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On Fiber Products of Rational Elliptic Surfaces with Section

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0. Introduction

Recently, simply connected, three dimensional, projective varieties with trivial canonical bundle have attracted the interest of physicists working on superstring theory and algebraic geometers working on the classification of threefolds and on algebraic cycles. The purpose of this note is to popularize a certain class of such 3-folds which is large enough to exhibit many of the phenomena which one wants to study, yet is special enough to be quite tractable. The members of this class are fiber products of relatively minimal, rational, elliptic surfaces with section. A nice attribute of these 3-folds is that many questions about them reduce to questions concerning the well studied surfaces from which they are built. To illustrate the point we shall take up the following three issues:

(i) How does one construct rigid 3-folds of Kodeira dimension zero?

(ii) What integers arise as topological Euler characteristic of 3-folds with $K=0$?

(iii) Are there examples of birational automorphisms of projective 3-folds with $K=0$ which are not biregular?

In general very little is known concerning (i), although two examples have been described [B1, §3], [S1]. However, when attention is restricted to resolutions of fiber products of rational elliptic surfaces with section we show in §7 that examples are easily produced and that subject to a mild restriction all such varieties with no non-trivial, first order deformations fit into four distinct classes.

It has been conjectured by Bogomolov that the third question should have a negative answer [Bo]. Contrary to the conjecture, investigation of the second question naturally leads to examples of (iii) (see (6.2)). The motivation for (ii) comes in part from questions raised by physicists working on superstring theory (see [SW] and references therein). They would like to find examples of compact, Kaehler threefolds with zero Ricci curvature and non-zero Euler characteristic of small absolute value, preferably 6. It follows from a fundamental theorem [B2] that the universal cover of such a manifold is a projective variety with trivial canonical bundle. Conversely, by Yau's proof of the Calabi conjecture [Y1], any quotient of a projective variety with trivial canonical class by the free action of a finite group admits a Kaehler metric with zero Ricci curvature.

The bulk of this paper is devoted to showing that considerations concerning fiber products of rational elliptic surfaces with section allow one to construct many projective, $K=0$, 3-folds having positive Euler charactgeristics. In fact we use two elementary constructions to produce examples having any desired even Euler characteristic between -8 and 92. Generally it has been difficult to find examples with a predetermined Euler characteristic in this range. In particular Euler number 6 does not seem to have been previously achieved, while only three Ricci flat Kaehler threefolds with Euler characteristic -6 have been found [Y2, appendix]. For other approaches to the problem of constructing compact, Kaehler 3-folds with zero Ricci curvature and Euler characteristic of small absolute value the reader is referred to [Hi], [HW], [Y2], [SW], [B2, §3], [Hu], [AGKM].

Several individuals have indicated in their comments on the initial version of this paper that physicists would be most interested in a $K=0$ projective 3-fold with Euler characteristic ± 6 having either no rational curves or, what seems more accessible, a finite, non-trivial fundamental group [Y2, appendix]. This has motivated me to add a final section in which certain fixed point free group actions on special fiber products of relatively minimal, rational, elliptic surfaces with section are introduced. There is no good general technique for isolating from a family of varieties those which admit a fixed point free automorphism. The technique employed here is to use the theory of principal homogeneous spaces for elliptic curves over function fields to explicitly construct the quotients. All the elliptic surfaces which are involved in this somewhat delicate procedure are simply connected, but their desingularized fiber products are not. Unfortunately for the physicists, the projective varieties with Euler characteristic $\leqq 6$ which arise from these considerations have Euler characteristic zero.

The fiber products with which we will deal tend to have ordinary double point singularities. We shall need to take small resolutions. Basic facts concerning these are collected in §1. Frequently it is difficult to determine if a small resolution of a projective variety is projective (or Kaehler). This is an important consideration, since the additional complex manifolds which turn up when we ignore it are likely to be of less interest to both physicists and algebraic geometers. As explained in §3, most of these difficulties are easily dealt with in the case of fiber products of elliptic surfaces. Those elliptic surfaces which form the building blocks for our constructions are described in §4 and the threefolds with desired Euler characteristics are produced in §5 and §6. Finally, §8 is devoted to interesting variations of Hodge structure which come from certain families of fiber products of rational elliptic surfaces with section.

Unless the contrary is specifically indicated, the base field for all algebraic varieties in this paper is the complex numbers.

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Contents

1. Preliminaries Concerning Small Resolutions

This paragraph summarizes without proof preliminary material on small resolution of ordinary double points on three dimensional, complex analytic spaces. A double point, w, on a threefold hypersurface, X, is said to be ordinary (or a node) if its projectivized tangent cone is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Equivalently, there are analytic coordinates (x, y, u, v) such that X is locally defined by $xy - uv = 0$ (cf. [M, p. 7]). It is always possible to find a smooth surface, S, on some neighborhood X_0 of w which passes through w. Given such a surface, the tangent plane $T_w S$ is the cone over a line in one of the two rulings on $\mathbf{P}^1 \times \mathbf{P}^1$. By a linear change of variables we may assume that $x = u = 0$ is the equation of $T_w S$ and that $x' = x + \text{h.o.t.}$ and $u' = u + \text{h.o.t.}$ generate the ideal of S in the local ring at w. Since S is contained in X_0, there exist functions y' and v' such that $xy - uv = x'y' - u'v'$. Furthermore, x', y', u', v' is a new system of coordinates, so after a local analytic change of coordinates we may actually assume that S coincides with its tangent space. Observe that the map $(x' : u')$: $X_0 - w \to \mathbf{P}^1$ is defined after shrinking X_0 if necessary. The closure of the graph is a complex submanifold, $\hat{X}_0$, of $X_0 \times \mathbf{P}^1$ called the blow up of X_0 along S. Projection of $\hat{X}_0$ onto the first factor is an isomorphism above $X_0 - w$. The fiber over w is isomorphic to $\mathbf{P}^1$. Gluing $X - w$ and $\hat{X}_0$ along $X_0 - w$ gives the desired small resolution of the ordinary double point on X. If S is replaced by any smooth surface S' whose tangent space gives a line in the same ruling of the projectivized tangent cone then the blow ups of X_0 along S and S' are isomorphic. However if the rulings are different, the two blow ups are not isomorphic over X_0. Finally suppose that $\hat{X}_0$ is blown up along the exceptional $\mathbf{P}^1$. In this case one checks easily that the resulting manifold is isomorphic to the usual blow up of X_0 at the point w.

If X is a projective threefold with only ordinary double point singularities, then by performing the above process in an analytic neighborhood of each node one arrives at a complex manifold $\hat{X}$. The canonical sheaf of $\hat{X}$ and the pullback of the dualizing sheaf of X [H, III.7] are invertible sheaves on $\hat{X}$ which are isomorphic outside the codimension two exceptional locus. Hence they are isomorphic. The topological Euler characteristics are related by $e(\hat{X}) = e(X) + {}^{\#}(\text{nodes})$.

The question of whether $\hat{X}$ admits a Kaehler metric is subtle and in practice frequently difficult to decide. When we wish to show that $\hat{X}$ does not admit a Kaehler metric, we shall attempt to find an effective curve supported on the fibers above the nodes which is orthogonal to $H^2(\hat{X}, \mathbf{R})$. This suffices since no effective curve could be orthogonal to the Kaehler class. Should we wish to show that $\hat{X}$ is projective, and hence Kaehler, we shall attempt to realize $\hat{X}$ as the result of successively blowing up a sequence of closed, non-singular, codimension one algebraic subvarieties, at least one of which passes through any given node. This suffices since the blow up of a projective variety along a closed subvariety is again projective [H, II.7]. These criteria turn out to be admirably suited for our purposes. A more systematic discussion of the projectivity, or equivalently, the Kaehlerness of $\hat{X}$ will appear in [W].

2. Fiber products of Elliptic Surfaces

In this section we describe the threefolds to be investigated and relate them to special hypersurfaces in $\mathbf{P}^2 \times \mathbf{P}^2$. Let r: $Y \to \mathbf{P}^1$ and r': $Y' \to \mathbf{P}^1$ denote two relatively minimal, rational, elliptic surfaces with sections. Let $x = (x_0, x_1, x_2)$ denote homogenous coordinates on $\mathbf{P}^2$. Then there exist homogeneous cubic polynomials $a(x)$ and $b(x)$ without common factors such that Y is the blow up of $\mathbf{P}^2$ at the base locus of the pencil associated to the rational map $r(x) = (a(x)$: $b(x))$ [MP, Prop. 6.1]. Write S (resp. S') for the images of the singular fibers of Y (resp. Y') in $\mathbf{P}^1$. It is immediate that the fiber product p: $W = Y \times_{\mathbf{P}^1} Y' \to \mathbf{P}^1$ is non-singular except at points in the fibers over $S'' = S \cap S'$. In order that the singularities of W be no worse than ordinary double points, we shall assume until §9 that the singular fibers of r and r' above the points in S'' are either irreducible nodal rational curves or cycles of smooth rational curves. In the notation of Kodaira [BPV, V.7] such fibers are of type I_b, where $b > 0$ denotes the number of irreducible components. Such fibers will also be called "semi-stable". Now the singularities of W occur precisely at the points (q, q') in $r^{-1}(s) \times r'^{-1}(s)$ where both q and q' are singular points in the fibers of the corresponding elliptic surfaces. A local computation shows that the singularities are indeed ordinary double points.

By choosing homogeneous cubics $a'(x')$ and $b'(x')$ which give the pencil on $\mathbf{P}^2$ corresponding to r', one may explicitly write the equation of a bicubic hypersurface in $\mathbf{P}^2 \times \mathbf{P}^2$ which is birational to W as $a(x)\, b'(x') - a'(x')\, b(x) = 0$. Such determinantal hypersurfaces form a very special subclass of bicubics in $\mathbf{P}^2 \times \mathbf{P}^2$. For a general choice of cubics a, b, a', b', the threefold has 81 nodes at the points (x, x') where a, b, a', and b' vanish simultaneously. The inverse image of these nodes in W give 81 sections of the morphism p.

