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Exposita note

Cournot-Walras and Cournot equilibria in mixed markets: A comparison*

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Summary. In this paper, we show that, in markets with a continuum of traders and atoms, the set of Cournot-Walras equilibria and the set of Cournot equilibria may be disjoint. We show also that, when the preferences of the traders are represented by Cobb-Douglas utility functions, the set of Cournot-Walras equilibria and the set of Cournot equilibria have a nonempty intersection.

1. Introduction

In two recent papers, Codognato and Gabszewicz (1991), (1993) have analyzed the oligopolistic behaviour of economic agents in the framework of an exchange economy with a particular structure of initial endowments, corresponding to a situation of homogeneous oligopoly. More precisely, they consider an exchange economy in which some agents behave strategically and have a "corner" on a particular commodity while the remaining agents behave competitively and share the initial endowments of the other goods. The strategic traders use quantity strategies, manipulating the fraction of their holdings they send to the market for trade. In this way, a strategic agent can exert a partial control over the equilibrium prices via his individual supply. A concept of Cournot-Walras equilibrium is introduced which is the natural counterpart of the same concept defined by Gabszewicz and Vial (1972) for an economy with production.

Some years ago, another line of research, dealing with a noncooperative theory of exchange, was opened by Shapley and Shubik (1977). In this approach, all traders are assumed to behave strategically "à la Cournot": Agents send out quantity signals which indicate how much of each commodity they are willing to sell on the market. Trade then takes place mediated by prices. The noncooperative equilibrium resulting from this exchange is called a Cournot equilibrium. It is worthwhile to stress that, in this class of models, no difference is made among the agents as far as their strategic behaviour is concerned: All of them are players in the trade game. It

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is along this line of research that some developments in the noncooperative theory of exchange are articulated (see, for instance, Dubey and Shubik (1978), Dubey, Mas-Colell and Shubik (1980), Sahi and Yao (1989) and Amir et al. (1990)).

In the present paper, we undertake a first investigation of the relationship between the Cournot-Walras and the Cournot equilibrium in exchange economies. To this end, we consider a mixed exchange economy "à la Shitovitz" (see Shitovitz (1973)) which has the same structure as the exchange economy analyzed by Codognato and Gabszewicz (1991), (1993) and in which the traders who have a corner on a particular commodity are represented by atoms while the remaining traders are represented by an atomless continuum. As in Codognato and Gabszewicz (1991), (1993), we can define, for this mixed market, a concept of Cournot-Walras equilibrium which is the noncooperative equilibrium of the game in which the atoms are the strategic agents while the atomless sector behave competitively. One can also, for the same mixed market, adapt the definition of a Cournot equilibrium proposed by Sahi and Yao (1989) in the framework of an exchange economy with a finite number of agents. A reasonable conjecture is that the set of Cournot-Walras equilibria and the set of Cournot equilibria would coincide in this mixed market, to the extent that, one could expect in both situations, the agents in the nonatomic sector to loose their strategic power, while the atoms still remain strategically "significant". The present paper invalidates this conjecture: We provide an example in which this equivalence does not obtain. Does it imply that no relationship exists, between the Cournot and Cournot-Walras equilibria, in the mixed model? At least for the case of Cobb-Douglas utility functions, we are able to prove that the set of Cournot-Walras equilibria and the set of Cournot equilibria have a nonempty interesection.

In Section 2, we define the notions of Cournot-Walras equilibrium and of Cournot equilibrium for the mixed exchange economy outlined above. In Section 3, we compare the equilibrium concepts. Finally, Section 4 is devoted to some concluding remarks.

2. The model

We shall be working in the space R^l_+. The dimension l represents the number of different commodities traded in the market. We shall denote by $x=(x^1,\dots,x^l)$ a vector of R^l_+. Let $(T,\mathscr{T},\mu)$ be a measure space of the economic agents, where T denotes the set of traders, $\mathscr{T}$ a σ-field of subsets of T (the set of coalitions), and μ a measure on $\mathscr{T}$. An atom of the measure space $(T,\mathscr{T},\mu)$ is a coalition S with $\mu(S)>0$ such that, for each coalition $R\subseteq S$, we have either $\mu(R)=0$ or $\mu(S\backslash R)=0$. We denote by T_1 the set of atoms and by $T_0=T\backslash T_1$ the atomless sector. A commodity bundle is a point in R^l_+. An assignment (of commodity bundles to traders) is an integrable function $\boldsymbol{x}$ from T to R^l_+. All integrals are with respect to $t, t\in T$. There is a fixed assignment $\boldsymbol{w}$. We assume that

