda(x+) = \mu(x+)$ for all $x \in \mathbb{R}$. Then the "natural" quasi fuzzy topology on $R|\sim$ is the one generated by the subbasis

$$\{L_x | x \in \mathbb{R}\} \cup \{R_x | x \in \mathbb{R}\}$$

where for each $x \in \mathbb{R}$ and $[\lambda] \in R|\sim$

$$L_x([\lambda]) := 1 - \lambda(x-) \quad \text{and} \quad R_x([\lambda]) := \lambda(x+).$$

In the literature $\mathbb{R}(I)$ usually denotes the space $R|\sim$ equipped with this quasi fuzzy topology.

Let now $D(\mathbb{R})$ denote the set of all distribution functions on $\mathbb{R}$ and let $\mathcal{B}$ be the collection of fuzzy sets on $D(\mathbb{R})$ defined by

$$\mathcal{B} := \{\mathcal{L}_x | x \in \mathbb{R}\} \cup \{\mathcal{R}_x | x \in \mathbb{R}\}$$

where for each $x \in \mathbb{R}$ and $F \in D(\mathbb{R})$

$$\mathcal{L}_x(F) = F(x) \quad \text{and} \quad \mathcal{R}_x(F) = 1 - F(x+).$$

Let $D_0(\mathbb{R})$ denote $D(\mathbb{R})$ equipped with $\Gamma_0(\mathbb{R})$:= the quasi fuzzy topology generated by $\mathcal{B}$, then we have the following result, the proof of which is straightforward.

PROPOSITION 2.1. $\mathbb{R}(I)$ and $D_0(\mathbb{R})$ are homeomorphic and the canonical homeomorphism is given by

$$D_0(\mathbb{R}) \to \mathbb{R}(I) : F \to [1-F].$$

In view of the fact that $D_0(\mathbb{R})$ is a more natural model than $\mathbb{R}(I)$, we shall refer to $D_0(\mathbb{R})$ as the Hutton fuzzy real line.

The in-between space we obtain using the technique of saturating a quasi fuzzy topology for the constants in order to obtain a fuzzy topology [9], [14]. This space has also been suggested in [21]. Thus we take as underlying set again $D(\mathbb{R})$ but now equipped with $\Gamma(\mathbb{R})$:= the fuzzy topology generated by $\mathcal{B}$. We shall denote this space simply $D(\mathbb{R})$ and, in this work, refer to it as the fuzzy real line.

The finest space is that of U. Höhle [1], [2] who considers on $D(\mathbb{R})$ the structure $\Gamma_1(\mathbb{R})$:= the translation closed fuzzy topology generated by $\mathcal{B}$. This space we shall denote $D_1(\mathbb{R})$ and we shall refer to as the Höhle fuzzy real line.
Remark that, U. Höhle usually only considers the subspace $D^+(\mathbb{R})$ of $D(\mathbb{R})$ consisting of those distribution functions F for which $F(0) = 0$.

Each of these three spaces can be viewed as the "model" of a certain concept of "fuzzy real line" in its corresponding category.

The results of the fourth section will reveal fundamental arguments, apart from those of [16], why $D(\mathbb{R})$ and $D_1(\mathbb{R})$ are maybe more natural and canonical spaces to consider than $D_0(\mathbb{R})$.

3. The probabilistic aspect

In classical probability theory on $\mathbb{R}$ there is advantage in being able to work with both distribution functions and probability measures on $\mathbb{R}$. We shall therefore translate the concepts of the fuzzy real lines into the language of probability measures on $\mathbb{R}$.

Let ξ denote the usual bijection from $M(\mathbb{R})$ to $D(\mathbb{R})$ i.e.

$$\xi : M(\mathbb{R}) \to D(\mathbb{R}) : P \to \xi(P) := F_P \qquad (3.1)$$

where F_P is the distribution function defined by $F_P(x) := P(]-\infty,x[$, $\forall x \in \mathbb{R}$. Let A be the subbasis for $T(\mathbb{R})$ defined by

$$A := \{]-\infty,a[\,|\,a \in \mathbb{R}\} \cup \{]a,+\infty[\,|\,a \in \mathbb{R}\}. \qquad (3.2)$$

