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The Gap Phenomena of Yang-Mills Fields over the Complete Manifold

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1. Introduction

It is a well known problem whether a sourceless $SU(2)$ gauge field over S^4 is self-dual (or anti-self-dual). So it is very interesting to study the relation between the sourceless fields and the self-dual (or anti-self-dual) fields.

In [5], we proved the following

Theorem A. *Let M be a four-dimensional self-dual compact Riemannian manifold with positive scalar curvature R, and assume that there is a sourceless $SU(N)$ gauge field over M. If the norm of the field strength is less than $\frac{R}{12}$, then this field is a self-dual gauge field.*

Bourguignon, Lawson [2] and Min-Oo [4] also discussed the related problem and obtained a better theorem [2]:

Theorem B. *Let M be a four-dimensional self-dual compact Riemannian manifold with positive scalar curvature R, and there is a sourceless $SU(N)$ gauge field over M. If the norm of the anti-self-dual part of the field strength is less than $\frac{R}{12}$, then this field is a self-dual gauge field.*

In this paper we generalize the above theorems to the non-compact case. We have

Theorem. *Let M be a four-dimensional complete self-dual Riemannian manifold, R the scalar curvature of M, f the strength of a Yang-Mills field over M with the gauge group $SU(N)$, and $|f^-|$ the norm of the anti-self-dual part of f. If*

(1) $|f^-|<\frac{R}{12}$,

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(2) *there is a point* $x_0 \in M$ *such that*

$$\int_{B_k} |f^-|^2 \, d\,\mathrm{vol} = o(k^2),$$

where B_k *is a geodesic ball with the center* x_0 *and radius* k, *(for example, if the action of this field is finite), then this Yang-Mills field must be self-dual.*

Corollary. *Let* M *be a four-dimensional conformally flat, non-compact but complete Riemannian manifold with positive scalar curvature* R, *and with a sourceless* $SU(N)$ *gauge field over* M. *If the norm of the field strength of* M *is less than* $\frac{R}{12}$, *and the action of this field is finite, then the field is a vacuum one.*

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2. Notations

In this paper, let M be a four-dimensional complete Riemannian manifold. In a local coordinate system, the metric of M is defined by

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu, \qquad (\mu, \nu = 0, 1, 2, 3), \tag{1}$$

and the *Riemannian curvature tensor* and the *conformal curvature tensor* are $R_{\nu\mu\alpha\beta}$ and

$$\begin{aligned} C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - \tfrac{1}{2}(g_{\mu\alpha} R_{\nu\beta} - g_{\mu\beta} R_{\nu\alpha} + g_{\nu\beta} R_{\mu\alpha} - g_{\mu\alpha} R_{\nu\beta}) \\ -\frac{R}{6}(g_{\mu\beta} g_{\nu\alpha} - g_{\mu\alpha} g_{\nu\beta}). \end{aligned} \tag{2}$$

respectively. $R_{\mu\alpha} = g^{\nu\beta} R_{\mu\nu\alpha\beta}$ and R are the *Ricci curvature tensor* and the *scalar curvature* of M respectively.

To any 2nd order covariant tensor $T_{\mu\nu}$, we can apply the $*$-operator

$$T^*_{\mu\nu} = \tfrac{1}{2} \eta_{\mu\nu}{}^{\alpha\beta} T_{\alpha\beta} \tag{3}$$

where

$$\eta_{\mu\nu\alpha\beta} = \sqrt{\det |g_{\lambda\delta}|}\; \varepsilon_{\mu\nu\alpha\beta}$$

and $\varepsilon_{\mu\nu\alpha\beta}$ is the complete skew-symmetric tensor with $\varepsilon_{0123} = 1$. From this, we can define the following expressions for the pair $(\mu\nu)$ of indices as

$$T^+_{\mu\nu} = \tfrac{1}{2}(T_{\mu\nu} + T^*_{\mu\nu}), \qquad T^-_{\mu\nu} = \tfrac{1}{2}(T_{\mu\nu} - T^*_{\mu\nu}). \tag{4}$$