$$\boldsymbol{w}(t)=(\boldsymbol{w}^1(t),0,\dots,0),\quad \text{for all } t\in T_1,$$

$$\boldsymbol{w}(t)=(0,\boldsymbol{w}^2(t),\dots,\boldsymbol{w}^l(t)),\quad \text{for all } t\in T_0,$$

$$\int_T \boldsymbol{w}(t)d\mu \gg 0.$$

For each trader t, a continous utility function $U_t(x)$ is defined on R^l_+, satisfying the following assumptions: $x > y \Rightarrow U_t(x) > U_t(y)$ and $U_t(x) \geq U_t(y) \Rightarrow \forall \alpha \in (0,1)$, $U_t(\alpha x + (1-\alpha)y) > U_t(y)$. An allocation is an assignment $\boldsymbol{x}$ for which $\int_T \boldsymbol{x}(t)d\mu = \int_T \boldsymbol{w}(t)d\mu$. A price system is a vector $p \in R^l_{++}$.

Now, we proceed to the definition of the Cournot-Walras equilibrium. A *strategy* for an atom $t \in T_1$ is a real number e in the interval $S_t = [0, \boldsymbol{w}^1(t)]$. A *strategy profile* is a real valued integrable function $\boldsymbol{e}$ defined on T_1 such that, for all $t \in T_1$, $\boldsymbol{e}(t) \in S_t$. Given a price system p, let us consider the problem, for all $t \in T_0$,

$$\max_{x^1,\ldots,x^l} U_t(x^1,\ldots,x^l) \quad \text{s.t.} \sum_{h=1}^{l} p^h x^h = \sum_{h=2}^{l} p^h \boldsymbol{w}^h(t).$$

Under the assumptions on the utility functions stated above, there exists a unique solution to this problem which we denote by $x(t,p)$. Given the same price p and a strategy profile $\boldsymbol{e}$, let us consider the problem, for all $t \in T_1$,

$$\max_{x^2,\ldots,x^l} U_t(\boldsymbol{w}^1(t) - \boldsymbol{e}(t), x^2,\ldots,x^l) \quad \text{s.t.} \sum_{h=2}^{l} p^h x^h = p^1 \boldsymbol{e}(t).$$

Under the assumptions on the utility functions stated above, there exists a unique solution to this problem and, for all $t \in T_1$, we represent by $x(t,p)$ the vector $(\boldsymbol{w}^1(t) - \boldsymbol{e}(t), x^2(t,p),\ldots,x^l(t,p))$, where $(x^2(t,p),\ldots,x^l(t,p))$ denotes this unique solution. Let $\boldsymbol{x}(\cdot,p)$ be the function on T with values in R^l_+ defined by $\boldsymbol{x}(\cdot,p) = x(t,p)$. We shall assume that, for all $p \in R^l_{++}$, $\boldsymbol{x}(\cdot,p)$ is an assignment. Given a strategy profile $\boldsymbol{e}$, we denote by $p(\boldsymbol{e})$ a price system such that

$$\int_{T_0} \boldsymbol{x}^1(t,p(\boldsymbol{e}))d\mu = \int_{T_1} \boldsymbol{e}(t)d\mu,$$

$$\int_{T} \boldsymbol{x}^h(t,p(\boldsymbol{e}))d\mu = \int_{T_0} \boldsymbol{w}^h(t)d\mu, \quad h = 2,\ldots,l.$$

We assume that, for all strategy profiles $\boldsymbol{e}, p(\boldsymbol{e})$ exists and is unique. We denote by $\boldsymbol{e}\backslash e(\tau)$ the strategy profile which coincides with $\boldsymbol{e}$ for all $t \in T_1$ except for $t = \tau$ with $e(\tau) \in S_\tau, e(\tau) \neq \boldsymbol{e}(\tau)$. Given a strategy profile $\boldsymbol{e}$, it is easy to verify that the assignment $\boldsymbol{x}(\cdot,p(\boldsymbol{e}))$ is an allocation. A *Cournot-Walras equilibrium* is a pair $(\tilde{\boldsymbol{e}}, \tilde{\boldsymbol{x}})$, consisting of a strategy profile $\tilde{\boldsymbol{e}}$ and an allocation $\tilde{\boldsymbol{x}}$ such that