The dualizing sheaf of W is most readily computed by regarding W as the hypersurface in $Y \times Y'$ obtained by pulling back the diagonal in $\mathbf{P}^1 \times \mathbf{P}^1$ via the map $r\times r'$. For a relatively minimal, regular elliptic surface, $r\colon Y\to\mathbf{P}^1$, the canonical sheaf is given by

$$\omega_Y \simeq r^*\mathcal{O}_{\mathbf{P}^1}(p_g(Y)-1)\otimes\mathcal{O}_Y(\sum_i (m_i-1)\,F_i)$$

where the second term is the contribution of the multiple fibers [BPV, V12.1 and 12.2]. An easy computation using the adjunction formula reveals that the dualizing sheaf ω_W is trivial exactly when $p_g(Y)=p_g(Y')=0$ and there are no multiple fibers. Of course these conditions are fulfilled when both Y and Y' are rational and have sections. However it follows from Castelnuovo's rationality criterion [BPV, VI2.1] and the vanishing of the Tate-Shafarevich group for rational elliptic surfaces with section [Sa, Thm. 3] or [La] that this is the only case when the conditions for triviality of ω_W are fulfilled. In this case any small resolution has trivial canonical bundle (§1). In any other case involving fiber products of regular elliptic surfaces the Kodaira dimension of a non-singular model of W is one provided that W has at worst ordinary double point singularities (or more generally canonical singularities).

Since the topological Euler characteristic of a smooth fiber of p is zero, the Euler characteristic, $e(W)$, is the sum of the Euler characteristics of the singular fibers. For a singular fiber of type $I_{b'}\times I_b$, the Euler characteristic is $b\times b'$ by the Kuenneth formula. Note that for the fiber over any s not in S'', $e(p^{-1}(s))=0$ since one factor in the product is an elliptic curve. It follows that $e(W)$ is equal to the number of nodes. Also if $\hat{W}$ is any small resolution of W the Euler characteristic is $2(^{\#}$ nodes of $W)$.

(2.1) *Remark.* Let $r\colon Y\to\mathbf{P}^1$ and $r'\colon Y'\to\mathbf{P}^1$ denote rational (or more generally simply connected) elliptic surfaces with sections. The fiber product W may be seen to be simply connected by arguing as in [S2, Lemma 1.1]. If W has at worst ordinary double point singularities, then a local computation shows that any small resolution $\hat{W}$ is also simply connected.

All elliptic surfaces in the remainder of this paper are assumed to be rational and, until §9, to have a section.

3. Small Resolution of Fiber Products

In this paragraph we show that it is generally easy to determine when W has or does not have a small projectile resolution.

(3.1) **Lemma.** (i) *If for all s in S'', neither $r^{-1}(s)$ nor $r'^{-1}(s)$ is irreducible, then W has a small projective resolution.*

(ii) *If $r=r'$, W has a small projective resolution.*

(iii) *If S does not equal S', and if for some s in S'' either $r^{-1}(s)$ or $r'^{-1}(s)$ is irreducible, then W has no small projective resolution.*

Proof. Recall that each node of W lies in a fiber over some s in S''. In case (i), every component of every such fiber is a smooth surface, which is in fact isomorphic to $\mathbf{P}^1\times\mathbf{P}^1$. Every node of W lies on such a Weil divisor. A small projective resolution is obtained by successively blowing up any sequence of these Weil divisors which contains all nodes (§1). In particular, if S'' is not empty, W has several non-isomorphic projective resolutions. By the considerations of (i), case (ii) will be settled if nodes in fibers over those points s for which $r^{-1}(s)$ is irreducible, can be resolved. But these singularities are all resolved by blowing up the diagonal.

To prove (iii) suppose that $g\colon \hat{W} \to W$ is small projective resolution and that $F = g^{-1}(w)$ for a node, w, in the fiber over s. A contradiction will be established by showing that no divisor on $\hat{W}$ meets F. Write η for the generic point of $\mathbf{P}^1$ (in the sense of schemes [H]) and Pic for the group of *Cartier* divisors modulo rational equivalence [H, II.6]. The first hypothesis implies that the generic fibers of r and r' are not isogenous. In particular $\mathrm{Pic}(p^{-1}(\eta)) \simeq \mathrm{Pic}(r^{-1}(\eta)) \times \mathrm{Pic}(r'^{-1}(\eta))$. Since $\mathrm{Pic}(Y)$ surjects to $\mathrm{Pic}(r^{-1}(\eta))$ and the same holds for $\mathrm{Pic}(Y')$, it follows that the pullback of $\mathrm{Pic}(W)$ in $\mathrm{Pic}(\hat{W})$ surjects onto the Picard group of the generic fiber, $\mathrm{Pic}(W_\eta)$. The second hypothesis implies that every component of the fiber $p^{-1}(s)$ is a Cartier divisor on W. From the exact sequence

$$\left.\begin{array}{r}\text{Free abelian group}\\ \text{on components of sing.}\\ \text{fibers of } \hat{W}.\end{array}\right\} \to \mathrm{Pic}(\hat{W}) \to \mathrm{Pic}(W_\eta) \to 0 \tag{3.2}$$

we see that $\mathrm{Pic}(\hat{W})$ is generated by the pullback of $\mathrm{Pic}(W)$ and components of fibers other than those over s. Of course the pullback of $\mathrm{Pic}(W)$ is orthogonal to F. Thus $\mathrm{Pic}(\hat{W})$ is orthogonal to F which contradicts the projectivity of $\hat{W}$.

I have learned from Hirzebruch that many constructions of threefolds in $\mathbf{P}^4$ with very large numbers of nodes involve fiber products of the affine plane with itself over the affine line [HW, IV]. The reader may answer positively a question of Hirzebruch [Hi] by applying the ideas of (3.1(i and ii)) and some evident symmetries to produce a small projective resolution of Hirzebruch's remarkable quintic threefold with 126 nodes. J. Werner has independently used quite similar considerations to produce small projective resolutions of certain other nodal threefolds described in [HW].

4. Construction of Certain Rational Elliptic Surfaces

In order to use (3.1(i, ii)) to construct examples of projective threefolds with trivial canonical sheaf and a diversity of Euler characteristics, it is necessary to have examples of rational elliptic surfaces with section with various sorts of singular fibers. First of all we shall make use of certain elliptic modular surfaces associated to torsion free congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$. Each of these has four singular fibers, all of which are of type I_b for various b. Several people have pointed out to me that this is actually a complete list of rational elliptic surfaces with exactly four semi-stable singular fibers (see Beauville [B3] for the proof and further discussion). For each of these surfaces the following table gives the level of the associated congruence subgroup and the number

Table 1. Certain rational elliptic modular surfaces

Level	Congruence *subgroup*	Number of components in singular fibers	n
3	$\Gamma(3)$	3, 3, 3, 3	4
4	$\Gamma_1(4) \cap \Gamma(2)$	4, 4, 2, 2	0
5	$\Gamma_1(5)$	5, 5, 1, 1	4
6	$\Gamma_1(6)$	6, 3, 2, 1	2
8	$\Gamma_0(8) \cap \Gamma_1(4)$	8, 2, 1, 1	2
9	$\Gamma_0(9) \cap \Gamma_1(3)$	9, 1, 1, 1	4

of components in each of the four singular fibers as well as the invariant n to be introduced in §6.

For most of our examples we shall use the following lemma, whose proof depends on the existence of the rational, elliptic modular surface of level 9.

(4.1) **Lemma.** *Given a set of at least four positive integers* $\{b_1, \ldots, b_j\}$ *satisfying*

(i) $b_1 = b_2 = b_3 = 1$,

(ii) $b_1 + \ldots + b_j = 12$

there exists a rational elliptic surface with section having exactly the singular fibers $I_{b_1}, \ldots, I_{b_j}$.

Proof. Let $k = {}^{\#}\{b_i \colon b_i > 1\}$ and $l = j - k - 3$. An elliptic surface with section and twelve I_1 singularities is given by a Lefschetz pencil of cubics on $\mathbf{P}^2$. Thus we may assume $k > 0$. By Kodaira's theory [BPV, V.11.1] it suffices to produce a degree twelve holomorphic map $J \colon \mathbf{P}^1 \to \mathbf{P}^1$ with the following ramification:

$$\begin{aligned} &J^{-1}(0) = 4 \text{ points, each with ramification index } 3, \\ &J^{-1}(1) = 6 \text{ points, each with ramification index } 2, \\ &J^{-1}(\infty) = j \text{ points with ramification indices } b_1, \ldots, b_j, \\ &J^{-1}(\alpha_i) \text{ has } 11 \text{ points for } 1 \leqq i \leqq k-1, \\ &J^{-1}(\beta_i) \text{ has } 11 \text{ points for } 1 \leqq i \leqq l, \end{aligned} \tag{4.2}$$

where $\{\alpha_1, \ldots, \alpha_{k-1}, \beta_1, \ldots, \beta_l, 0, 1\}$ are pairwise distinct complex numbers. Note that by the Hurwitz formula,

$$\begin{aligned} 2(g(\mathbf{P}^1)) - 2 &\geqq 12(-2) + 4(2) + 6(1) + \sum (b_i - 1) + k - 1 + l \\ &= -10 + 12 - j + k - 1 + l = -2 \end{aligned}$$

J can have no further ramification. In the language of Kodaira, J will be the functional invariant of an elliptic surface. In order to completely determine the associated rational elliptic surface with section, one must also specify one of finitely many compatible homological invariants. With J as above there is exactly one such which gives only type I_b singular fibers in the corresponding elliptic surface [BPV, V.11 and V.10, Table 6].

