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Convolution Conditions for Convexity, Starlikeness and Spiral-Likeness*

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1. Introduction

The convolution or Hadamard product of two power series $f(z)=\sum_{n=0}^{\infty} a_n z^n$ and $g(z)=\sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f*g)(z)=\sum_{n=0}^{\infty} a_n b_n z^n$. The Pólya-Schoenberg conjecture [3] generated a great deal of intrinsic interest in properties of convolutions. The proof of the conjecture [7] increased rather than decreased the work done on Hadamard products and led to several generalizations ([6, 10]) of the conjecture. In addition, a wide range of applications to extremal problems in univalent functions ([5, 8]) has created extrinsic interest in convolutions.

In this note we give characterizations for convex, starlike, and spiral-like functions in terms of convolutions. A function f, analytic in $|z|<R\leq 1$ and normalized by $f(0)=f'(0)-1=0$, is said to be *convex of order* $\alpha(0\leqq\alpha<1)$ if $\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha\,(|z|<R)$, is *starlike of order* α if $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\,(|z|<R)$ and is *spiral-like* if for some real λ, $|\lambda|<\pi/2$, we have $\operatorname{Re}\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\}>0\ (|z|<R)$. For each of these classes, $\mathscr{F}$, we find a function g, depending on $\mathscr{F}$, such that $\frac{1}{z}(f*g)\neq 0$ is both necessary and sufficient for f to be in $\mathscr{F}$. More generally, given a specific function $\varphi(z,f,f',\dots,f^{(n)})$ with $\operatorname{Re}\varphi(0)>0$, our method shows how one can often construct a function g_φ such that $\operatorname{Re}\varphi(z)>0$ for $|z|<R$ if and only if $\frac{1}{z}(f*g_\varphi)\neq 0$ for $|z|<R$.

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The conditions obtained are used to determine the radius of convexity of functions whose coefficients form a totally monotone sequence.

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2. Convolution Conditions

In the sequel, f will be analytic in $|z|<1$ and normalized by $f(0)=f'(0)-1=0$. In addition, α will satisfy $0\leqq\alpha<1$.

Theorem 1. *The function f is convex of order α in $|z|<R\leqq 1$ if and only if*

$$\frac{1}{z}\left[f * \frac{z+\frac{x+\alpha}{1-\alpha}z^2}{(1-z)^3}\right] \neq 0 \qquad (|z|<R,\ |x|=1).$$

Proof. The function f is convex of order α in $|z|<R$ if and only if

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)}\right\}>\alpha \qquad (|z|<R). \tag{1}$$

Since $\frac{(zf')'}{f'}=1$ at $z=0$, (1) is equivalent to

$$\frac{\frac{(zf')'}{f'}-\alpha}{1-\alpha} \neq \frac{x-1}{x+1} \qquad (|z|<R,\ |x|=1,\ x\neq -1)$$

which simplifies to

$$(1+x)(zf')'+(1-2\alpha-x)f'\neq 0. \tag{2}$$

Setting $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$, we have

$$(zf')'=1+\sum_{n=2}^{\infty} n^2 a_n z^{n-1}=f' * \left(\sum_{n=1}^{\infty} n z^{n-1}\right)=f' * \frac{1}{(1-z)^2},$$

so that the left hand side of (2) may be expressed as

$$f' * \left(\sum_{n=1}^{\infty} [1-2\alpha-x+(1+x)n]\, z^{n-1}\right)=f' * \left(\frac{1-2\alpha-x}{1-z}+\frac{1+x}{(1-z)^2}\right)$$
$$=f' * \left(\frac{2-2\alpha+(x+2\alpha-1)\,z}{(1-z)^2}\right).$$

Thus (2) is equivalent to

$$\frac{1}{z}\left[zf' * \frac{z+\dfrac{x+2\alpha-1}{2-2\alpha}z^2}{(1-z)^2}\right] \neq 0. \tag{3}$$

Since $zf' * g = f * zg'$, we can write (3) as

$$\frac{1}{z}\left[f * \frac{z+\dfrac{x+\alpha}{1-\alpha}z^2}{(1-z)^3}\right] \neq 0 \qquad (|z|<R,\ |x|=1).$$

Remark. The case $x=-1$ in the convolution condition for Theorem 1 as well as the analogous results contained in Theorems 2, 3 and 4 is equivalent to stating $f' \neq 0$ for $|z|<1$, which is a necessary condition for univalence.

Theorem 2. *The function f is starlike of order α in $|z|<R \leqq 1$ if and only if*

$$\frac{1}{z}\left[f * \frac{z+\dfrac{x+2\alpha-1}{2-2\alpha}z^2}{(1-z)^2}\right] \neq 0 \qquad (|z|<R,\ |x|=1).$$

Proof. Since f is starlike of order α if and only if $g(z)=\int_0^z \frac{f(\zeta)}{\zeta}\,d\zeta$ is convex of order α, we have

$$\frac{1}{z}\left[g * \frac{z+\dfrac{x+\alpha}{1-\alpha}z^2}{(1-z)^3}\right] = \frac{1}{z}\left[f * \frac{z+\dfrac{x+2\alpha-1}{2-2\alpha}z^2}{(1-z)^2}\right].$$

Thus the result follows from Theorem 1.

Remark. The special cases of $\alpha=0$ in Theorem 1 and $\alpha=1/2$ in Theorem 2 are contained in [6].