$$\tilde{\boldsymbol{x}}(t) = \boldsymbol{x}(t,p(\tilde{\boldsymbol{e}})), \quad \text{for all } t \in T,$$
$$U_t(\boldsymbol{w}^1(t) - \tilde{\boldsymbol{e}}(t), \boldsymbol{x}^2(t,p(\tilde{\boldsymbol{e}})),\ldots,\boldsymbol{x}^l(t,p(\tilde{\boldsymbol{e}}))) \geq$$
$$\geq U_t(\boldsymbol{w}^1(t) - e(t), \boldsymbol{x}^2(t,p(\tilde{\boldsymbol{e}}\backslash e(t))),\ldots,\boldsymbol{x}^l(t,p(\tilde{\boldsymbol{e}}\backslash e(t)))),$$

for all $t \in T_1$ and for all $e(t) \in S_t$.

Now, we proceed to the definition of the Cournot equilibrium. A *strategy* for a trader $t \in T$ is an $l \times l$ matrix $B = (b_{ij})$ such that

(i) $$b_{ij} \geq 0, \quad i,j = 1,\ldots,l,$$

(ii) $$\sum_{i=1}^{l} b_{ji} \leq \boldsymbol{w}^j(t), \quad j = 1,\ldots,l.$$

The strategy set of a trader $t \in T$ is the set of all matrices B satisfying (i) and (ii) and we denote it by Q_t. A *strategy profile* is a function $\boldsymbol{B}(t) = (\boldsymbol{b}_{ij}(t))$ defined on T such that, for all $t \in T$, $\boldsymbol{B}(t) \in Q_t$ and such that $\boldsymbol{b}_{ij}(t)$ $i, j = 1, \ldots, l$, are real valued integrable functions on T. Given a strategy profile $\boldsymbol{B}$, we denote by $p(\boldsymbol{B})$ a price vector which is the solution, if it exists, of the following system of equations

$$\sum_{i=1}^{l} p^i \int_T \boldsymbol{b}_{ij}(t) d\mu = p^j \left(\sum_{i=1}^{l} \int_T \boldsymbol{b}_{ji}(t) d\mu \right), \quad j = 1, \ldots, l.$$

Given a strategy profile $\boldsymbol{B}$, if $p(\boldsymbol{B})$ exists, the final holdings of the traders correspond to the assignment determined by

$$\boldsymbol{x}^j(t, \boldsymbol{B}(t), p(\boldsymbol{B})) = \boldsymbol{w}^j(t) - \sum_{i=1}^{l} \boldsymbol{b}_{ji}(t) + \sum_{i=1}^{l} \boldsymbol{b}_{ij}(t) \frac{p^i(\boldsymbol{B})}{p^j(\boldsymbol{B})}, \quad \text{for all } t \in T,\ j = 1, \ldots, l.$$

If $p(\boldsymbol{B})$ does not exist, we impose that

$$\boldsymbol{x}^j(t, \boldsymbol{B}(t), p(\boldsymbol{B})) = \boldsymbol{w}^j(t), \quad \text{for all } t \in T,\ j = 1, \ldots, l.$$

It is easy to verify that the assignment $\boldsymbol{x}(t, \boldsymbol{B}(t), p(\boldsymbol{B}))$ is an allocation. We denote by $\boldsymbol{B} \backslash B(\tau)$ a strategy profile which coincides with $\boldsymbol{B}$ for all $t \in T$ except for $t = \tau$ with $B(\tau) \in Q_\tau$, $B(\tau) \neq \boldsymbol{B}(\tau)$. A *Cournot equilibrium* is a pair $(\hat{\boldsymbol{B}}, \hat{\boldsymbol{x}})$, consisting of a strategy profile $\hat{\boldsymbol{B}}$ and an allocation $\hat{\boldsymbol{x}}$ such that

$$\hat{\boldsymbol{x}}(t) = \boldsymbol{x}(t, \hat{\boldsymbol{B}}(t), p(\hat{\boldsymbol{B}})),$$

$$U_t(\boldsymbol{x}(t, \hat{\boldsymbol{B}}(t), p(\hat{\boldsymbol{B}}))) \geq U_t(\boldsymbol{x}(t, B(t), p(\hat{\boldsymbol{B}} \backslash B(t)))),$$