We call $T^+_{\mu\nu}$ or $T^-_{\mu\nu}$ the *self-dual* or *anti-self-dual* part of $T_{\mu\nu}$ respectively. Similarly, we may define these expressions for the pairs of indices $(\mu\nu)$ or $(\alpha\beta)$ of the conformal curvature tensor $C_{\mu\nu\alpha\beta}$ and obtain

$$C_{\mu\nu\alpha\beta} = C^{+\,+}_{\mu\nu\alpha\beta} + C^{+\,-}_{\mu\nu\alpha\beta} + C^{-\,+}_{\mu\nu\alpha\beta} + C^{-\,-}_{\mu\nu\alpha\beta}.$$

It is evident that $C^{+-}=C^{-+}=0$ (to be brief, we shall suppress the lower indices).

If we have $C^{++}=0$ or $C^{--}=0$, then the Riemannian manifold M is said to be *self-dual* or *anti-self-dual* respectively [1].

Now let us study a gauge field the base manifold of which is a self-dual with $R>0$ whose gauge group is $SU(N)$.

Let X_a be a basis of the Lie algebra $SU(N)'$ and C^a_{bc} the structure constants. We can define the Cartan-Killing inner product in the $SU(N)'$ by

$$\langle A, B\rangle = -\operatorname{Tr}(AB) \tag{5}$$

where A, $B \in SU(N)'$, and let

$$G_{ab} = \langle X_a, X_b\rangle. \tag{6}$$

Then we define the *norm* $\|f\|$, $\|f^+\|$, $\|f^-\|$ of the field strength $f = \frac{1}{2} f_{\mu\nu}\, dx^\mu \wedge dx^\nu$ and its self-dual parts and anti-self-dual parts by

$$\begin{aligned} \|f\|^2 &= g^{\mu\alpha} g^{\nu\beta} \langle f_{\mu\nu}, f_{\alpha\beta}\rangle = g^{\mu\alpha} g^{\nu\beta} f^a_{\mu\nu} f^b_{\alpha\beta} G_{ab}, \\ \|f^+\|^2 &= g^{\mu\alpha} g^{\nu\beta} \langle f^+_{\mu\nu}, f^+_{\alpha\beta}\rangle, \\ v = \|f^-\|^2 &= g^{\mu\alpha} g^{\nu\beta} \langle f^-_{\mu\nu}, f^-_{\alpha\beta}\rangle \end{aligned} \tag{7}$$

respectively. The field is said to be *self-dual* if $f^- = 0$, and *anti-self-dual* if $f^+ = 0$.

3. Proof of the Theorem

The outline of the proof is that we first compute the Laplacian of the function $\sqrt{v}$. With a complicated calculation we can prove $\Delta(\sqrt{v}) \geqq 0$. We know a theorem of Yau [6]:

If M is a non-compact Riemannian manifold, u is a non-negative subharmonic function on M, and $\int_M u^2\, d\,\mathrm{vol} < \infty$, then u must be a constant function.

We would like to use this theorem to prove $v = \text{const.}$ and then prove $v=0$, i.e., the field is self-dual. However it is not a priori evident that $\sqrt{v}$ is of class C^1 or C^2 while Yau's theorem requires the smoothness of this function. Therefore we must use the standard trick of approximating $\sqrt{v}$ with $\sqrt{v+\varepsilon}$, for some $\varepsilon > 0$.

3.1. Now we compute the Laplacian of $\sqrt{v+\varepsilon}$. By the Bianchi identities, sourceless condition and Ricci identities, we obtain