Now put

$$\Sigma_A(\mathbb{R}) := \{\delta_A | A \in A\} \qquad (3.3)$$

and let $\Delta_A(\mathbb{R})$ (resp. $\Delta_A^0(\mathbb{R})$ and $\Delta_A^1(\mathbb{R})$) denote the fuzzy (resp. quasi fuzzy and translation closed fuzzy) topology generated by $\Sigma_A(\mathbb{R})$.
The space $(M(\mathbb{R}),\Delta_A(\mathbb{R}))$ (resp. $(M(\mathbb{R}),\Delta_A^0(R))$ and $(M(\mathbb{R}),\Delta_A^1(\mathbb{R}))$) shall then simply be denoted $M_A(\mathbb{R})$ (resp. $M_A^0(\mathbb{R})$ and $M_A^1(\mathbb{R})$). Analogously if $E(\mathbb{R}) \subset M(\mathbb{R})$, then as a subspace it shall be denoted $E_A(\mathbb{R})$ (resp. $E_A^0(\mathbb{R})$ and $E_A^1(\mathbb{R})$).

PROPOSITION 3.1. The following hold

1° $M_A^0(\mathbb{R})$ is homeomorphic to $D_0(\mathbb{R})$

2° $M_A(\mathbb{R})$ is homeomorphic to $D(\mathbb{R})$

3° $M_A^1(\mathbb{R})$ is homeomorphic to $D_1(\mathbb{R})$.

In each case the canonical homeomorphism is ξ.

Proof. This follows at once from the fact that for any $x \in \mathbb{R}$: $\xi^{-1}(L_x) = \delta_{]-\infty,x[}$ and $\xi^{-1}(R_x) = \delta_{]x,+\infty[}$. □

From this we see that there is a striking resemblance between the fuzzy real line and the spaces introduced in [12].

$M_A(\mathbb{R})$ is constructed in a similar way as $M(\mathbb{R})$ by extending only a specific subbasis of $T(\mathbb{R})$ namely A.
The only subspaces of the fuzzy real line, studied so far, essentially, are the open and closed fuzzy unit interval [4], [20].
The closed fuzzy unit interval [4] consists of all $F \in D(\mathbb{R})$ fulfilling $F(0) = 0$ and $F(1+) = 1$. We shall denote this subset $D(I)$.
The open fuzzy unit interval [20] consists of all $F \in D(\mathbb{R})$ fulfilling $F(0+) = 0$ and $F(1) = 1$. This subset we shall denote $D(\mathring{I})$. Further if either $D(I)$ or $D(\mathring{I})$ are equipped with the fuzzy topology induced by $\Gamma(\mathbb{R})$ (resp. $\Gamma_0(\mathbb{R})$ and $\Gamma_1(\mathbb{R})$) then we shall also denote them simply $D(I)$ and $D(\mathring{I})$ (resp. $D_0(I)$, $D_0(\mathring{I})$, $D_1(I)$ and $D_1(\mathring{I})$).

If we now denote $M_A^X(\mathbb{R})$ (resp. $M_A^{0,X}(\mathbb{R})$, $M_A^{1,X}(R)$) the set $M^X(\mathbb{R})$ equipped with the structure induced by $\Delta_A(\mathbb{R})$ (resp. $\Delta_A^0(\mathbb{R})$, $\Delta_A^1(\mathbb{R})$) then we have the following result.

PROPOSITION 3.2. *The following hold*

1° $M_A^{0,I}(\mathbb{R})$ *is homeomorphic to* $D_0(I)$
$M_A^{0,\mathring{I}}(\mathbb{R})$ *is homeomorphic to* $D_0(\mathring{I})$

2° $M_A^{I}(\mathbb{R})$ *is homeomorphic to* $D(I)$
$M_A^{\mathring{I}}(\mathbb{R})$ *is homeomorphic to* $D(\mathring{I})$

3° $M_A^{1,I}(\mathbb{R})$ *is homeomorphic to* $D_1(I)$
$M_A^{1,\mathring{I}}(\mathbb{R})$ *is homeomorphic to* $D_1(\mathring{I})$.

Again in each case the canonical homeomorphism is induced by ξ.

Proof. By straightforward verification. □

Let $\lambda(T(\mathbb{R}))$ denote the quasi fuzzy topology $\lambda(T(\mathbb{R})) := \{1_G \mid G \in T(\mathbb{R})\}$. Then we have the following result, the verification of which again causes no problem.