$$\text{for all } t \in T \text{ and for all } B(t) \in Q_t.$$

3. A comparison between the equilibrium concepts

The definitions of Cournot-Walras equilibrium and of Cournot equilibrium given above lead naturally to a comparison between these equilibrium concepts. In the model proposed by Sahi and Yao (1989), all traders behave strategically but those belonging to the nonatomic sector have a negligible influence on the prices. Therefore, the strategic behaviour of the nonatomic sector could be considered as a competitive behaviour. On the other hand, in the definition of the Cournot-Walras equilibrium, the nonatomic sector is *supposed* to behave competitively while the atoms have a strategic power. Thus, the conjecture spontaneously emerges: Given a Cournot-Walras equilibrium $(\tilde{\boldsymbol{e}}, \tilde{\boldsymbol{x}})$, does there exist a strategy profile $\tilde{\boldsymbol{B}}$ such that $(\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{x}})$ is a Cournot equilibrium? Conversely, given a Cournot equilibrium $(\hat{\boldsymbol{B}}, \hat{\boldsymbol{x}})$, does there exist a strategy profile $\hat{\boldsymbol{e}}$ such that $(\hat{\boldsymbol{e}}, \hat{\boldsymbol{x}})$ is a Cournot-Walras equilibrium?

We shall first show by an example that this conjecture is not true: The set of Cournot-Walras equilibria does not necessarily coincide with the set of Cournot equilibria. To this end, consider an exchange economy with two goods, embodying two atoms A_1 and A_2, each of measure $\mu(A_r) = 1$, $r = 1, 2$, and an atomless continuum of traders represented by the unit interval $[0, 1]$ with Lebesgue measure

μ. The initial endowments of the traders are

$$w(A_r) = (1, 0), \quad r = 1, 2,$$
$$w(t) = (0, 1), \quad t \in [0, 1].$$

Furthermore, we set

$$U_{A_r}(x) = x^1 x^2, \quad r = 1, 2,$$
$$U_t(x) = x^1 + \ln x^2, \quad t \in [0, 1].$$

Moreover, we normalize the price vector by letting the price of commodity 2 be equal to one. The Cournot-Walras equilibrium of this exchange economy is the pair $(\tilde{e}, \tilde{x})$ with (see the Appendix)

$$\tilde{e}(A_r) = \frac{-1 + \sqrt{13}}{6}, \quad r = 1, 2,$$

$$\tilde{x}(A_r) = \left(\frac{7 - \sqrt{13}}{6}, \frac{-1 + \sqrt{13}}{4 + 2\sqrt{13}} \right), \quad r = 1, 2,$$

$$\tilde{x}(t) = \left(\frac{-1 + \sqrt{13}}{3}, \frac{3}{2 + \sqrt{13}} \right), \quad t \in [0, 1].$$

The price corresponding to the Cournot-Walras equilibrium is

$$p(\tilde{e}) = \left(\frac{3}{2 + \sqrt{13}}, 1 \right).$$

The Cournot equilibrium is the pair $(\hat{B}, \hat{x})$ with (see the Appendix)

$$\hat{B}(A_r) \text{ such that } \hat{b}_{12}(A_r) = \frac{1}{3}, \quad r = 1, 2,$$

$$\hat{B}(t) \text{ such that } \hat{b}_{21}(t) = \frac{2}{5}, \quad t \in [0, 1],$$

$$\hat{x}(A_r) = \left(\frac{2}{3}, \frac{1}{5} \right), \quad r = 1, 2,$$

$$\hat{x}(t) = \left(\frac{2}{3}, \frac{3}{5} \right), \quad t \in [0, 1].$$

The price corresponding to the Cournot equilibrium is

$$p(\hat{B}) = \left(\frac{3}{5}, 1 \right).$$

First, notice that the allocation corresponding to the Cournot-Walras equilibrium differs from that corresponding to the Cournot equilibrium. This unambiguously shows that the set of Cournot-Walras equilibria and the set of Cournot equilibria are disjoint. Moreover, we stress the fact that, for the particular exchange economy considered in our example, the concept of Cournot equilibrium derived from the model proposed by Sahi and Yao (1989) coincides with the concept to Cournot equilibrium derived from the models proposed by Dubey and Shubik (1978) and Amir et al. (1990). This means that the set of Cournot-Walras equilibria and the set of Cournot equilibria, derived from the models proposed in the literature quoted above, may be disjoint.