$$\begin{aligned} \Delta \sqrt{v+\varepsilon} = g^{\alpha\beta} (\sqrt{v+\varepsilon})_{\|\alpha\beta} &= \frac{1}{2\sqrt{v+\varepsilon}} g^{\alpha\beta} v_{\|\alpha\beta} \\ &- \frac{1}{4} \frac{1}{(v+\varepsilon)^{3/2}} g^{\alpha\beta} v_{\|\alpha} \cdot v_{\|\beta} \end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{\sqrt{v+\varepsilon}} g^{\alpha\beta}(g^{\mu\lambda}g^{\nu\delta}f_{\mu\nu\|\alpha}^{-a} f_{\lambda\delta}^{-b}\, G_{ab})_{\|\beta} \\
&\quad -\frac{1}{4}\frac{1}{(v+\varepsilon)^{3/2}} g^{\alpha\beta} v_{\|\alpha}\cdot v_{\|\beta} \\
&=\frac{1}{\sqrt{v+\varepsilon}} g^{\alpha\beta}g^{\mu\lambda}g^{\nu\delta} f_{\mu\nu\|\alpha\beta}^{-a} f_{\lambda\delta}^{-b}\, G_{ab} \\
&\quad +\frac{1}{\sqrt{v+\varepsilon}} g^{\alpha\beta}g^{\mu\lambda}g^{\nu\delta} f_{\mu\nu\|\alpha}^{-a} f_{\lambda\delta\|\beta}^{-b}\, G_{ab} \\
&\quad -\frac{1}{4}\frac{1}{(v+\varepsilon)^{3/2}} g^{\alpha\beta} v_{\|\alpha}\cdot v_{\|\beta} \\
&=-\frac{2}{\sqrt{v+\varepsilon}} g^{\alpha\beta}g^{\mu\lambda}g^{\nu\delta} f_{\nu\alpha\|\mu\beta}^{-a} f_{\lambda\delta}^{-b}\, G_{ab} \\
&\quad +\frac{1}{\sqrt{v+\varepsilon}} g^{\alpha\beta}g^{\mu\lambda}g^{\nu\delta} f_{\mu\nu\|\alpha}^{-a} f_{\lambda\delta\|\beta}^{-b}\, G_{ab} \\
&\quad -\frac{1}{4}\frac{1}{(v+\varepsilon)^{3/2}} g^{\alpha\beta} v_{\|\alpha}\cdot v_{\|\beta} \\
&=-\frac{2}{\sqrt{v+\varepsilon}} g^{\alpha\beta}g^{\mu\lambda}g^{\nu\delta}(f_{\nu\alpha\|\beta\mu}^{-a} \\
&\quad +f_{\rho\alpha}^{-a} R^{\rho}_{\nu\mu\beta}+f_{\nu\rho}^{-a} R^{\rho}_{\alpha\mu\beta}+f_{\nu\alpha}^{-c}\, C^{a}_{cd} f^{d}_{\mu\beta}) f_{\lambda\delta}^{-b}\, G_{ab} \\
&\quad +\frac{1}{\sqrt{v+\varepsilon}} g^{\alpha\beta}g^{\mu\lambda}g^{\nu\delta} f_{\mu\nu\|\alpha}^{-a} f_{\lambda\delta\|\beta}^{-b}\, G_{ab} \\
&\quad -\frac{1}{4}\frac{1}{(v+\varepsilon)^{3/2}} g^{\alpha\beta} v_{\|\alpha}\cdot v_{\|\beta} \\
&=-\frac{2}{\sqrt{v+\varepsilon}} g^{\alpha\beta}g^{\mu\lambda}g^{\nu\delta} R^{\rho}_{\nu\mu\beta}\, f_{\rho\alpha}^{-a} f_{\lambda\delta}^{-b}\, G_{ab} \\
&\quad -\frac{2}{\sqrt{v+\varepsilon}} g^{\alpha\beta}g^{\mu\lambda}g^{\nu\delta} R^{\rho}_{\alpha\mu\beta}\, f_{\nu\rho}^{-a} f_{\lambda\delta}^{-b}\, G_{ab} \\
&\quad -\frac{2}{\sqrt{v+\varepsilon}} g^{\alpha\beta}g^{\mu\lambda}g^{\nu\delta}\, C^{a}_{cd} f_{\nu\alpha}^{-c} f^{d}_{\mu\beta} f_{\lambda\delta}^{-b}\, G_{ab} \\
&\quad +\frac{1}{\sqrt{v+\varepsilon}} g^{\alpha\beta}g^{\mu\lambda}g^{\nu\delta} f_{\mu\nu\|\alpha}^{-a} f_{\lambda\delta\|\beta}^{-b}\, G_{ab} \\
&\quad -\frac{1}{4}\frac{1}{(v+\varepsilon)^{3/2}} g^{\alpha\beta} v_{\|\alpha}\cdot v_{\|\beta}.
\end{aligned} \tag{8}
$$