Choose loops $\delta_0, \delta_1, \delta_\infty, \delta_{\alpha_i}$ $1 \leqq i \leqq k-1$, δ_{β_i} $1 \leqq i \leqq l$ around $0, 1, \infty, \alpha_i$ $1 \leqq i \leqq k-1$, β_i $1 \leqq i \leqq l$ which generate $\pi_1(\mathbf{P}^1 - \{0, 1, \infty, \alpha_1, \ldots, \alpha_{k-1}, \beta_1, \ldots, \beta_l\}, *)$ subject only to the relation

$$(4.3) \qquad \delta_0 \delta_1 \delta_{\beta_l} \ldots \delta_{\beta_1} \delta_\infty \delta_{\alpha_1} \ldots \delta_{\alpha_{k-1}} = 1.$$

By covering space theory and the Riemann Existence Theorem [F, Prop. 1.2], to construct J it suffices to produce a homomorphism

$$(4.4) \qquad \chi: \pi_1(\mathbf{P}^1 - \{0, 1, \infty, \alpha_1, \ldots, \alpha_{k-1}, \beta_1, \ldots, \beta_l\}, *) \to \mathscr{S}_{12}$$

whose image is a transitive subgroup of the permutations on twelve elements and which also satisfies

a) $\chi(\delta_{\alpha_n})$, $\chi(\delta_{\beta_m})$ are transpositions for all n and m.
b) $\chi(\delta_0)$ is conjugate to $(1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)$,
c) $\chi(\delta_1)$ is conjugate to $(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)$,
d) $\chi(\delta_\infty)$ is a product of j disjoint cycles of lengths $b_1, \ldots, b_j$.

Example. $j=4$, $b_4=9$. Then $k=1$ and $l=0$. Then a homomorphism χ_0: $\pi_1(\mathbf{P}^1 = \{0, 1, \infty\}, *) \to \mathscr{S}_{12}$ having the desired properties exists. In fact the map J: $\mathbf{P}^1 \to \mathbf{P}^1$ associated to χ_0 is the J-function of the unique rational elliptic modular surface of level 9.

The homomorphism χ_0 of the example will be used to construct χ in the general case. Set $\chi(\delta_0) = \chi_0(\delta_0)$ and $\chi(\delta_1) = \chi_0(\delta_1)$. The problem now is to find transpositions $\chi(\delta_{\alpha_1}), \ldots, \chi(\delta_{\alpha_{k-1}}), \chi(\delta_{\beta_1}), \ldots, \chi(\delta_{\beta_l})$ and a product, $\chi(\delta_\infty)$, of j disjoint cycles of lengths $b_1, \ldots, b_j$, such that

$$(4.5) \qquad \chi(\delta_{\beta_l}) \ldots \chi(\delta_{\beta_1}) \chi(\delta_\infty) \chi(\delta_{\alpha_1}) \ldots \chi(\delta_{\alpha_{k-1}}) = \chi_0(\delta_\infty).$$

Because $\pi_1(\mathbf{P}^1 - \{0, 1, \alpha_1, \ldots, \alpha_{k-1}, \beta_l, \ldots, \beta_1\} *)$ is a free group modulo the one relation (4.3), such a χ would extend uniquely to a homomorphism to $\mathscr{S}_{12}$. Furthermore the image would be transitive, since it would contain the image of χ_0.

To construct permutations which satisfy (4.5) we are free to assume $b_j \geqq b_{j-1} \geqq \ldots \geqq b_{j-k-1} > 1$. For $1 \leqq i \leqq k$, set $p_i = \sum_{1 \leqq n \leqq i} b_{j-n+1}$. Observe that $9 = p_k + l$ and $p_i > p_{i-1} + 1$. Set

$$\kappa = (1, \ldots, p_1)(p_1 + 1, \ldots, p_2) \ldots (p_{k-1} + 1, \ldots, p_k)$$

and note that the following identity among cycles holds

$$\begin{aligned} &\kappa(p_1, p_1 + 1)(p_2, p_2 + 1) \ldots (p_{k-1}, p_{k-1} + 1) \\ &\quad = (1, 2, \ldots, p_1, p_1 + 2, \ldots, p_2, p_2 + 2, \ldots, \ldots, p_{k-1}, \\ &\qquad p_{k-1} + 2, \ldots, p_k, p_{k-1} + 1, p_{k-2} + 1, \ldots, p_1 + 1). \end{aligned}$$

Left multiplication by $(p_1 + 1, 9)(p_1 + 1, p_k + l - 1) \ldots (p_1 + 1, p_k + 2)(p_1 + 1, p_k + 1)$ gives a 9-cycle, ν. Choose $\sigma \in \mathscr{S}_{12}$ so that $\sigma \nu \sigma^{-1} = \chi_0(\delta_\infty)$. Then set $\chi(\delta_\infty) = \sigma \kappa \sigma^{-1}$, $\chi(\delta_{\alpha_i}) = \sigma(p_i, p_i + 1)\sigma^{-1}$, $1 \leqq i \leqq k-1$; $\chi(\delta_{\beta_i}) = \sigma(p_1 + 1, p_k + i)\sigma^{-1}$ $1 \leqq i \leqq l$. Now (4.5) is satisfied and the lemma is proved.

5. Euler Characteristics

Using (3.1(i)) and rational elliptic surfaces constructed in (4.1) it is a simple matter to produce projective threefolds with trivial canonical sheaf having any even Euler characteristic greater than or equal to zero and less than 100 except 2, 4, 6, 10, 14, 22, 88, 94.

To illustrate the point we construct an example with Euler characteristic 96. According to (4.1) there exists a rational elliptic surface, Y, with section and with one I_6, one I_3, and three I_1 singular fibers. Similarly, there exists Y' with one I_7, one I_2, and three I_1 singular fibers. By choosing an appropriate isomorphism between the base curves we may arrange that the I_6 and I_7 fibers map to the same point of $\mathbf{P}^1$ and that the I_3 and I_2 fibers similarly map to one point. There is still enough freedom in the choice of isomorphism to insure that S'' consists only of these two points. The fiber product, W, has $42+6=48$ nodes. Any small resolution has the desired Euler characteristic, 96, and some small resolutions are projective (3.1(i)).

The Euler characteristic 88 can be achieved by taking a projective small resolution (3.1(ii)) of the self-fiber product of Y where Y has one I_6, one I_2, and four I_1 singular fibers.

Take $Y=Y'$ to be the modular surface of level 9 in Table 1. The resulting fiber product has a small projective resolution with Euler characteristic 168. Because every semi-stable elliptic surface has at least four singular fibers [MP], this is the largest Euler characteristic which can be obtained from this construction. It appears to be the largest known Euler characteristic for any 3-fold with trivial canonical bundle. However, the set of such Euler characteristics is not even known to be bounded.

6. Families of Kummer Surfaces

It does not seem possible to obtain projective varieties with non-zero Euler characteristic less than eight by taking small resolutions of fiber products of rational elliptic surfaces with section. In order to reach smaller Euler characteristics we consider the family of Kummer surfaces associated to the fiber products. The inversion homomorphism in the generic fiber of r (resp. r') extends uniquely to a biregular involution of Y (resp. Y'). This gives an involution ι of W which we wish to lift to a small resolution. We shall assume either that $r=r'$ or that for all s in S'' neither $r^{-1}(s)$ nor $r^{-1}(s)$ is irreducible. Then the considerations of §3 show that small projective resolutions, $\hat{W}$, of W exist. Extra care must be taken in the choice of resolution to guarantee that the birational isomorphism ι is biregular on $\hat{W}$. For s in S'' write $b(s)$, or just b if confusion is unlikely, for the number of irreducible components of $r^{-1}(s)$. Label these components $A_0, \ldots, A_{b-1}$ with subscripts mod b so that $\iota A_{-n}=A_n$. The same notation with a prime added will be used for the components of the fibers of r'. Suppose that W'' is a projective, partial, small resolution of W to which ι lifts. Let D denote the strict transform of some smooth component of a fiber $p^{-1}(s)$ with s in S''. It is important to know when the variety obtained from W''

by first blowing up the Weil divisor D and then blowing up the strict transform of ιD is isomorphic to the variety gotten by first blowing up W'' along ιD then blowing up the strict transform of D. Of course if D does not meet the singular locus of W'' then neither does ιD. In this case both D and ιD are Cartier divisors and blowing up has no effect [H, II.7.14]. If D and ιD meet only at nodes of W'' then their tangent spaces give planes in the same rulings of the tangent cones at each of these nodes. In this case the order in which the blow ups are taken does not affect the isomorphism class of the resulting variety (§1). The only remaining possibility is that the intersection of D and ιD contains a $\mathbf{P}^1$ which passes through a node of W''. In this case the tangent spaces of the two divisors give planes in different rulings of the tangent cone and the varieties obtained by blowing up D and ιD in different orders are not isomorphic over W.

(6.1) **Lemma.** *Let W be as above. Suppose in addition that for each s in S'' for which $b(s)=2$ (resp. $b'(s)=2$) then $b'(s)$ (resp. $b(s)$) is even. Then there exists a small projective resolution $\hat{W}$ of W to which ι lifts.*

Proof. First of all resolve the singularities in the fibers where both $b(s)$ and $b'(s)$ are at least two. If $b(s)=b'(s)=2$, then all four components of $p^{-1}(s)$ are stable under the involution and blowing up any one component resolves all singularities in the fiber. If either $b(s)$ or $b'(s)>2$ consider a sequence $D_1, \iota D_1, D_2, \iota D_2, \ldots, D_m, \iota D_m$, where each D_k is a component in the fiber $p^{-1}(s)$. When both $b(s)$ and $b'(s)>2$, choose each D_k of the form $A_n \times A'_{n'}$ so that $2n$ and $2n'$ are not zero and every node of $p^{-1}(s)$ is contained in some D_k or ιD_k. From the considerations above, if W'' is obtained from W by blowing up first D_1, then the strict transform of ιD_1, and so on through the whole sequence, then ι lifts to W'' and all nodes in the fiber are resolved. When $b(s)=2$ and $b'(s)$ is even and greater than two, choose the D_k in the above sequence in the form $A_0 \times A'_{n'}$ with $2n'$ not 0. Again ι lifts to the partial resolution W''. If for some s in S'' $b(s)=1$ or $b'(s)=1$, then by assumption $r=r'$. In this case the final step in the resolution process is to blow up the diagonal. This resolves all remaining nodes and it is evident that the involution lifts to the resulting small projective resolution.