We have just shown that the equivalence between the Cournot-Walras equilibrium and the Cournot equilibrium, in general, does not hold. Nevertheless, we could ask whether there exists a class of mixed markets for which the equivalence holds. We are actually able to show that there exists a class of mixed exchange economies for which the intersection between the set of Cournot-Walras equilibria and the set of Cournot equilibria is nonempty. Consider an exchange economy with l goods, embodying two atoms A_1 and A_2, each of measure $\mu(A_r) = 1$, $r = 1, 2$, and an atomless continuum of traders represented by the unit interval $[0, 1]$ with Lebesgue measure μ. The initial endowments of the traders are

$$w(A_r) = (1, 0, \dots, 0), \quad r = 1, 2,$$
$$w(t) = (0, 1, \dots, 1), \quad t \in [0, 1].$$

Furthermore, we set

$$U_{A_r}(x) = x^{1^{\alpha_1}} x^{2^{\alpha_2}} \cdots x^{l^{\alpha_l}}, \quad r = 1, 2,$$
$$U_t(x) = x^{1^{\alpha_1}} x^{2^{\alpha_2}} \cdots x^{l^{\alpha_l}}, \quad t \in [0, 1],$$
$$\text{where } \alpha_h > 0, \quad h = 1, \dots, l, \sum_{h=1}^{l} \alpha_h = 1.$$

Moreover, we normalize the price vector by letting $\sum_{h=1}^{l} p^h = 1$. The symmetric Cournot-Walras equilibrium, i.e., the equilibrium at which all the strategic traders of the same type (same endowments and preferences) have the same strategic behaviour, of this exchange economy is the pair $(\tilde{e}, \tilde{x})$ with (see Codognato (1993))

$$\tilde{e}(A_r) = \frac{\sum_{h=2}^{l} \alpha_h}{\alpha_1 + 1}, \quad r = 1, 2,$$

$$\tilde{x}(A_r) = \left(\frac{2\alpha_1}{\alpha_1 + 1}, \frac{\alpha_1}{2}, \dots, \frac{\alpha_1}{2} \right), \quad r = 1, 2,$$

$$\tilde{x}(t) = \left(\frac{2 - 2\alpha_1}{\alpha_1 + 1}, 1 - \alpha_1, \dots, 1 - \alpha_1 \right), \quad t \in [0, 1].$$

The price corresponding to the symmetric Cournot-Walras equilibrium is

$$p(\tilde{e}) = \left(\frac{\alpha_1^2 + \alpha_1}{2 - \alpha_1 + \alpha_1^2}, \frac{2\alpha_2}{2 - \alpha_1 + \alpha_1^2}, \dots, \frac{2\alpha_l}{2 - \alpha_1 + \alpha_1^2} \right).$$

We can show (see Codognato (1993)) that the pair $(\hat{\boldsymbol{B}}, \hat{\boldsymbol{x}})$ with

$$\hat{\boldsymbol{B}}(A_r) \text{ such that } \hat{\boldsymbol{b}}_{1j}(A_r) = \frac{\alpha_j}{\alpha_1 + 1}, \quad j = 2, \ldots, l, \quad r = 1, 2,$$

$$\hat{\boldsymbol{B}}(t) \text{ such that } \hat{\boldsymbol{b}}_{ij}(t) = \alpha_j, \quad i = 2, \ldots, l, \quad j = 1, \ldots, l, \quad t \in [0, 1],$$

$$\hat{\boldsymbol{x}}(A_r) = \left(\frac{2\alpha_1}{\alpha_1 + 1}, \frac{\alpha_1}{2}, \ldots, \frac{\alpha_1}{2}\right), \quad r = 1, 2,$$

$$\hat{\boldsymbol{x}}(t) = \left(\frac{2 - 2\alpha_1}{\alpha_1 + 1}, 1 - \alpha_1, \ldots, 1 - \alpha_1\right), \quad t \in [0, 1],$$

is a symmetric Cournot equilibrium, i.e., an equilibrium at which all the traders of the same type (same endowments and preferences) have the same strategic behaviour. The price corresponding to this equilibrium is

$$p(\hat{\boldsymbol{B}}) = \left(\frac{\alpha_1^2 + \alpha_1}{2 - \alpha_1 + \alpha_1^2}, \frac{2\alpha_2}{2 - \alpha_1 + \alpha_1^2}, \ldots, \frac{2\alpha_l}{2 - \alpha_1 + \alpha_1^2}\right).$$