We denote the sum of the first and second terms in the right hand side by I, the third term by II, and the remaining terms by III, then

$$\Delta(\sqrt{v+\varepsilon})=\mathrm{I}+\mathrm{II}+\mathrm{III}. \tag{9}$$

First, from the definition of the conformal curvature tensor, we have

$$\begin{aligned}
\mathrm{I} = & -\frac{2}{\sqrt{v+\varepsilon}}\Big(C^{\rho\delta\lambda\alpha}\langle f_{\rho\alpha}^-, f_{\lambda\delta}^-\rangle + \tfrac{1}{2}(g^{\rho\lambda}R^{\delta\alpha} \\
& - g^{\rho\alpha}R^{\delta\lambda} + g^{\delta\alpha}R^{\rho\lambda} - g^{\delta\lambda}R^{\rho\alpha})\langle f_{\rho\alpha}^-, f_{\lambda\delta}^-\rangle \\
& + \frac{R}{6}(g^{\rho\alpha}g^{\delta\lambda} - g^{\rho\lambda}g^{\delta\alpha})\langle f_{\rho\alpha}^-, f_{\lambda\delta}^-\rangle\Big) \\
& - \frac{2}{\sqrt{v+\varepsilon}}\, g^{\nu\delta}R^{\rho\lambda}\langle f_{\nu\rho}^-, f_{\lambda\delta}^-\rangle \\
= & -\frac{2}{\sqrt{v+\varepsilon}}\, C^{\rho\delta\lambda\alpha}\langle f_{\rho\alpha}^-, f_{\lambda\delta}^-\rangle + \tfrac{1}{3}R\frac{v}{\sqrt{v+\varepsilon}}. \qquad (10)
\end{aligned}$$

If we adapt the normal coordinate system about a point x of M such that $g_{\mu\nu}|_x = \delta_{\mu\nu}$, then, by the self-dual condition of M, we have

$$C_{\rho\delta\lambda\alpha} = C^{+\,+}_{\rho\delta\lambda\alpha}$$

and

$$\mathrm{I} = \frac{2}{\sqrt{v+\varepsilon}}\, C^{+\,+}_{\rho\delta\alpha\lambda}\langle f_{\rho\alpha}^-, f_{\lambda\delta}^-\rangle + \tfrac{1}{3}R\cdot\frac{v}{\sqrt{v+\varepsilon}}. \qquad (11)$$

Through a straightforward but complicated computation, we know that the first term on the right hand vanishes. So

$$\mathrm{I} = \frac{R}{3}\cdot\frac{v}{\sqrt{v+\varepsilon}}. \qquad (12)$$

Secondly, in the normal coordinate system,

$$\mathrm{II} = \frac{2}{\sqrt{v+\varepsilon}}\langle [f_{\delta\alpha}^-, f_{\alpha\mu}], f_{\mu\delta}^-\rangle. \qquad (13)$$

Since the Cartan-Killing inner product is invariant under the adjoint representation, we have

$$\begin{aligned}
\mathrm{II} & = \frac{2}{\sqrt{v+\varepsilon}}\langle [f_{\mu\delta}^-, f_{\delta\alpha}^-], f_{\alpha\mu}\rangle \\
& = \frac{2}{\sqrt{v+\varepsilon}}(\langle [f_{\mu\delta}^-, f_{\delta\alpha}^-], f_{\alpha\mu}^+\rangle + \langle [f_{\mu\delta}^-, f_{\delta\alpha}^-], f_{\alpha\mu}^-\rangle).
\end{aligned}$$