(6.2) **Remark.** Contrary to a conjecture of Bogomolov [Bo] (see also [B1, §6] for a counterexample to another aspect of this conjecture) which asserts that birational automorphisms of varieties with trivial canonical sheaf are biregular, it is possible in the present situation to construct small projective resolutions to which the involution does not lift. For example suppose that S'' consists of a single point s with $b(s)=2$ and $b'(s)=3$. Then a non-singular $\hat{W}$ may be constructed by first blowing up $A_0 \times A'_0$ followed by the strict transform of $A_0 \times A'_1$. The involution clearly does not lift to $\hat{W}$. In fact when $S''=\{s\}$, $b(s)=2$, and $b'(s)=1 \bmod 2$ it is not posible to construct a small projective resolution to which inversion lifts.

A counter-example of a different sort is obtained by considering the case where W is a self-fiber product of Y (i.e. $r=r'$) and Y has a singular fiber of type I_n, $n>1$. Let $Y^{\#} \subset Y$ denote the complement of the singular points in the fibers of r. Then $Y^{\#}$ and hence $W^{\#}=Y^{\#} \times_{\mathbf{P}^1} Y^{\#}$ is a group scheme over

$\mathbf{P}^1$. There is an obvious injective homomorphism γ: $\mathrm{GL}_2(\mathbf{Z}) \to \mathrm{Aut}(W^{\#}/\mathbf{P}^1)$. Any group scheme automorphism raised to an appropriate power will stabilize the connected component of the identity in every fiber of $p|_{W^{\#}}$. For $s \in S$, the connected component of $(p|_{W^{\#}})^{-1}(s)$ is $\mathbf{C}^* \times \mathbf{C}^*$. An automorphism $(t_1, t_2) \to (t_1^a t_2^b, t_1^c t_2^d)$ extends to a rational map $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$ which collapses some divisor $t_i = 0$ or $t_i = \infty$ unless one member of each pair $\{a, b\}$, $\{c, d\}$ is zero – in particular unless the automorphism is of finite order. It follows that the subgroup of $\mathrm{im}(\gamma)$ which extends to biregular automorphisms of W (or $\hat{W}$ or $\tilde{W}$) is finite.

Suppose that $\hat{W}$ is a small projective resolution of some W satisfying the hypotheses of (3.1) and that ι is biregular on $\hat{W}$. In order to show that the induced action on global holomorphic 3-forms is trivial, it suffices by Serre duality to investigate the action on $H^3(\hat{W}, \mathcal{O})$. Since inversion acts by -1 on $R^1 r_* \mathcal{O} = \mathcal{O}_{\mathbf{P}}(-1)$, the action on $H^3(\hat{W}, \mathcal{O}) = H^3(W, \mathcal{O}) = H^1(\mathbf{P}^1, R^2 p_* \mathcal{O}) = H^1(\mathbf{P}^1, (R^1 r_* \mathcal{O})^{\otimes 2})$ is trivial.

The fixed locus of inversion on $\hat{W}$ is a smooth curve which will be denoted, B. Write W^* for the blow up of $\hat{W}$ along B. The quotient of W^* by the involution is a smooth variety V *. Standard formulas which compute the canonical sheaf of blow ups and branched covers show that V^* has trivial canonical sheaf. It is an easy exercise to compute the topological Euler characteristic of V^*. One finds $e(V^*) = (e(\hat{W}) + 3e(B))/2$.

It remains to compute $e(B)$. Write I for the identity section of Y and $(I+C)$ for the smooth curve on Y fixed by inversion. The even integer $n = 6 - e(C)$ ranges from 0, when all the two torsion in the generic fiber is rational, to 12. If the only singular fibers are of type I_b, then n is the number of fibers with b odd, as one sees by the Hurwitz formula. Suppose that for each s in S'', one of the two numbers $b(s)$ or $b'(s)$ is even. Then the fiber product $(I+C) \times_{\mathbf{P}^1} (I+C)$ is non-singular, disjoint from the singularities of W, and may be identified with B. In general let k denote the number of s in S'' for which both $b(s)$ and $b'(s)$ are odd. In this case B corresponds to the strict transform of $(I+C) \times_{\mathbf{P}^1} (I+C)$ in $\hat{W}$. By applying the Hurwitz formula to B viewed in the obvious way as a branched cover of $\mathbf{P}^1$ one finds $e(B) = [32 - 4(n+n') + 2k]$. This proves

(6.3) **Lemma.** *Let W be a fiber product of rational elliptic surfaces with section satisfying the hypotheses of* (3.1). *Then the topological Euler characteristic of any variety V^* constructed as above is given by*

$$e(V^*) = 3(16 - 2(n+n') + k) + \sum_{s \in S''} b(s)\, b'(s). \tag{6.4}$$

The following table illustrates possible ways of choosing elliptic surfaces r: $Y \to \mathbf{P}^1$ and r': $Y' \to \mathbf{P}^1$ so that the projective variety V^* arising from the above construction has the specified Euler characteristic. As an example consider the fourth line which asserts that all Euler characteristics congruent to 6 mod 12 between -42 and 30 are achievable by the indicated construction. Specifically, r and r' are to be chosen so that S'' consists of a single point, s_1, and $r^{-1}(s_1)$ is a type I_3 fiber while $r'^{-1}(s_1)$ is a type I_5 fiber. There is still some freedom in choosing the other singular fibers of r and r' and hence in choosing the

Table 2. Euler characteristics

$e(V^*)$	Range of h	$^{\#}S''$	$b(s_1)$	$b(s_2)$	$b(s_3)$	$b(s_4)$	Range of even *integers*		h
							n	n'	
$12h$	$-8\leqq h\leqq 4$	0					[0, 12]	[0, 12]	$h=4-\frac{n+n'}{2}$
$2+12h$	$0\leqq h\leqq 3$	1	5	7			[4, 8]	[4, 6]	$h=7-\frac{n+n'}{2}$
$4+12h$	$-6\leqq h\leqq 4$	1	2	2			[0, 10]	[0, 10]	$h=4-\frac{n+n'}{2}$
$6+12h$	$-4\leqq h\leqq 2$	1	3	5			[2, 10]	[4, 8]	$h=5-\frac{n+n'}{2}$
$8+12h$	$-5\leqq h\leqq 4$	1	2	4			[0, 10]	[0, 8]	$h=4-\frac{n+n'}{2}$
$10+12h$	$-2\leqq h\leqq 2$	2	5	3	2	2	[4, 6]	[2, 8]	$h=5-\frac{n+n'}{2}$

invariants n and n'. In fact n may take any even value from 2 through 10, while n' may be any of the numbers 4, 6, or 8. This is easy to see using §4. For instance, if r corresponds to the modular surface of level 6 in Table 1, then $n=2$. If r' has seven I_1 fibers, which is possible by (4.1), then $n'=8$. In this case $h=0$ and the resulting threefold V^* has Euler characteristic 6. Of course, many other combinations also yield $e(V^*)=6$. The other entries in the table are constructed in an equally straightforward manner using only the methods of §4 to produce the necessary elliptic surfaces and the considerations of §3 and §6 to guarantee that V^* is projective.

The largest value that $e(V^*)$ can take is 100. This occurs when $r=r'$ is the elliptic modular surface of level 8 from Table 1.

(6.5) *Remark.* The threefolds V^* are simply connected. The argument is similar to the proof of [S2, Lemma 1.1] but easier since the general fiber is simply connected. The section on the level of fundamental groups which is necessary for this argument is supplied by the $\mathbf{P}^1$-bundle in V^* which corresponds to the identity section of $\hat{W}$.

7. Rigid Varieties

As is well known, a complex manifold X has no infinitesimal deformations exactly when $H^1(X,\mathscr{T})=0$, where $\mathscr{T}$ denotes the sheaf of germs of holomorphic vector fields. The following proposition indicates when this phenomenon occurs for blow ups of the nodal fiber products which we have been considering.

(7.1) **Proposition.** *Let* $r\colon Y\to \mathbf{P}^1$ *and* $r'\colon Y'\to \mathbf{P}^1$ *denote relatively minimal rational elliptic surfaces with section. Suppose that the fibers of* r *and* r' *above all points* *in S'' are semi-stable. Then the fiber product $Y \times_{\mathbf{P}^1} Y'$ has only ordinary double point singularities. The projective variety, $\tilde{W}$, obtained by blowing up the nodes has no infinitesimal deformations, exactly when all fibers of r and r′ are semi-stable and one is in one of the following four cases, each of which actually occurs:*

(i) *Y and Y' appear in Table 1 and are isogenous. In particular $S=S'=S''$ has 4 elements.*

(ii) *Y and Y' appear in Table 1, each has at least one I_1 fiber, and the map r' has been modified by an automorphism of $\mathbf{P}^1$ so that $^{\#}(S'')=3$. Furthermore the singular fibers of r and r' which do not lie above points of S'' are of type I_1.*

(iii) *Y and Y' are not isogenous and $S=S'=S''$ has 5 elements.*

(iv) *Y appears in Table 1, $S=S''$, $^{\#}(S')=5$, and the singular fiber of r' which does not map to S'' has type I_1.*

Proof. It is convenient to replace $\tilde{W}$ by any small resolution $\hat{W}$ of $Y \times_{\mathbf{P}} Y'$. It is not difficult to show $H^1(\tilde{W}, \mathcal{T}_{\tilde{W}}) \simeq H^1(\hat{W}, \mathcal{T}_{\hat{W}})$. For this we write E for the exceptional curve in $\hat{W}, f\colon \tilde{W} \to \hat{W}$ for the blow up centered at E, and $Q = f^{-1}(E)$. As the normal bundle of each component of E is $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$, it suffices to show

(1) $R^1 f_* \mathcal{T}_{\tilde{W}} \simeq 0$,

(2) There is an exact sequence $0 \to f_* \mathcal{T}_{\tilde{W}} \to \mathcal{T}_{\hat{W}} \to \mathcal{N}_{E/\hat{W}} \to 0$.