PROPOSITION 3.3. *The following hold*

1° $\mathcal{D}_A^0(\mathbb{R})$ <u>is homeomorphic to</u> $(\mathbb{R},\lambda(\mathcal{T}(\mathbb{R})))$

2° $\mathcal{D}_A(\mathbb{R})$ <u>is homeomorphic to</u> $(\mathbb{R},\omega(\mathcal{T}(\mathbb{R})))$

3° $\mathcal{D}_A^1(\mathbb{R})$ <u>is homeomorphic to</u> $(\mathbb{R},\omega(\mathcal{T}(\mathbb{R})))$.

It is surprising that in none of the many previous papers dealing with any of the three fuzzy real lines, it was ever questioned why the only somewhat naturally definable subspaces were fuzzy intervals. The reason herefore lies in the fact that the relation between $\mathbb{R}(I)$ and $M(\mathbb{R})$ uptill now was never used, especially not in papers dealing with the Hutton fuzzy real line which usually deal with very general lattices L instead of with I. The framework which we have proposed now makes it possible to consider a large class of subspaces of the fuzzy real line in a canonical way. There is however a more fundamental problem, with the very nature of the fuzzy real lines, in particular again with the Hutton fuzzy real line. From the literature, this space has always been considered a natural fuzzy extension of the topological space $\mathbb{R}$. Its structure was obtained however by extending only the subbasis A of $\mathcal{T}(\mathbb{R})$.

A crucial question then is the independence of the chosen subbasis. In [12] we showed that the extension process is not subbasis-independent and that by considering the entire topology $\mathcal{T}(\mathbb{R})$, one obtains an in many ways more natural fuzzy extension of $\mathbb{R}$.

Nevertheless the basic concept of the fuzzy real lines is important, as follows from [4], [5], [20].

Consequently we were faced with the problem whether there exists a more canonical way of obtaining these spaces. This problem is not so important for the Höhle fuzzy real line since there the structure is indeed obtained from a different point of view using the concept of statistical metrics. This method however is only meaningful in the subcategory of translation closed fuzzy topological spaces. Nevertheless for both the Höhle fuzzy real line and the fuzzy real line we were able to discover, for the former an alternative, and for the latter its fundamental defining structure. For the Hutton fuzzy real line this works only if one somewhat alters the space thus obtaining a new quasi fuzzy topological space. This shall be shown in the next chapter, but first we still give some preliminary results.

THEOREM 3.4. *If X and Y are Borel subsets of $\mathbb{R}$ and $f : X \to Y$ is continuous and monotone then $\hat{f} : M_A^X(\mathbb{R}) \to M_A^Y(\mathbb{R})$ is continuous.*

Proof. By straightforward verification. □

It is sometimes advantageous to use the following alternative description of subspaces $M_A^X(\mathbb{R})$. For any Borel subset $X \subset \mathbb{R}$ put

$$A|_X := \{A \cap X | A \in A\}$$

where A is as defined in (3.2) and put

$$\Sigma_{A|_X}(X) := \{\delta_B^X | B \in A|_X\}$$

where for any $B \in A|_X : \delta_B^X : M(X) \to I : P \to P(B)$. Denote by $M_{A|_X}(X)$ the fuzzy topological space with underlying set $M(X)$ and subbasis $\Sigma_{A|_X}(X)$ (see also [12]).

THEOREM 3.5. *If X is a Borel subset of $\mathbb{R}$ then the map*

$$\theta : M_{A|_X}(X) \to M_A^X(\mathbb{R})$$

where $\theta(P)(B) := P(B \cap X)$ for all $B \in \mathcal{B}(\mathbb{R})$ is a homeomorphism.

Proof. Using the fact that $\mathcal{B}(X) = \{B \cap X | B \in \mathcal{B}(\mathbb{R})\}$ the reader can easily verify this himself. □

THEOREM 3.6. [12] *The topological modification of $\Delta_A(\mathbb{R})$ coincides with the weak topology on $M(\mathbb{R})$.*

This result was first shown by U. Höhle in [3] in the context of translation closed fuzzy topologies. It also holds for the Hutton fuzzy real line [12].