It is easy to see that $[f_{\mu\delta}^-, f_{\delta\alpha}^-]$ are anti-self-dual, so $\langle [f_{\mu\delta}^-, f_{\delta\alpha}^-], f_{\alpha\mu}^+\rangle = 0$. Thus

$$\mathrm{II} = -\frac{2}{\sqrt{v+\varepsilon}}\mathrm{Tr}([f_{\mu\delta}^-, f_{\delta\alpha}^-] f_{\alpha\mu}^-) = -\frac{4}{\sqrt{v+\varepsilon}}\mathrm{Tr}(f_{\mu\delta}^- f_{\delta\alpha}^- f_{\alpha\mu}^-). \qquad (14)$$

Let $(f_{\mu\delta}^{-ij})$, $(f_{\delta\alpha}^{-jk})$ and $(f_{\alpha\mu}^{-ki})$ be the elements of the matrices $f_{\mu\delta}^{-}$, $f_{\delta\alpha}^{-}$ and $f_{\alpha\mu}^{-}$ respectively. Then

$$\mathrm{II} = -\frac{4}{\sqrt{v+\varepsilon}} f_{\mu\delta}^{-ij} f_{\delta\alpha}^{-jk} f_{\alpha\mu}^{-ki}. \tag{15}$$

If we denote the double indices $\binom{i}{\mu}$, $\binom{j}{\delta}$ and $\binom{k}{\alpha}$ by A, B and C respectively, then

$$\mathrm{II} = -\frac{4}{\sqrt{v+\varepsilon}} f_{AB}^{-} f_{BC}^{-} f_{CA}^{-}. \tag{16}$$

Consider the matrices

$$T = (f_{CA}^{-}), \qquad S = (f_{AB}^{-} f_{BC}^{-}).$$

It is clear that S, T are both Hermitian matrices and further S is semi-positive definite. Therefore, it is easy to prove that

$$\mathrm{Tr}\,(ST) \leqq \sqrt{\mathrm{Tr}\,(T^2)} \cdot \mathrm{Tr}\,(S).$$

Hence

$$\begin{aligned}
\mathrm{II} &\geqq -\frac{4}{\sqrt{v+\varepsilon}} \cdot \sqrt{f_{AB}^{-} \cdot f_{BA}^{-}} \cdot f_{CD}^{-} f_{DC}^{-} \\
&= -\frac{4}{\sqrt{v+\varepsilon}} \cdot \sqrt{f_{\mu\delta}^{-ij} \cdot f_{\delta\mu}^{-ji}} \cdot f_{\alpha\beta}^{-kl} f_{\beta\alpha}^{-lk} \\
&= -\frac{4}{\sqrt{v+\varepsilon}} \cdot \sqrt{-\mathrm{Tr}\,(f_{\mu\delta}^{-} \cdot f_{\mu\delta}^{-})} \cdot (-\mathrm{Tr}\,(f_{\alpha\beta}^{-} f_{\alpha\beta}^{-})) \\
&= -\frac{4}{\sqrt{v+\varepsilon}} \cdot v^{3/2}.
\end{aligned} \tag{17}$$

Finally, in the normal coordinate system,

$$\begin{aligned}
\mathrm{III} &= \frac{1}{\sqrt{v+\varepsilon}} \left(\langle f_{\mu\nu\|\alpha}^{-}, f_{\mu\nu\|\alpha}^{-} \rangle - \frac{1}{4(v+\varepsilon)} v_{\|\mu} \cdot v_{\|\mu} \right) \\
&= \frac{1}{\sqrt{v+\varepsilon}} \left(\langle f_{\mu\nu\|\alpha}^{-}, f_{\mu\nu\|\alpha}^{-} \rangle - \frac{1}{4(v+\varepsilon)} \sum_{\mu} (2 \langle f_{\lambda\delta\|\mu}^{-}, f_{\lambda\delta}^{-} \rangle)^2 \right).
\end{aligned} \tag{18}$$