The first assertion reduces to an exercise in Grothendieck's Theorem on formal functions [H, III.11] and the cohomology of various invertible sheaves on Q. The exact sequence (2) may be constructed from the standard exact sequence

$$0 \to f^* \Omega_{\hat{W}} \to \Omega_{\tilde{W}} \to \Omega_{\tilde{W}/\hat{W}} \to 0$$

in three steps. First apply $\mathcal{H}om_{\mathcal{O}_{\tilde{W}}}(\ ,\mathcal{O}_{\tilde{W}})$. Then observe that $\Omega_{\tilde{W}/\hat{W}} \simeq \Omega_{Q/E}$, whence $\mathcal{E}xt^1_{\mathcal{O}_{\tilde{W}}}(\Omega_{\tilde{W}/\hat{W}}, \mathcal{O}_{\tilde{W}}) \simeq \mathcal{T}_{Q/E} \otimes \mathcal{N}_{Q/\tilde{W}}$. Now apply f_*, using the exact sequence for the relative tangent bundle of the projective bundle, [F2, Appendix B, 5.8],

$$Q \simeq \operatorname{Proj}(\operatorname{Sym} \mathcal{N}^*_{E/\hat{W}})$$
$$0 \to \mathcal{N}_{Q/\tilde{W}} \to f^* \mathcal{N}_{E/\hat{W}} \to \mathcal{T}_{Q/E} \otimes \mathcal{N}_{Q/\tilde{W}} \to 0,$$

to compute $f_*(\mathcal{T}_{Q/E} \otimes \mathcal{N}_{Q/\tilde{W}}) \simeq \mathcal{N}_{E/\hat{W}}$.

If X is a threefold with trivial canonical bundle, the vanishing of $H^1(X, \mathcal{T})$ is equivalent to $H^2(X, \Omega_X)=0$ by Serre duality. Suppose that the cohomology of X has a Hodge decomposition (e.g. X is a Moishezon manifold [U]) and that $h^1(X, \mathcal{O}) = h^2(X, \mathcal{O}) = 0$. Then we may write

$$e(X) = 2h^{1,1}(X) - 2h^{1,2}(X). \tag{7.2}$$

This follows from $e(X) = 2(\chi(\mathcal{O}_X) - \chi(\Omega_X))$, $\chi(\mathcal{O}_X)=0$ (Riemann-Roch), and Hodge symmetry, etc. Now take $X = \hat{W}$ and let $\tilde{S} = S \cup S' - S''$. For any s in $\mathbf{P}^1$ let $b(s)$ (resp. $b'(s)$) denote the number of irreducible components in the fiber $r^{-1}(s)$ (resp. $r'^{-1}(s)$). It is easy to check that $H^1(\hat{W}, \mathcal{O}) \simeq H^2(\hat{W}, \mathcal{O}) = 0$, so that $h^{1,1}(\hat{W}) = \operatorname{rk} \operatorname{Pic} \hat{W}$ follows from the exponential sequence. We compute $\operatorname{rk} \operatorname{Pic} \hat{W}$ using

(3.2) and the fact that the only relations in Pic $\hat{W}$ among components of fibers arise from the fact that two fibers are linearly equivalent. This assertion, which is well known for surfaces [BPV, III 8.2], may be verified in our situation by pulling back to a general very ample divisor on W. Now (7.2) may be rewritten,

$$2\sum_{s\in S''} b(s)\,b'(s)=2[1+\sum_{s\in S''}(b(s)\,b'(s)-1)$$
$$+\sum_{s\in\tilde{S}}(b(s)\,b'(s)-1)+\mathrm{rk}(\mathrm{Pic}(W_\eta)]-2h^{1,2}(\hat{W})$$

where η is the generic point of $\mathbf{P}^1$ in the sense of schemes. This simplifies to,

$$h^{1,2}(\hat{W})=1-{}^{\#}(S'')+\sum_{s\in\tilde{S}}(b(s)\,b'(s)-1)+\mathrm{rk}\,\mathrm{Pic}\,W_\eta. \tag{7.3}$$

Since $\mathrm{rk}(\mathrm{Pic}(W_\eta))>1$, we must have ${}^{\#}(S'')>2$ when $h^{1,2}(\hat{W})=0$.

Suppose ${}^{\#}(S'')=3$. For (7.3) to hold with $h^{1,2}(\hat{W})=0$ we must have $\mathrm{rk}(\mathrm{Pic}\,Y_\eta)=\mathrm{rk}(\mathrm{Pic}\,Y'_\eta)=1$. Rational elliptic surfaces with only torsion sections have been classified [MP]. In fact the list of those having three or more semi-stable fibers coincides with the entries in Table 1. Take any two such surfaces Y and Y' (not necessarily distinct) each having at least one fiber of type I_1. Then it is possible to find an automorphism $a\colon \mathbf{P}^1\to\mathbf{P}^1$ such that exactly three critical values of r and r' match up (i.e. ${}^{\#}(S'')=3$) and those that do not match up have irreducible fibers. Since $\mathrm{rk}\,\mathrm{Pic}\,W_\eta=2$, by (7.3) any small resolution will be infinitesimally rigid. In most cases no small projective resolution exists. However $\tilde{W}$ is a rigid projective variety associated to W with Kodaira dimension zero. Using Table 1 it is not difficult to enumerate all varieties of this sort (type (ii)).

A further reformulation of (7.3) is useful to get an upper bound on ${}^{\#}(S)$. Set $d=1$ if Y_η and Y'_η are isogenous, and $d=0$ otherwise. Then

$$\mathrm{rk}\,\mathrm{Pic}\,W_\eta=\mathrm{rk}\,\mathrm{Pic}\,Y_\eta+\mathrm{rk}\,\mathrm{Pic}\,Y'_\eta+d.$$

Which may be rewritten as

$$\mathrm{rk}(\mathrm{Pic}\,W_\eta)=d+18-\sum_{s\in S} b(s)-\sum_{s\in S'}(b'(s'))+{}^{\#}S+{}^{\#}S'.$$

By Euler characteristic considerations, the quantity, $g=24-\sum_{s\in S} b(s)-\sum_{s'\in S'} b'(s')$, is zero if all fibers of r and r' are semi-stable and is positive otherwise. Rewrite (7.3) in the form below and notice that ${}^{\#}S''\leqq{}^{\#}S+{}^{\#}S'-{}^{\#}S''$.

$$h^{1,2}(\hat{W})=1-{}^{\#}(S'')+{}^{\#}(S)+{}^{\#}(S')+d+(g-6)$$
$$+\sum_{s\in\tilde{S}}(b(s)\,b'(s')-1). \tag{7.4}$$

Evidently, ${}^{\#}(S'')<6$ when $h^{1,2}(\hat{W})=0$. If ${}^{\#}(S'')=5$, then $d=g=0$ and $S=S'=S''$, which is case (iii). If ${}^{\#}(S'')=4$ and $d=1$, then $g=0$ and $S=S'=S''$, which is case (i) by [B3]. Besides the self fiber products, the other type (i) examples are the fiber product of level 3 with level 9 and level 4 with level 8. If $^{\#}(S'')=4$ and $d=0$, then either $S=S'=S''$ or we are in case (iv). If the former case were to occur, all fibers would be semi-stable, hence $g=0$. Since $\tilde{S}$ is empty in this case, (7.4) reads $h^{1,2}(\hat{W})=-1$. Hence only the second situation could possibly arise.

A complete enumeration of the numerous examples of types (iii) and (iv) would be a lengthy undertaking. We shall content ourselves with constructing a single example of each sort, thereby proving the non-obvious fact that such examples do occur. The starting point for the examples presented here was suggested by W. McCallum.

For complex numbers z not in $\{0, -1\}$ the Weierstrass model associated to

$$y^2=x(x-1)(x-(t^2-z))$$

is a double cover of $\mathbf{P}^1\times\mathbf{P}^1$ branched along sections $0, 1, \infty$, and (t^2-z). The final section intersects the first two transversely at $\pm z^{1/2}$ resp. $\pm(z+1)^{1/2}$. These give rise to a total of four A_1 singularities in the Weierstrass model and in fact to four I_2 fibers in the corresponding minimal elliptic surface which will be denoted Y_z. The remaining two sections have local intersection multiplicity 2 at infinity which contributes an I_4 fiber to Y_z. All other fibers are smooth. I claim that the generic fiber of Y_z does not have a cyclic subgroup which is rational over $C(t)$ of order $n>2$. The existence of such would give a map from the t-line to the coarse moduli space $X_0(n)$ such that the following diagram involving the j-function associated to Y_z commutes,

$$\begin{array}{ccc} \mathbf{P}^1 & \longrightarrow & X_0(n) \\ & {\scriptstyle j}\searrow \qquad \swarrow{\scriptstyle \text{can.}} & \\ & X_0(1) & \end{array}$$

Now j has four poles of order 2 and one of order 4. Since the canonical map has a pole of order n, n is 2 or 4. The latter case is ruled out using the fact that when $n=4$ the canonical map has degree $[\mathrm{SL}_2(\mathbf{Z})\colon \Gamma_0(4)]=6$. Because deg$\cdot$ $j=12$, j would have to have a pole of order 8 or two of order 4. This means that the only elliptic surfaces which are isogenous to Y_z are obtained by modding out by the action of a section of order two. As usual, we shall take the section at infinity to be the identity section. The resulting involution on Y_z is biregular with fixed points contained in the singular points of the fibers. It is easy to check that if one takes the quotient by the action of either of the sections 0 or 1 and desingularizes, then the resulting minimal elliptic surface has two I_1 fibers, two I_4 fibers, and one I_2 fiber. The quotient by the other two torsion subgroup gives rise to an elliptic surface with four I_1 fibers and one I_8 fiber, which shall be denoted Y_z'.