4. The order aspect and ... a fourth fuzzy real line

The topology of $\mathbb{R}$ is an order topology in the sense that if we put $S := \{(x,y) | x < y\}$ the "strict order relation" then the topology of $\mathbb{R}$ is generated by the left and right sections of S i.e. by the collection of all sets

$$S_l(x) := \{y|y < x\} \quad x \in \mathbb{R}$$
$$S_r(x) := \{y|x < y\} \quad x \in \mathbb{R}.$$

It is our purpose in this chapter to prove that the same is true for the fuzzy topologies $\Delta_A(\mathbb{R})$ and $\Delta_A^1(\mathbb{R})$. Precisely we shall prove that there exists an extension of S to a "fuzzy strict order relation" on $M(\mathbb{R})$ such that the left and right sections generate (in their respective categories) $\Delta_A(\mathbb{R})$ and $\Delta_A^1(\mathbb{R})$.

Recall the definition of the following t-norm

$$T_m(x,y) := (x+y-1) \vee 0 \qquad x,y \in I$$

and its dual

$$S_m(x,y) := (x+y) \wedge 1 \qquad x,y \in I.$$

<u>DEFINITION 4.1.</u> <u>For any</u> $P,Q \in M(\mathbb{R})$ <u>we define</u>

$$\rho(P,Q) := \sup_{x\in\mathbb{R}} P(]-\infty,x[) \wedge Q(]x,+\infty[).$$

<u>Example</u>. Let us illustrate this definition on some concrete probability measures.

Suppose that $N_1 := N(m_1,1)$ and $N_2 = N(m_2,1)$ are normal probability measures, then straightforward calculus shows that

$$\rho(N_1,N_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{m_2-m_1}{2}} e^{-\frac{x^2}{2}}\, dx$$

from which we conclude that

$$\rho(N_1,N_2) \to 1 \quad \text{iff} \quad m_2 - m_1 \to +\infty$$
$$\rho(N_1,N_2) \to 0 \quad \text{iff} \quad m_1 - m_2 \to +\infty$$

and

$$\rho(N_1,N_2) = \frac{1}{2} \quad \text{iff} \quad m_1 = m_2.$$

This example illustrates ρ and, with the results of Propositions 4.1 and 4.4 and Theorems 4.2 and 4.3, show that we can interpret $\rho(P,Q)$ as a "degree" with which P is strictly smaller than Q".

PROPOSITION 4.1. The fuzzy relation ρ is an extension of the strict order on $\mathbb{R}$, i.e. $\rho \circ (\varphi\times\varphi) = 1_S$.

Proof. This is straightforward. □

THEOREM 4.2. The fuzzy relation ρ is a linear T_m-strict order in the sense that it fulfills the following properties

1° (T_m-antireflexivity) For all $P,Q \in M(\mathbb{R})$:

$$T_m(\rho(P,Q),\rho(Q,P)) = 0$$

2° (T_m-transitivity) For all $P,Q \in M(\mathbb{R})$:

$$\rho(P,Q) \geq \sup_{R\in M(\mathbb{R})} T_m(\rho(P,R),\rho(R,Q))$$

3° (T_m-linearity) For all $P \neq Q \in M(\mathbb{R})$:

$$S_m(\rho(P,Q),\rho(Q,P)) > 0.$$

Proof. 1° It suffices to note that if $x \leq y$ then

$$P(]-\infty,x[) \wedge Q(]x,+\infty[) + Q(]-\infty,y[) \wedge P(]y,+\infty[)$$
$$\leq P(]-\infty,x[) + P(]x,+\infty[) \leq 1$$

and if $y \leq x$ then analogously

$$P(]-\infty,x[) \wedge Q(]x,+\infty[) + Q(]-\infty,y[) \wedge P(]y,+\infty[)$$
$$\leq Q(]y,+\infty[) + Q(]-\infty,y[) \leq 1$$

and consequently $\rho(P,Q) + \rho(Q,P) \leq 1$.

2° Let $P,Q,R \in M(\mathbb{R})$, then it suffices to show that for any $x,y \in \mathbb{R}$ there exists $z \in \mathbb{R}$ such that

$$P(]-\infty,x[) \wedge R(]x,+\infty[) + R(]-\infty,y[) \wedge Q(]y,+\infty[)$$
$$\leq P(]-\infty,z[) \wedge Q(]z,+\infty[) + 1.$$

Now if $x \leq y$ then

$$P(]-\infty,x[) \wedge R(]x,+\infty[) + R(]-\infty,y[) \wedge Q(]y,+\infty[)$$
$$\leq P(]-\infty,y[) + Q(]y,+\infty[)$$
$$\leq P(]-\infty,y[) \wedge Q(]y,+\infty[) + 1$$

i.e., $z := y$ will do, and if $y \leq x$ then

$$P(]-\infty,x[) \wedge R(]x,+\infty[) + R(]-\infty,y[) \wedge Q(]y,+\infty[)$$
$$\leq R(]y,+\infty[) + R(]-\infty,y[) \leq 1$$

i.e., any $z \in \mathbb{R}$ will do.