Theorem 3. *For $|z|<R \leqq 1$, λ real with $|\lambda|<\pi/2$ and $|x|=1$, we have*

$$\operatorname{Re}\left\{e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>0$$

if and only if

$$\frac{1}{z}\left[f * \frac{z+\dfrac{2x+1-e^{-2i\lambda}}{1+e^{-2i\lambda}}z^2}{(1-z)^2}\right] \neq 0.$$

Proof. We have $\operatorname{Re}\left\{e^{i\lambda}\left(1+\frac{zf''}{f'}\right)\right\}>0$ in $|z|<R$ if and only if

$$\frac{e^{i\lambda}\frac{(zf')'}{f'}-i\sin\lambda}{\cos\lambda} \neq \frac{x-1}{x+1}, \qquad (|z|<R,\ |x|=1,\ x\neq -1),$$

which simplifies to

$$(1+x)(zf')'+(e^{-2i\lambda}-x)f' \neq 0. \tag{4}$$

Observe that (4) can be obtained by substituting $e^{-2i\lambda}$ for $1-2\alpha$ in (2). The remainder of the argument is the same, with $e^{-2i\lambda}$ replacing $1-2\alpha$ in Theorem 1.

Theorem 4. *For $|z|<R\leqq 1$, λ real with $|\lambda|<\pi/2$, and $|x|=1$, we have*

$$\operatorname{Re}\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\}>0$$

if and only if

$$\frac{1}{z}\left[f * \frac{z+\dfrac{x-e^{-2i\lambda}}{1+e^{-2i\lambda}}z^2}{(1-z)^2}\right] \neq 0.$$

Proof. The result follows from Theorem 3 in the same manner that Theorem 2 followed from Theorem 1.

Remark. Although a function that satisfies the conditions of Theorem 4 for $|z|<1$ must be univalent [9], a function that satisfies the conditions of Theorem 3 need not be [2].

3. An Application

We will need the following result due to Ruscheweyh [4].

Theorem A. *Let $g(z,t)$ be analytic in the disk $|z|<1$ and continuous in the variable t on $[a,b]$. Denote by V functions of the form $f(z)=\int^{b} g(z,t)\,d\mu(t)$, where μ is a probability measure on $[a,b]$, and V^2 the subset of V in which μ is a step function with at most two jumps. If L_1 and L_2 are continuous linear functionals with $0\notin L_2(V^2)$, then to each $f\in V$ there corresponds an $f_0\in V^2$ such that*

$$\frac{L_1(f_0)}{L_2(f_0)}=\frac{L_1(f)}{L_2(f)}.$$

A sequence of real numbers $\{a_n\}$ is said to be *totally monotone* if

$$\Delta^0 a_n = a_n \geqq 0 \qquad (n\geqq 1)$$

and

$$\Delta^k a_n = \Delta^{k-1} a_n - \Delta^{k-1} a_{n+1} \geqq 0 \qquad (n \geqq 1,\, k \geqq 1).$$

Hausdorff [1] showed that a necessary and sufficient condition for the coefficients of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ to be totally monotone is that

$$f(z) = \int_0^1 \frac{z}{1-tz}\, d\mu(t)$$

for some probability measure $\mu(t)$ defined on $[0, 1]$. Such functions are known to be univalent in $|z| < 1$. Wirths [11] found the radius of starlikeness for this class. We have

Theorem 5. *The radius of convexity for functions of the form* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *whose coefficients are totally monotone is* $\sqrt{2}/2$.

Proof. Let $L_1(f) = zf'' + f'$ and $L_2(f) = f'$ in Theorem A. It follows that we need only consider f in the form

$$f(z) = \gamma \frac{z}{1-t_1 z} + (1-\gamma) \frac{z}{1-t_2 z} \qquad (t_1, t_2, \gamma \in [0, 1]).$$

It is an obvious geometrical fact that for A and B nonzero complex numbers, $|\arg A - \arg B| = \pi$ if and only if the line segment connecting A and B passes through the origin. Thus, from Theorem 1 with $\alpha = 0$, we see that proving the result is equivalent to showing that

$$F(t_1, t_2, x, z) = \left| \arg \frac{(1+xt_1 z)(1-t_2 z)^3}{(1+xt_2 z)(1-t_1 z)^3} \right| \neq \pi \qquad \left(|z| < \frac{\sqrt{2}}{2},\ |x| = 1 \right). \tag{5}$$

For each point z in the unit disk, $|\arg(1+tz)|$ is an increasing function of t, $0 \leqq t \leqq 1$. Thus

$$\max_{t_1, t_2 \in [0, 1]} \left| \arg \left(\frac{1+t_1 z}{1+t_2 z} \right) \right| = |\arg(1+z)|,$$

and for $|z| \leqq r$, we have

$$\begin{aligned} F(t_1, t_2, x, z) &\leqq \left| \arg \left(\frac{1+xt_1 z}{1+xt_2 z} \right) \right| + 3 \left| \arg \left(\frac{1-t_2 z}{1-t_1 z} \right) \right| \\ &\leqq \max_{|x|=1,\, |z|=r} |\arg(1+xz)| + 3 \max_{|z|=r} |\arg(1-z)| = 4 \sin^{-1} r. \end{aligned}$$

Note that $F(1, 0, -1, z) = 4 \sin^{-1} r$ when $\arg(1-z) = \sin^{-1} |z|$. Since $4 \sin^{-1} \left(\frac{\sqrt{2}}{2} \right) = \pi$, the result follows.

Corollary. *Functions of the form* $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ *whose coefficients are totally monotone are starlike of order* 1/2 *in the disk* $|z|<\sqrt{3}/2$.

Proof. Applying Theorem 2 with $\alpha=1/2$ instead of Theorem 1 with $\alpha=0$, we find that the 3 in the exponent of (5) can be replaced by a 2. From there it follows that f is starlike of order 1/2 when $3\sin^{-1}|z|<\pi$.

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