To produce a threefold of type (iv), we recall that the elliptic modular surface of full level 3 is given by the famous pencil [B3]

$$x_0^3+x_1^3+x_2^3-3t x_0 x_1 x_2=0,$$

and thus has bad fibers over infinity and the cube roots of unity. Write c for a primitive cube root of 1. The automorphism of $\mathbf{P}^1$, $t \to (t+(c^2/2))/(1+(c^2/2))$, takes these four points to $\{\infty, 1, -1, 3c^2/(2+c^2)\}$. Now the I_8 fiber of Y_z' lies over ∞. By applying the automorphism $t \to t/(z)^{1/2}$ to $\mathbf{P}^1$, we arrange that the I_1 fibers lie over $\{1, -1, (1+z^{-1})^{1/2}, -(1+z^{-1})^{1/2}\}$. Take $z = (9c^4(2+c^2)^{-2}-1)^{-1}$, then the fiber product of these two surfaces is of type (iv).

To produce an example of type (iii), consider the involution κ of $\mathbf{P}^1$ which fixes 1 and interchanges ∞ and -1, namely $t \to (t-3)/(-t-1)$. If $p=(-3)^{1/2}$, then $\kappa(p) = -p$. When $z=-1/4$, $1+z^{-1} = -3$, so we may consider $Y_{-1/4}$ to have bad reduction at $\{\infty, 1, -1, (-3)^{1/2}, -(-3)^{1/2}\}$. Write $Y_{-1/4}^{\kappa}$ for the base change of $Y_{-1/4}$ with respect to κ. Now $Y_{-1/4}$ and $Y_{-1/4}^{\kappa}$ have the same places of bad reduction, yet are not isomorphic since one has an I_2 fiber at infinity and the other has an I_4 fiber. Furthermore we have listed the fiber types for the minimal elliptic surfaces isogenous to $Y_{-1/4}$. From this we infer that $Y_{-1/4}$ and $Y_{-1/4}^{\kappa}$ are not isogenous, so we are in case (iii). By (3.1(i)) $Y_{-1/4} \times_{\mathbf{P}^1} Y_{-1/4}^{\kappa}$ has a small projective resolution, so we get a rigid projective 3-fold with $K=0$. However, if $Y_{-1/4}$ were replaced by $Y'_{-1/4}$, no small resolution would be projective.

(7.5) *Remark.* It may be shown that the list of examples of isomorphism classes of rigid threefolds constructed in (7.1) is finite.

The existence of the varieties constructure in this section raise interesting arithmetic questions. The first of these is to describe the Galois representation on $H^3(\tilde{W}_{\bar{Q}}, Q_l)$ and then to find correspondences between these varieties and modular varieties when the Tate conjecture implies such a relationship. However these issues will not be dealt with here.

8. Some Problems Concerning Moduli and the Period Map

In this section we raise two questions concerning moduli and the period map for simply connected, projective 3-folds with trivial dualizing sheaf and at worst ordinary double point singularities. If X is such a 3-fold write $\hat{X}$ for some choice of small resolution and $\tilde{X}$ for the blow up of X along the nodes. For any variety V write $\mathcal{T}_V = \operatorname{Hom}(\Omega_V^1, \mathcal{O}_V)$. From Hodge theory, the proof of (7.1) and [Fr, Lemma 3.1] we deduce easily

$$(8.1) \qquad H^1(\tilde{X}, \Omega_{\tilde{X}}^2) \simeq H^1(\hat{X}, \Omega_{\hat{X}}^2) \simeq H^1(\hat{X}, \mathcal{T}_{\hat{X}}) \simeq H^1(\tilde{X}, \mathcal{T}_{\tilde{X}}) \simeq H^1(X, \mathcal{T}_X).$$

Let $\mathcal{H}$ be an irreducible, open subvariety in the Hilbert scheme of $\mathbf{P}^N$ such that $\mathcal{H}$ parametrizes 3-folds with the properties listed above. If $X_h \subset \mathbf{P}^N$ corresponds to a general point $h \in \mathcal{H}$ and e is an integer satisfying $0 \leqq e \leqq \dim H^1(X_h, \mathcal{T}_{X_h})$ it is interesting to ask if there is an e-dimensional subvariety $\mathcal{H}_e \subset \mathcal{H}$ with the property that for a *general* closed point $p \in \mathcal{H}_e$ the Kodaira-Spencer map

$$(8.2) \qquad T_p \mathcal{H}_e \to H^1(X_p, \mathcal{T}_{X_p})$$

is an isomorphism. Note that (8.2) is defined since deformations of a *general* member of the family will be locally trivial. To get some feeling for our first question, observe that in the special case $e=0$, this reduces to asking for a point p in $\mathscr{H}$ with the property that $\tilde{X}_p$ is infinitesimally rigid. If $e=1$, we are asking for a one parameter family of 3-folds, $\tilde{X}$, with Hodge numbers $h^{3,0}(\tilde{X})=h^{2,1}(\tilde{X})=1$. I am not aware that interesting variations of Hodge structure with $h^{3,0}>0$ and $h^{2,1}$ small have previously been realized geometrically.

If for $\mathscr{H}$ we take the open subvariety of $\mathbf{P}H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(5))$ parametrizing quintic hypersurfaces with at worst ordinary double points, the question appears to be quite difficult. Indeed for many integers $e, 0 \leqq e \leqq 101 = h^1(X_h, \mathscr{T}_{X_h})$, I do not know how to construct even a single nodal quintic X with $e=h^1(X, \mathscr{T}_X) = h^{2,1}(\tilde{X})$, let alone an e-dimensional family. To show that there is some hope that the answer could be yes for all e, we sketch a means of constructing an $\mathscr{H}_e$ when $\mathscr{H}$ parametrizes fiber products of relatively minimal, semi-stable, rational elliptic surfaces with section.

For the general fiber product, W, ${}^{\#}S = {}^{\#}S' = 12$, $g=0$, $S''=\phi$, $d=0$, all fibers are irreducible so (7.4) and (8.1) yield $\dim H^1(W, \mathscr{T}_W)=19$. Now fix an integer e satisfying $2 \leqq e \leqq 18$ and write $e=l_1+l_2+2$ for integers $0 \leqq l_i \leqq 8$. Write $\bar{\mathscr{P}}_l$ for the open subvariety of $(\mathbf{P}^1)^l$ parametrizing l-tuples of distinct points on $\mathbf{P}^1$ with support disjoint from $\{0, 1, \infty\}$. Let $\beta \subset \bar{\mathscr{P}}_l \times \mathbf{P}^1$ denote the resulting universal family of degree l divisors. Using techniques described in [F, §1] or [CH, §1] and the proof of (4.1) it is possible to construct a dominant étale map $\phi: \mathscr{P}_l \to \bar{\mathscr{P}}_l$ and a morphism $J: \mathscr{P}_l \times \mathbf{P}^1 \to \mathscr{P}_l \times \mathbf{P}^1$ branched over $\mathscr{P}_l \times \{0, 1, \infty\}$ and $(\phi \times \mathrm{id})^* \beta$ as in 4.2 with invariants $j=l+4$ and $k=1$. We may also arrange that $\mathscr{P}_l$ is irreducible, that $J(\mathscr{P}_l \times \{0, 1, \infty\}) \subset \mathscr{P}_l \times \infty$ and that the ramification index of J along $\mathscr{P}_l \times \{\infty\}$ is $l+1$.

In addition we may arrange that $\mathscr{P}_l \times \mathbf{P}^1$ has a double cover with set theoretic branch locus $J^{-1}(\mathscr{P}_l \times \{0, 1, \infty\})$ when l is even and $J^{-1}(\mathscr{P}_l \times \{0, 1, \infty\}) - \mathscr{P}_l \times \{\infty\}$ when l is odd. Once more replacing $\mathscr{P}_l$ by a dense Zariski open it is possible to construct a family of relatively minimal, rational elliptic surfaces

$$\mathscr{Y}_l \xrightarrow{\pi_l} \mathscr{P}_l \times \mathbf{P}^1 \xrightarrow{\mathrm{pr}_1} \mathscr{P}_l$$

where π_l has a section, there are $l+3$ type I_1 singular fibers, one I_{9-l} singular fiber over $\mathscr{P}_l \times \{\infty\}$ and the modular invariant is the given function J. Indeed following [L, §1] begin with the constant elliptic surface over $\mathscr{P}_l \times \mathbf{P}^1$ whose Weierstrass model in homogeneous coordinates $(t:s)$ on $\mathbf{P}^1$ is given by

$$y^2 z = x^3 + 3t(s-t)s^2 x z^2 + 2t(s-t)^2 s^3 z^3 \tag{8.3}$$

and whose modular invariant j: $\mathbf{P}^1 \to \mathbf{P}^1$ is easily computed to be the identity, $(j=t/s)$. Now pull back this Weierstrass model via J and twist by the double cover just mentioned. The singular fibers of the relatively minimal model over the generic point of $\mathscr{P}_l$ are determined by the local behaviour on $\mathbf{P}^1$ of the coefficients in the Weierstrass equation. By [MP, Table 1.1] they are of the desired type.

Given a closed point $p \in \mathscr{P}_l$ write J_p: $\mathbf{P}^1 \to \mathbf{P}^1$ for the modular invariant of $(pr_1 \circ \pi_l)^{-1}(p)$. Due to branch locus considerations there are at most finitely many other closed points p' such that J_p and $J_{p'}$ differ by automorphisms of $\mathbf{P}^1$. To see that this implies that $(pr_1 \circ \pi_l)^{-1}(p)$ is isomorphic to at most finitely many other fibers in the family recall that a rational surface has at most one minimal elliptic pencil with section. Indeed the elliptic pencil determines the canonical class [BPV, Chap. V, Cor. 12.3]. From this one can deduce that the Kodaira-Spencer map [BPV, p. 31]

$$T_p \mathscr{P}_l \hookrightarrow H^1((\mathrm{pr}_1 \circ \pi_l)^{-1}(p), \mathscr{T}) \tag{8.4}$$

is injective for a general point p.