3° Let $P \neq Q \in M(\mathbb{R})$ and $S_m(\rho(P,Q),\rho(Q,P)) = 0$ i.e.,

$$\rho(P,Q) = \rho(Q,P) = 0. \tag{4.1}$$

This means that for all $x \in \mathbb{R}$:

$$P(]-\infty,x[) \wedge Q(]x,+\infty[) = 0 \quad \text{and} \tag{4.2}$$

$$Q(]-\infty,x[) \wedge P(]x,+\infty[) = 0. \tag{4.3}$$

If we now put

$$A_-(P) := \{x | P(]-\infty,x[) = 0\}$$
$$A_+(P) := \{x | P(]x,+\infty[) = 0\}$$
$$A_-(Q) := \{x | Q(]-\infty,x[) = 0\}$$
$$A_+(Q) := \{x | Q(]x,+\infty[) = 0\}$$

then $A_-(P)$ and $A_-(Q)$ are intervals extending to $-\infty$ and $A_+(P)$ and $A_+(Q)$ are intervals extending to $+\infty$. Further it follows from (4.2) and (4.3) that

$$A_-(P) \cup A_+(Q) = \mathbb{R} \quad \text{and} \tag{4.4}$$

$$A_-(Q) \cup A_+(P) = \mathbb{R} \tag{4.5}$$

and that none of the sets $A_-(P)$, $A_+(P)$, $A_-(Q)$ or $A_+(Q)$ can be empty, otherwise either $P(\mathbb{R}) = 0$ or $Q(\mathbb{R}) = 0$ which is preposterous.

Consequently we have for instance by (4.4)

$$-\infty < \inf A_+(Q) \leqslant \sup A_-(P) < +\infty. \tag{4.6}$$

Case 1 : $\inf A_+(Q) < \sup A_-(P)$. Then there exist $a < b$ such that $P(]-\infty,b[) = Q(]a,+\infty[) = 0$ and consequently for any $x \in]a,b[$ we have $P(]-\infty,x[) = Q(]x,+\infty[) = 1$ which implies that $\rho(P,Q) = 1$ contradicting (4.1).

Case 2 : $\inf A_+(Q) = \sup A_-(P) =: a$. Then obviously

$$P([a,+\infty[) = Q(]-\infty,a]) = 1 \tag{4.7}$$

Now if $P(]a,+\infty[) = 0$ then $P = P_a$ and it follows from $Q \neq P$ and from (4.7) that there exists $b < a$ such that $Q(]-\infty,b[) > 0$. Consequently $Q(]-\infty,b[) \wedge P(]b,+\infty[) > 0$, which implies that $\rho(Q,P) > 0$, which again contradicts (4.1). If on the other hand $P(]a,+\infty[) > 0$, then for some $b > a$ also $P(]b,+\infty[) > 0$, and it follows again that

$Q(]-\infty,b[) \wedge P(]b,+\infty[) > 0$, which implies that $\rho(Q,P) > 0$ and which once more contradicts (4.1). □

THEOREM 4.3. The fuzzy relation ρ is a linear "weak" $\wedge$-strict order in the sense that it fulfills the following properties

1° (weak $\wedge$-antireflexivity) For all $P,Q \in M(\mathbb{R})$:

$$\rho(P,Q) \wedge \rho(Q,P) \leqslant \frac{1}{2}$$

2° (weak $\wedge$-transitivity) For all $P,Q \in M(\mathbb{R})$:

$$\rho(P,Q) \vee \frac{1}{2} \geqslant \sup_{R \in M(\mathbb{R})} \rho(P,R) \wedge \rho(R,Q)$$

3° ($\wedge$-linearity) For all $P \neq Q \in M(\mathbb{R})$:

$$\rho(P,Q) \vee \rho(Q,P) > 0.$$

Proof. 1° This is an immediate consequence of Theorem 4.2 since for any $a,b \in I$: $T_m(a,b) = 0 \Rightarrow a \wedge b \leqslant \frac{1}{2}$.