First, we notice that there exists a symmetric Cournot equilibrium which coincides with the symmetric Cournot-Walras equilibrium: This implies that the set of Cournot-Walras equilibria and the set of Cournot equilibria intersect. Furthermore, it is interesting to look at the nature of the strategy profiles which lead, through two different mechanisms, to the same allocation. The strategies corresponding to the Cournot equilibrium are related to the "weights" assigned by the utility function to the goods which are demanded. The strategies corresponding to the Cournot-Walras equilibrium are also related to the "weights" of the goods but, in this case, the atoms are not allowed to spread their offers over the markets. Therefore, their supplies of the good which they control correspond to the sum of the "weights" of the goods which they buy.

4. Conclusion

In this paper, we have shown that, in a mixed market with a continuum of traders and atoms, the set of Cournot-Walras equilibria and the set of Cournot equilibria may be disjoint. On the other hand, we have seen that, when the preferences of the traders are represented by Cobb-Douglas utility functions, the set of Cournot-Walras equilibria and the set of Cournot equilibria have a nonempty intersection. At an intuitive level, this result is due to the fact that loglinear utility functions allow the traders to consider separately the "weight" of each good when making their choices. This is no longer true when loglinear utility functions are replaced by general utility functions and, in particular, as we have seen in the first example, by quasilinear utility functions.

In a recent paper, Gabszewicz and Michel (1992) have extended the notion of Cournot-Walras equilibrium proposed by Codognato and Gabszewicz (1991), (1993) to a general class of exchange economies. In particular, this extension allows for *all* traders being strategic on the markets, which is the normal assumption considered in the literature inspired by Shapley and Shubik (1977). This invites spontaneously to extend the analysis proposed in the present paper to a comparison between the concept of oligopoly equilibrium introduced by Gabszewicz and Michel (1992) and the concept of Cournot equilibrium.

Appendix

In this Appendix, we give the detailed computations concerning the two goods exchange economy of Section 3. Given a price system $(p^1, 1)$, consider the problem, for all $t \in [0, 1]$,

$$\max_{x^1, x^2} x^1 + \ln x^2 \quad \text{s.t. } p^1 x^1 + x^2 = 1.$$

The solution to this problem is

$$x(t, p) = \left(\frac{1 - p^1}{p^1}, p^1\right), \quad t \in [0, 1].$$

Given a price system $(p^1, 1)$ and the strategy profile e, consider the problem

$$\max_{x^2} (1 - e(A_r))x^2 \quad \text{s.t. } x^2 = p^1 e(A_r), \quad r = 1, 2.$$

The solution to this problem is

$$x^2(A_r, p) = p^1 e(A_r), \quad r = 1, 2.$$

Therefore,

$$x(A_r, p) = (1 - e(A_r), p^1 e(A_r)), \quad r = 1, 2.$$

Given the strategy profile e, $p(e)$ is determined as the solution to the equation

$$\int_0^1 x^1(t, p) d\mu = e(A_1) + e(A_2),$$

which gives $p(e) = \left(\frac{1}{e(A_1) + e(A_2) + 1}, 1\right)$. Thus we obtain

$$x(A_r, p(e)) = \left(1 - e(A_r), \frac{e(A_r)}{e(A_1) + e(A_2) + 1}\right), \quad r = 1, 2,$$

$$x(t, p(e)) = \left(e(A_1) + e(A_2), \frac{1}{e(A_1) + e(A_2) + 1}\right), \quad t \in [0, 1].$$

The Cournot-Walras equilibrium obtains as the simultaneous solution to the problems

$$\max_{e(A_1)} (1 - e(A_1))\left(\frac{e(A_1)}{e(A_1) + e(A_2) + 1}\right),$$

$$\max_{e(A_2)} (1 - e(A_2))\left(\frac{e(A_2)}{e(A_1) + e(A_2) + 1}\right).$$

The first order conditions give us the following system of equations

$$-(e(A_1))^2 - 2e(A_1)e(A_2) - 2e(A_1) + e(A_2) + 1 = 0,$$
$$-(e(A_2))^2 - 2e(A_2)e(A_1) - 2e(A_2) + e(A_1) + 1 = 0,$$

which has as a solution

$$\tilde{e}(A_r) = \frac{-1+\sqrt{13}}{6}, \quad r = 1, 2.$$

Accordingly, we obtain $p(\tilde{e}) = \left(\frac{3}{2+\sqrt{13}}, 1\right)$ and

$$\tilde{\boldsymbol{x}}(A_r) = \left(\frac{7-\sqrt{13}}{6}, \frac{-1+\sqrt{13}}{4+2\sqrt{13}}\right), \quad r = 1, 2,$$

$$\tilde{\boldsymbol{x}}(t) = \left(\frac{-1+\sqrt{13}}{3}, \frac{3}{2+\sqrt{13}}\right), \quad t \in [0, 1].$$