A family of 3-folds parametrized by $\mathscr{P}_{l_1} \times \mathscr{P}_{l_2} \times \mathrm{Aut}(\mathbf{P}^1, \{\infty\})$ is given by the hypersurface, $\mathscr{W}$, in $\mathscr{Y}_{l_1} \times \mathscr{Y}_{l_2} \times \mathrm{Aut}(\mathbf{P}^1, \{\infty\})$ obtained by pulling back the tautological family of graphs $\Gamma \subset \mathbf{P}^1 \times \mathbf{P}^1 \times \mathrm{Aut}(\mathbf{P}^1, \{\infty\})$ via the map

$$(pr_2 \circ \pi_{l_1}, pr_2 \times \pi_{l_2}, \mathrm{id}) : \mathscr{Y}_{l_1} \times \mathscr{Y}_{l_2} \times \mathrm{Aut}(\mathbf{P}^1, \{\infty\}) \to \mathbf{P}^1 \times \mathbf{P}^1 \times \mathrm{Aut}(\mathbf{P}^1, \{\infty\}. \tag{8.5}$$

The variety $W = \mathscr{W}_{(p_1, p_2, \gamma)}$ obtained by intersecting $\mathscr{W}$ with the fiber of

$$(pr_1 \circ \pi_{l_1}, pr_1 \circ \pi_{l_2}, \mathrm{id}) : \mathscr{Y}_{l_1} \times \mathscr{Y}_{l_2} \times \mathrm{Aut}(\mathbf{P}^1, \{\infty\}) \to \mathscr{P}_{l_1} \times \mathscr{P}_{l_2} \times \mathrm{Aut}(\mathbf{P}^1, \{\infty\})$$

over the point (p_1, p_2, γ) is a fiber product of the elliptic surfaces $Y_1 = (pr_1 \circ \pi_{l_1})^{-1}(p_1)$ and $Y_2 = (pr_1 \circ \pi_{l_2})^{-1}(p_2)$. For a general choice of (p_1, p_2, γ) W will have $l_1 \cdot l_2$ nodes in the fiber over $\infty \in \mathbf{P}^1$ and no other singularities. To show that the Kodaira-Spencer map for a general point (p_1, p_2, γ),

$$T_{p_1} \mathscr{P}_{l_1} \times T_{p_2} \mathscr{P}_{l_2} \times T_\gamma \,\mathrm{Aut}(\mathbf{P}^1, \{\infty\}) \to H^1(\mathscr{W}_{(p_1, p_2, \gamma)}, \mathscr{T}) \tag{8.6}$$

is an isomorphism consider the exact sequence

$$0 \to \mathscr{T}_W \to \mathscr{T}_{Y_1 \times Y_2 | W} \to \mathscr{I}_{W_{\mathrm{sing}}} \mathscr{N}_{W/Y_1 \times Y_2} \to 0. \tag{8.7}$$

One computes

$$H^P(W, \mathscr{T}_{Y_1 \times Y_2 | W}) \simeq \bigoplus_{i \in \{1,2\}} H^P(W, pr_i^* \mathscr{T}_{Y_i}) \simeq \bigoplus_{i \in \{1,2\}} H^P(Y_i, \mathscr{T}_{Y_i})$$

for $P \in \{0, 1\}$. When $P = 0$, this vector space is zero. Since for a general point (p_1, p_2, γ) deformations are locally trivial we get a well defined injective map $T_\gamma \mathrm{Aut}(\mathbf{P}^1, \{\infty\}) \to H^0(W, \mathscr{I}_{W_{\mathrm{sing}}} \mathscr{N}_{W/Y_1 \times Y_2}) \to \ker(H^1(W, \mathscr{T}_W) \to H^1(W, \mathscr{T}_{Y_1 \times Y_2 | W}))$. Again using the local triviality of the deformations, the product of Kodaira-Spencer maps (8.4) factors

$$\begin{array}{ccc} T_{p_1} \mathscr{P}_{l_1} \oplus T_{p_2} \mathscr{P}_{l_2} & \hookrightarrow & H^1(Y_1, \mathscr{T}_{Y_1}) \oplus H^1(Y_2, \mathscr{T}_{Y_2}) \\ \downarrow{\scriptstyle\alpha} & & \| \wr \\ H^1(W, \mathscr{T}_W) & \to & H^1(W, \mathscr{T}_{Y_1 \times Y_2 | W}), \end{array}$$

so α is injective. Since $^{\#}(S'') = 1$, $^{\#}(S) = l_1 + 4$, $^{\#}(S') = l_2 + 4$, $d = 0$, $g = 0$, and all fibers over $\tilde{S}$ are irreducible, (7.4) and (8.1) yield $\dim H^1(W, \mathscr{T}_W) = l_1 + l_2 + 2$.

This shows that (8.6) is an isomorphism. Thus it is possible to construct an $\mathscr{H}_e$ with the desired properties when $2 \leqq e \leqq 18$ as the image of $\mathscr{P}_{l_1} \times \mathscr{P}_{l_2} \times \mathrm{Aut}(\mathbf{P}^1, \infty)$ in an appropriate Hilbert scheme. Minor modifications of the above procedure involving replacing $\mathrm{Aut}(\mathbf{P}^1, \{\infty\})$ by $\mathrm{Aut}(\mathbf{P}^1)$ (respectively $\mathrm{Aut}(\mathbf{P}^1, \{0, \infty\})$) allow one to treat the cases $e=19$ (respectively $e=1$). The case $e=0$ was treated in §7.

Our second question relates to the image of the period map for the middle dimensional cohomology of non-singular Moishezon 3-folds with trivial canonical bundle. Let $2q$ denote the rank of the third cohomology. Recall the definition of the relevant classifying space for marked polarized Hodge structures, $D(q, \langle , \rangle)$. Begin with a free rank $2q$ $\mathbf{Z}$-module $H_{\mathbf{Z}}$ and a non-degenerate, alternating, integer valued form $\langle , \rangle$ such that $(H_{\mathbf{Z}}, \langle , \rangle)$ is isomorphic to the middle cohomology of our 3-fold modulo torsion, equipped with the usual intersection pairing. Then $D(q, \langle , \rangle)$ is the parameter space for flags $F^3 \subset F^2 \subset H_{\mathbf{C}} := H_{\mathbf{Z}} \otimes \mathbf{C}$ satisfying

i) $\dim F^3 = 1$

ii) $F^2 = (F^2)^{\perp}$

iii) $i\langle , ^{-}\rangle$ is positive definite on F^3

iv) $-i\langle , ^{-}\rangle$ is positive definite on $H^{2,1} := F^2 \cap (\bar{F}^3)^{\perp}$

Also write $\mathscr{S}(q, \langle , \rangle)$ for the Siegel space parametrizing subspaces $G \subset H_{\mathbf{C}}$ satisfying

v) $G = G^{\perp}$

vi) $i<, ^{-}>$ is positive definite on G.

The real analytic map $D(q, \langle , \rangle) \to \mathscr{S}(q, <, >)$ which associates to the filtration $F^{\bullet}$ the subspace $G = F^3 + ((F^3)^{\perp} \cap \bar{F}^2)$ is surjective with fiber $\mathbf{P}_{\mathbf{C}}^{q-1}$.

Now suppose that a positive rank, saturated subgroup $H'_{\mathbf{Z}} \subset H_{\mathbf{Z}}$ has been fixed on which $\langle , \rangle$ is non-degenerate. Write $H''_{\mathbf{Z}}$ for $(H'_{\mathbf{Z}})^{\perp}$ and $\langle , \rangle'$ (respectively $\langle , \rangle''$) for the restriction of $\langle , \rangle$ to $H'_{\mathbf{Z}}$ (respectively $H''_{\mathbf{Z}}$). Define a map

$$\psi_{H'_{\mathbf{Z}}}\colon \mathscr{S}(q', \langle , \rangle') \times D(q-q', \langle , \rangle'') \to D(q, \langle , \rangle).$$

by $\psi_{H'_{\mathbf{Z}}}(G, (F^{\bullet})'') \to F^3 = (F^3)''$ and $F^2 = \bar{G} + (F^2)''$.

Define the parameter space for reducible Hodge structures, $R(q, \langle , \rangle) = \bigcup \operatorname{im} \psi_{H'_{\mathbf{Z}}}$, where the union is taken over all saturated, positive rank subgroups on which $\langle , \rangle$ is non-degenerate. Given a complex analytic family of Moishezon 3-folds with $K \equiv 0$ parametrized by a simply connected base, Δ, it is interesting to ask how the image of the period map, $\wp\colon \Delta \to D(q, \langle , \rangle)$ meets $R(q, \langle , \rangle)$. Intersection points correspond to 3-folds whose intermediate Jacobian contains a subtorus with associated Hodge type (2, 1), (1, 2). It is precisely for such 3-folds that the generalized Hodge conjecture concerning the image of the Abel-Jacobi homomorphism on 1-cycles algebraically equivalent to zero is interesting. Now if Δ is a Kuranishi family $\dim \Delta \leqq \dim H^1(X, \mathscr{T}_X) = q-1$ (8.1). On the other hand $\dim \mathscr{S}(q, \langle , \rangle) = q(q+1)/2$, whence $\dim D(q, <, >) = q(q+1)/2 + (q-1)$ and

$$\begin{aligned}\dim(\operatorname{im} \psi_{H'_{\mathbf{Z}}}) &= q'(q'+1)/2 + (q-q')(q-q'+1)/2 + (q-q'-1)\\ &\leqq 1 + (q-1)\, q/2 + q - 2 = q(q+1)/2 - 1.\end{aligned}$$

(The inequality arises because the first expression is maximized for $1 \leqq q' \leqq q-1$ when $q'=1$.) In other words the codimension of every component of $R(q, \langle , \rangle)$ is at least q. Thus a naive dimension count would lead one to suspect $\wp(\Delta) \cap R(q, \langle , \rangle)$ is empty. However, large families of reducible Hodge structures may be produced by considering non-singular quintic hypersurfaces in $\mathbf{P}^4$ which are invariant under a suitable finite group action. Unfortunately it is difficult to have much insight into this situation because the dimensions of the parameter spaces are so large ($q=102$). Perhaps it would be worthwhile to study in detail a situation where q is as small as possible, yet the threefolds in question are not rigid (i.e. $q=2$). As alluded to above ($\mathscr{H}_e$ with $e=1$), an interesting test case may be constructed as follows: Let $\pi\colon Y\to\mathbf{P}^1$ denote the rational elliptic modular surface of level 9. Choose an inhomogeneous coordinate on $\mathbf{P}^1$ so that $\pi^{-1}(\infty)$ is the I_9 fiber and $\pi^{-1}(0)$ is an I_1 fiber. Given two copies of Y identify the base curves via $\gamma\in\mathrm{Aut}(\mathbf{P}^1,\{0,\infty\})$ and take a small resolution $\hat{W}_\gamma$ of the fiber product. By varying γ we get a one parameter family of Moishezon 3-folds with trivial canonical bundle, non-constant period map, and invariant $q=2$. It would be interesting to describe the intersection of the image of the period map for this family with the set $R(2,\langle,\rangle)$ of reducible Hodge structures. Is it empty? Is it finite? Or does it give rise to a dense subset of the parameter space $\mathrm{Aut}(\mathbf{P}^1,\{0,\infty\})$?