2° Let $P,Q,R \in M(\mathbb{R})$ be such that

$$\rho(P,R) \wedge \rho(R,Q) > \alpha > \frac{1}{2} \tag{4.8}$$

then there exist $x,y \in \mathbb{R}$ such that

$$P(]-\infty,x[) \wedge R(]x,+\infty[) \wedge R(]-\infty,y[) \wedge Q(]y,+\infty[) > \alpha \tag{4.9}$$

Now if $y < x$ then $R(]-\infty,y[) \wedge R(]x,+\infty[) \leqslant \frac{1}{2}$, which together with (4.9) implies $\alpha < \frac{1}{2}$, in contradiction with (4.8). Consequently $x \leqslant y$ and it follows by (4.9) that

$$P(]-\infty,x[) \wedge Q(]x,+\infty[) > \alpha$$

which implies $\rho(P,Q) > \alpha$.

3° This again is an immediate consequence of Theorem 4.2 since for any $a,b \in I$: $S_m(a,b) > 0 \Leftrightarrow a \vee b > 0$. □

PROPOSITION 4.4. The fuzzy relation ρ is open, i.e. it is an open fuzzy subset of $M_A(\mathbb{R}) \times M_A(\mathbb{R})$.

Proof. This is an immediate consequence of the definitions of ρ and the product fuzzy topology. □

From this we conclude that ρ not only is a set theoretical extension of the strict order relation on $\mathbb{R}$ but also a

topological one.
We shall now prove that $\Delta_A(\mathbb{R})$ is indeed an "order fuzzy topology.
Let us denote the left and right sections of ρ by

$$\rho_l(P) : M(\mathbb{R}) \to I : Q \to \rho(Q,P) \tag{4.10}$$

$$\rho_r(P) : M(\mathbb{R}) \to I : Q \to \rho(P,Q) \tag{4.11}$$

and the fuzzy topology generated by

$$\{\rho_l(P) | P \in M(\mathbb{R})\} \cup \{\rho_r(P) | P \in M(\mathbb{R})\} \tag{4.12}$$

of all sections, by $\Delta_<(\mathbb{R})$.
The space $(M(\mathbb{R}), \Delta_<(\mathbb{R}))$ shall then be denoted $M_<(\mathbb{R})$.
If $E(\mathbb{R}) \subset M(\mathbb{R})$ then as a subspace it shall be denoted $E_<(\mathbb{R})$.

THEOREM 4.5. <u>The fuzzy topology of the fuzzy real line is generated by the linear</u> T_m<u>-strict order</u> ρ, <u>i.e.</u>

$$\Delta_A(\mathbb{R}) = \Delta_<(\mathbb{R}).$$

Proof. First, for any $x \in \mathbb{R}$, we have

$$\rho_l(P_x) = \delta_{]-\infty,x[} \quad \text{and} \quad \rho_r(P_x) = \delta_{]x,+\infty[}.$$

Second, for any $P \in M(\mathbb{R})$, we have

$$\rho_l(P) = \sup_{x\in\mathbb{R}} \delta_{]-\infty,x[} \wedge P(]x,+\infty[) \quad \text{and}$$

$$\rho_r(P) = \sup_{x\in\mathbb{R}} \delta_{]x,+\infty[} \wedge P(]-\infty,x[). \qquad \square$$

Let us now see whether this result can be obtained also for the Hutton- and Höhle fuzzy real lines.
In consistence with our previous notation we shall denote by $\Delta^0_<(\mathbb{R})$ and $\Delta^1_<(\mathbb{R})$ the quasi and the translation closed fuzzy topology generated by the collection of (4.12).
$(M(\mathbb{R}), \Delta^0_<(\mathbb{R}))$ shall then be denoted by $M^0_<(\mathbb{R})$ and $(M(\mathbb{R}), \Delta^1_<(\mathbb{R}))$ by $M^1_<(\mathbb{R})$, and analogously if $E(\mathbb{R}) \subset M(\mathbb{R})$ then as subspace of $M^0_<(\mathbb{R})$ (resp. $M^1_<(\mathbb{R})$) it shall be denoted $E^0_<(\mathbb{R})$ (resp. $E^1_<(R)$).

THEOREM 4.6. <u>The translation closed fuzzy topology of the Höhle fuzzy real line is generated by the linear</u> T_m<u>-strict order</u> ρ, <u>i.e.</u> $\Delta^1_A(\mathbb{R}) = \Delta^1_<(\mathbb{R})$.