Now, we compute the Cournot equilibrium. Given a strategy profile $\boldsymbol{B}, p(\boldsymbol{B})$ is determined as a solution to the equation

$$\int_0^1 \boldsymbol{b}_{21}(\tau) d\mu = p^1(\boldsymbol{b}_{12}(A_1) + \boldsymbol{b}_{12}(A_2)),$$

which gives $p(\boldsymbol{B}) = \left(\frac{\int_0^1 \boldsymbol{b}_{21}(\tau) d\mu}{\boldsymbol{b}_{12}(A_1) + \boldsymbol{b}_{12}(A_2)}, 1\right)$. Thus we obtain

$$\boldsymbol{x}(A_r, \boldsymbol{B}(A_r), p(\boldsymbol{B})) = \left(1 - \boldsymbol{b}_{12}(A_r), \frac{\boldsymbol{b}_{12}(A_r)\int_0^1 \boldsymbol{b}_{21}(\tau) d\mu}{\boldsymbol{b}_{12}(A_1) + \boldsymbol{b}_{12}(A_2)}\right), \quad r = 1, 2,$$

$$\boldsymbol{x}(t, \boldsymbol{B}(t), p(\boldsymbol{B})) = \left(\boldsymbol{b}_{21}(t)\left(\frac{\boldsymbol{b}_{12}(A_1) + \boldsymbol{b}_{12}(A_2)}{\int_0^1 \boldsymbol{b}_{21}(\tau) d\mu}\right), 1 - \boldsymbol{b}_{21}(t)\right), \quad t \in [0, 1].$$

The Cournot equilibrium obtains as the simultaneous solution to the problems

$$\max_{b_{12}(A_1)} (1 - b_{12}(A_1))\left(\frac{b_{12}(A_1)\int_0^1 b_{21}(\tau) d\mu}{b_{12}(A_1) + b_{12}(A_2)}\right),$$

$$\max_{b_{12}(A_2)} (1 - b_{12}(A_2))\left(\frac{b_{12}(A_2)\int_0^1 b_{21}(\tau) d\mu}{b_{12}(A_1) + b_{12}(A_2)}\right),$$

$$\max_{b_{21}(t)} b_{21}(t)\left(\frac{b_{12}(A_1) + b_{12}(A_2)}{\int_0^1 b_{21}(\tau) d\mu}\right) + \ln(1 - b_{21}(t)), \quad t \in [0, 1].$$

The first order conditions give us the following system of equations

$$-(b_{12}(A_1))^2 - 2b_{12}(A_1)b_{12}(A_2) + b_{12}(A_2) = 0,$$
$$-(b_{12}(A_2))^2 - 2b_{12}(A_2)b_{12}(A_1) + b_{12}(A_1) = 0,$$
$$-b_{21}(t)(b_{12}(A_1) + b_{12}(A_2)) - \int_0^1 b_{21}(\tau) d\mu + b_{12}(A_1) + b_{12}(A_2) = 0, \quad t \in [0, 1].$$

From the first two equations, which correspond to the first order conditions of the atoms, we obtain

$$\hat{\boldsymbol{b}}_{12}(A_r) = \frac{1}{3}, \quad r = 1, 2.$$

Substituting this value into the equation which corresponds to the first order condition of each trader t and imposing that $b_{21}(\tau) = b_{21}(t)$, $\tau \in [0, 1]$, we obtain

$$\hat{\boldsymbol{b}}_{21}(t) = \frac{2}{5}, \quad t \in [0, 1].$$

Accordingly, we obtain $p(\hat{\boldsymbol{B}}) = \left(\frac{3}{5}, 1\right)$ and

$$\hat{\boldsymbol{x}}(A_r) = \left(\frac{2}{3}, \frac{1}{5}\right), \quad r = 1, 2,$$

$$\hat{\boldsymbol{x}}(t) = \left(\frac{2}{3}, \frac{3}{5}\right), \quad t \in [0, 1].$$

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