9. Fixed-point Free Group Actions

In this section we describe how the theory of principal homogeneous spaces for elliptic curves over functions fields naturally leads to fixed-point free group actions on certain fiber products of relatively minimal, rational, elliptic surfaces with section. We shall make use of the following

Theorem 9.1. *Let $q_0\colon T_0\to\mathbf{P}^1$ be a relatively minimal rational elliptic surface with section. Let m_i, $(1\leqq i\leqq n)$ be a collection of positive integers and $P_i\in\mathbf{P}^1$, $(1\leqq i\leqq n)$ a collection of points over which q_0 is smooth. Then there is a projective elliptic surface $q\colon T\to\mathbf{P}^1$ which is a principal homogeneous space for q_0 and whose multiple fibers are exactly $q^{-1}(P_i)$ with multiplicity m_i.*

Proof. See [Sa, Thm. 3 and 4] or [La] or [Ogg, Thm. 2(b)].

Fix an inhomogeneous parameter t on $\mathbf{P}^1$. We shall make use of four classes of relatively minimal rational elliptic surfaces with section, $q_0\colon T_0\to\mathbf{P}^1$, such that q_0 is smooth over ∞ and the fiber over 0 is described in the following table. Examples of types 3, 4 and 6 may be found in [MP, §4]. It is not difficult to construct examples of type 2, say as double covers of $\mathbf{P}^1\times\mathbf{P}^1$ whose branch

Table 3

Type	Kodaira type of $q_0^{-1}(0)$	Order of local monodromy at 0
2	I_0^*	2
3	IV^*	3
4	III^*	4
6	II^*	6

locus consists of one member of each ruling and the graph of a general degree three map from $\mathbf{P}^1$ to itself. Given an elliptic surface q_0: $T_0 \to \mathbf{P}^1$ of type m above, we let q: $T \to \mathbf{P}^1$ denote a relatively minimal, projective, elliptic surface which arises from a principal homogeneous space associated to q_0 and has exactly one multiple fiber – namely $q^{-1}(\infty)$ which has multiplicity m. Let r: $Y \to \mathbf{P}^1$ (resp. r_0: $Y_0 \to \mathbf{P}^1$) denote the relatively minimal elliptic surface associated to the pullback of q: $T \to \mathbf{P}^1$ (resp. q_0: $T_0 \to \mathbf{P}^1$) via ϕ_m: $\mathbf{P}^1 \to \mathbf{P}^1$, $\phi_m(t) = t^m$. As the local monodromies for r and r_0 around 0 and ∞ are trivial the corresponding fibers are smooth (criterion of Néron, Ogg, and Safarevic) [BPV, V.7 and V.10, Table 6]. The topological Euler characteristics of Y and Y_0 are equal to the sum of the Euler characteristics of the singular fibers, which in each case is $m(12 - e(q_0^{-1}(0))) = 12$. This equality explains why in the table above we choose only those finite monodromy singular fibers which appear with a star in Kodaira's list. Since $K_{Y_0} = K_Y = -\text{Fiber}$ [BVP, V12.1, 12.2] these surfaces are rational by Castelnuovo's criterion. This means that the Tate-Shafarevich group for Y_0 vanishes [La] or [Sa, Thm. 3] whence the principal homogeneous space Y has a section. Thus $Y \simeq Y_0$. The (biregular) action of the cyclic group σ: $\mathbf{Z}/m \hookrightarrow \mathrm{Aut}(Y)$ corresponding to the cover Y/T has fixed points only in the fiber $r^{-1}(0)$.

Now let q_0'': $T_0'' \to \mathbf{P}^1$ be a second elliptic surface of the same type m. Exactly as before we associate to q_0'' an elliptic surface q'': $T'' \to \mathbf{P}^1$ with a multiplicity m fiber at ∞ and a relatively minimal rational elliptic surface with section r'': $Y'' \to \mathbf{P}^1$ which possesses a natural $\mathbf{Z}/m$ action with quotient birational to T''. By

$$q_0'\colon\ T_0' \to \mathbf{P}^1, \quad (\text{resp. } q'\colon T' \to \mathbf{P}^1), \ (\text{resp. } r'\colon Y' \to \mathbf{P}^1)$$

we denote the pullback of q_0'', (resp. q''), (resp. r'') via the automorphism of $\mathbf{P}^1$, $t \to t^{-1}$. Now we may choose an isomorphism σ': $\mathbf{Z}/m \xrightarrow{\sim} \mathrm{Gal}(Y'/T')$ so that $\mathbf{Z}/m$ acts on $W = Y \times_{\mathbf{P}^1} Y'$ via $\sigma \times \sigma'$: $\mathbf{Z}/m \to \mathrm{Aut}(Y \times Y')$. As the fixed points on Y lie in $r^{-1}(0)$ and those on Y' in $r'^{-1}(\infty)$, the action on W is fixed point free. The quotient, U, is birational to $T \times_{\mathbf{P}^1} T'$. It has fibers of multiplicity m over 0 and ∞, whose reductions are hyperelliptic surfaces. If q_0 and q_0'' are semi-stable away from 0 then W will have a small resolution, $\hat{W}$, to which the fixed point free group action lifts. By Riemann-Roch the holomorphic Euler characteristic of the quotient, $\chi(\mathcal{O}_{\hat{U}})$, is zero. This implies $h^0(\omega_{\hat{U}}) = 1$, whence $\omega_{\hat{U}} \simeq \mathcal{O}_{\hat{U}}$.

From the point of view of superstring theory it would be desirable to find a projective $\hat{U}$ with topological Euler characteristic ± 6. Under the semi-stability hypotheses on q_0 and q_0'' made above, $e(\hat{U}) = 2\,^{\#}(\text{nodes of } U)$. If the Kodaira types of $q^{-1}(s)$ and $(q')^{-1}(s)$ are I_b and $I_{b'}$, then $e(q^{-1}(s) \times (q')^{-1}(s)) = 2bb'$. The requirement $0 < e(\hat{U}) \leqq 6$ forces b or $b' = 1$. We shall argue that $\hat{U}$ cannot be projective in this situation. Indeed if $\hat{U}$ were projective, there would be a divisor D whose intersection with each exceptional $\mathbf{P}^1$ is non-zero. The pullback, D_W, of D to the covering space $\hat{W}$ is invariant with respect to $\mathrm{Gal}(\hat{W}/\hat{U})$. Furthermore by the arguments of §3, D_W gives rise via the standard composition

$$\mathrm{Pic}(\hat{W}) \longrightarrow \mathrm{Pic}(W_\eta) \xrightarrow{\ \alpha\ } \mathrm{Hom}(r^{-1}(\eta),\ \ r'^{-1}(\eta))$$

[F2, 16.1.2] to a (non-zero) isogeny f. Now f gives rise to an isogeny F of Néron models which we may restrict to get an isogeny F_0 between the (smooth) fibers $r^{-1}(0)$ and $r'^{-1}(0)$. Given a function g, it will be convenient to denote its graph by $\langle g\rangle$. Since σ' acts trivially on $H^0((r')^{-1}(0),\Omega)$ but σ acts by a non-trivial character on $H^0(r^{-1}(0),\Omega)$, the image of $\sum_{0\leqq i\leqq m-1}\langle\sigma'(i)\circ F_0\circ\sigma(-i)\rangle$ $\in\operatorname{Pic}(r^{-1}(0)\times(r')^{-1}(0))$ in $\operatorname{Hom}(r^{-1}(0),(r')^{-1}(0))$ is zero. From this we may conclude that $\sum_{0\leqq i\leqq m-1}\langle\sigma'(i)\circ f\circ\sigma(-i)\rangle$ has zero image in $\operatorname{Hom}(r^{-1}(\eta),(r')^{-1}(\eta))$. The point here is that the image is detected in the étale cohomology over the algebraic closure of the base field. Now the cycle class map from Pic to cohomology is compatible with specialization [F2, 20.3.5]. By the proper base change theorem specialization on cohomology is an isomorphism. It is now clear that when we sum the conjugates of D_W with respect to $\operatorname{Gal}(\hat{W}/\hat{U})$ and apply α we get zero. This contradiction shows that $\hat{U}$ is not projective if b or $b'=1$.

One may be tempted to apply the methods of §6 to $\hat{U}$ in an attempt to produce more threefolds with trivial canonical sheaf. However in cases 3, 4, and 6 inversion, i_Y, on Y does not descend to T. The point here is that although i_Y does commute with a generator $\gamma\in\operatorname{Gal}(Y/T_0)$, a generator of $\operatorname{Gal}(Y/T)$ is of the form $\tau\circ\gamma^j$ where τ is a translation which does not commute with i_Y. Consequently i_Y does nor normalize $\operatorname{Gal}(Y/T)$. Thus there is no notion of inversion on U in cases 3, 4, or 6. In case 2 one verifies that if inversion on W descends to U then it has some isolated fixed points which live in the multiple fibers. This means that the desingularization of the quotient by this $\mathbf{Z}/2$ action will contain exceptional $\mathbf{P}^2$'s which contibute to the canonical divisor, whence $K\neq 0$.

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