Proof. Since $\Delta^1_A(\mathbb{R})$ is obtained saturating $\Delta_A(\mathbb{R})$ for translations and $\Delta^1_<(\mathbb{R})$ is obtained saturating $\Delta_<(\mathbb{R})$ for translations this follows at once from the previous theorem.

□

We thus conclude that $M_A(\mathbb{R})$ and $M_A^1(\mathbb{R})$ both in a natural way are indeed "ordered fuzzy topological spaces". This however is not the case for $M_A^0(\mathbb{R})$ as we shall see. For $\alpha \in I$ and $x \leq y$ we put

$$P_{x,y}^{\alpha} := \alpha P_x + (1-\alpha) P_y \tag{4.13}$$

PROPOSITION 4.7. For any $\alpha \in I$ and $x < y \in \mathbb{R}$ we have

$$\rho_r(P_{x,y}^{\alpha}) = (\alpha \vee \delta_{]y,+\infty[}) \wedge \delta_{]x,+\infty[}$$

$$\rho_l(P_{x,y}^{1-\alpha}) = (\alpha \vee \delta_{]-\infty,x[}) \wedge \delta_{]-\infty,y[}.$$

PROPOSITION 4.8. $\mathcal{D}_<^0(\mathbb{R})$ is homeomorphic to $(\mathbb{R}, \omega(\mathcal{T}(\mathbb{R})))$.

Proof. It is clear that for any $x \in \mathbb{R}$:

$$\delta_{]-\infty,x[} \circ \varphi = 1_{]-\infty,x[} \quad \text{and} \quad \delta_{]x,+\infty[} \circ \varphi = 1_{]x,+\infty[},$$

so that it suffices to show that $\Delta_<^0(\mathbb{R})|_{\mathcal{D}(\mathbb{R})}$ contains the constants. Consider hereto, for any $\alpha \in I$, the open fuzzy set

$$u_\alpha := \sup_{\substack{x,y\in\mathbb{R} \\ x<y}} \rho_r(P_{x,y}^{\alpha}) \wedge \rho_l(P_{x,y}^{1-\alpha}),$$

then, using Proposition 4.7, for any $x \in \mathbb{R}$ we have $\mu_\alpha(P_x) = \alpha$, i.e. $\mu_\alpha \circ \varphi = \alpha$. □

THEOREM 4.9. The quasi fuzzy topology of the Hutton fuzzy real line is strictly coarser than the quasi fuzzy topology generated by ρ, i.e. $\Delta_A^0(\mathbb{R}) \subsetneq \Delta_<^0(\mathbb{R})$.

Proof. The same argument as in the proof of Theorem 4.5. shows that $\Delta_A^0(\mathbb{R}) \subset \Delta_<^0(\mathbb{R})$. That the inclusion is indeed strict follows from Propositions 3.3 and 4.8. □

The next diagram pictures the situation of all the spaces considered so far. Note all inclusions are strict.

$M_A^0(\mathbb{R})$	$\subset$	$M_A(\mathbb{R})$	$\subset$	$M_A^1(\mathbb{R})$
$\cap$ Thm 4.9		‖ Thm 4.5		‖ Thm 4.6
$M_<^0(\mathbb{R})$	$\subset$	$M_<(\mathbb{R})$	$\subset$	$M_<^1(\mathbb{R})$

PROPOSITION 4.10. The following hold

1° $\Delta^0_<(\mathbb{R})$ contains all constant fuzzy sets with constant value larger than $\frac{1}{2}$

2° $\sup_{P \in M(\mathbb{R})} \mu(P) \geq \frac{1}{2}$ for all $\mu \in \Delta^0_<(\mathbb{R})$.

Proof. 1° Let us assume $\alpha \geq \frac{1}{2}$ and let again

$$\mu_\alpha := \sup_{\substack{x,y \in \mathbb{R} \\ x<y}} \rho_r(P^\alpha_{x,y}) \wedge \rho_l(P^{1-\alpha}_{x,y}).$$

For any $P \in M(\mathbb{R})$, making use of Proposition 4.7, with

$$T(x) := P(]x,+\infty[) \wedge (P(]-\infty,x[\vee \alpha)$$

$$S(y) := P(]-\infty,y[) \wedge (P(]y,+\infty[\vee \alpha)$$

we have

$$\mu_\alpha(P) = \sup_{x\in\mathbb{R}} T(x) \wedge \sup_{\substack{x<y \\ y\in\mathbb{R}}} S(y))$$

$$\geq \sup_{x\in\mathbb{R}} T(x) \wedge \alpha \geq \alpha.$$

Conversely if for some $x \in \mathbb{R}$ we have $T(x) > \alpha$ then $P(]x,+\infty[) > \alpha$ and $P(]-\infty,x[) > \alpha$, thus $\alpha < \frac{1}{2}$ in contradiction with our assumption. Thus it follows that also

$$\mu_\alpha(P) = \sup_{x\in\mathbb{R}} (T(x) \wedge \sup_{\substack{x<y \\ y\in\mathbb{R}}} S(y)) \leq \alpha.$$

2° Let $\mu := \inf_{k\in K} \rho_r(Q_k) \wedge \inf_{j\in J} \rho_l(R_j) \in \Delta^0_<(\mathbb{R})$, where K and J without loss of generality, may be supposed non empty. $\mathbb{R}$ being a complete separable metric space we can find, for each $j \in J$ and $k \in K$,

(i) $x_j \in \mathbb{R}$ such that $R_j(]x_j,+\infty[) \geq \frac{1}{2}$

(ii) $y_k \in \mathbb{R}$ such that $Q_k(]-\infty,y_k[) \geq \frac{1}{2}$.

Putting $x := \min_{j\in J} x_j$ and $y := (\max_{k\in K} y_k) \vee (x+1)$, then from Proposition 4.7, (i) and (ii),

$$\mu(P^{\frac{1}{2}}_{x,y}) = \inf_{k\in K}(\tfrac{1}{2} \vee Q_k(]-\infty,x[)) \wedge Q_k(]-\infty,y[))$$

$$\wedge \inf_{j\in J}((\tfrac{1}{2} \vee R_j(]y,+\infty[)) \wedge R_j(]x,+\infty[)) \geq \inf_{k\in K}(\tfrac{1}{2} \wedge Q_k(]-\infty,y_k[))$$

$$\wedge \inf_{j\in J}(\tfrac{1}{2} \wedge R_j(]x_j,+\infty[)) = \tfrac{1}{2}. \qquad \square$$

THEOREM 4.11. *If H is a tight subset of $M(\mathbb{R})$ then the fuzzy topologies induced on it by respectively $\Delta_<(\mathbb{R})$ and $\Delta_<^0(\mathbb{R})$ coincide.*

Proof. It suffices to show that $\Delta_<^0(\mathbb{R})|_H$ contains the constants. Let $\alpha \in]0,1[$ and $\varepsilon \in I_0$ such that

$$\varepsilon < \alpha \wedge (1-\alpha) \tag{4.14}$$

By tightness of H we can find $x,y \in \mathbb{R}$ such that

$$P(]x,y[) > 1 - \varepsilon \quad \forall P \in H, \tag{4.15}$$

then, from Proposition 4.7, (4.14) and (4.15) we have for any $P \in H$: $\rho_r(P^{\alpha}_{x,y}) \wedge \rho_l(P^{1-\alpha}_{x,y})(P) = \alpha$. □

COROLLARY 4.12. *If X is a bounded Borel subset of $\mathbb{R}$ then the fuzzy topologies induced on $M^X(\mathbb{R})$ by $\Delta_<(\mathbb{R})$ and $\Delta_<^0(\mathbb{R})$ respectively coincide, i.e. $M_<^X(\mathbb{R}) = M_<^{0,X}(\mathbb{R})$.*

COROLLARY 4.13. *The fuzzy topologies induced on any fuzzy interval, by $\Delta_<(\mathbb{R})$ and $\Delta_<^0(\mathbb{R})$ respectively, coincide.*

Remarks. 1° Tightness in Theorem 4.11 is only sufficient and not necessary as can be seen from Proposition 4.8.
2° Theorem 4.11 may be interpreted by saying that $M_<^0(\mathbb{R})$ only lacks the constants "at infinity".
3° The usual subbase of a linearly ordered topological space is obtained taking sections over all elements. This is what was done to obtain $\Delta_<^0(\mathbb{R})$. The quasi fuzzy topology $\Delta_A^0(\mathbb{R})$ was obtained however taking sections only over the elements of $\mathcal{D}(\mathbb{R})$. It is noteworthy that saturation by constants eliminates this difference.

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