Observe that

$$\begin{aligned}
\langle f_{\lambda\delta\|\mu}^{-}, f_{\lambda\delta}^{-} \rangle^2 &= (-\mathrm{Tr}\,(f_{\lambda\delta\|\mu}^{-} \cdot f_{\lambda\delta}^{-}))^2 \\
&= f_{\lambda\delta}^{-ij} \cdot f_{\lambda\delta}^{-ji} \cdot f_{\alpha\beta\|\mu}^{-kl} \cdot f_{\alpha\beta}^{-lk} \\
&= (f_{\lambda\delta\|\mu}^{-ij} \cdot \overline{f_{\alpha\beta\|\mu}^{-lk}}) \cdot (f_{\alpha\beta}^{-lk} \cdot \overline{f_{\lambda\delta}^{-ij}}).
\end{aligned} \tag{19}$$

Introduce the notations $A = \begin{pmatrix} i & j \\ \lambda & \delta \end{pmatrix}$, $B = \begin{pmatrix} l & k \\ \alpha & \beta \end{pmatrix}$. Then

$$\langle f_{\lambda\delta\|\mu}^{-}, f_{\lambda\delta}^{-} \rangle^2 = (f_{A\|\mu}^{-} \cdot \overline{f_{B\|\mu}^{-}}) \cdot (f_{B}^{-} \cdot \overline{f_{A}^{-}}). \tag{20}$$

Let

$$P=(P_{AB}), \qquad Q=(Q_{AB}),$$

where

$$P_{AB}=f_{A\|\mu}^{-}\cdot\overline{f_{B\|\mu}^{-}}$$
$$Q_{AB}=f_A^{-}\cdot\overline{f_B^{-}}.$$

It is clear that both P and Q are semi-positive definite Hermitian matrices. Hence it is easy to prove that

$$\operatorname{Tr}(PQ)\leqq\operatorname{Tr}P\cdot\operatorname{Tr}Q.$$

It follows from (19) that

$$\begin{aligned}\langle f_{\lambda\delta\|\mu}^{-},f_{\lambda\delta}^{-}\rangle^2&=P_{AB}\cdot Q_{BA}=\operatorname{Tr}(PQ)\\&\leqq\operatorname{Tr}P\cdot\operatorname{Tr}Q=(f_{A\|\mu}^{-}\overline{f_{A\|\mu}^{-}})\cdot(f_B^{-}\overline{f_B^{-}})\\&=(f_{\lambda\delta\|\mu}^{-ij}\cdot\overline{f_{\lambda\delta\|\mu}^{-ij}})\cdot(f_{\alpha\beta}^{-lk}\cdot\overline{f_{\alpha\beta}^{-lk}})\\&=\langle f_{\lambda\delta\|\mu}^{-},f_{\lambda\delta\|\mu}^{-}\rangle\cdot\langle f_{\alpha\beta}^{-},f_{\alpha\beta}^{-}\rangle.\end{aligned}$$

Hence from (18), we obtain

$$\mathrm{III}\geqq\frac{1}{\sqrt{v+\varepsilon}}\cdot\langle f_{\mu\nu\|\alpha}^{-},f_{\mu\nu\|\alpha}^{-}\rangle\cdot\left(1-\frac{v}{v+\varepsilon}\right)\geqq0$$

$$\Delta(\sqrt{v+\varepsilon})\geqq4\left(\frac{R}{12}-\sqrt{v}\right)\frac{v}{\sqrt{v+\varepsilon}}. \tag{21}$$

From the condition of the theorem, we have $\Delta(\sqrt{v+\varepsilon})\geqq0$.

3.2. Now we want to prove $v=0$. Even though we have proved $\sqrt{v+\varepsilon}$ is subharmonic, we still cannot apply the Yau's theorem to $\sqrt{v+\varepsilon}$ directly. We do not know whether $\int_M(v+\varepsilon)\,d\,\mathrm{vol}<\infty$, because $\mathrm{vol}(M)$ may be infinite. Now we follow the proof of Yau's theorem [6] with some modifications as Hildebrandt did in [3].

As Yau [6] pointed out, for every $R>0$, we can find a Lipschitz function η on M such that $\eta(x)=1$ for $x\in B_R$, $\eta(x)=0$ on $M\setminus B_{2R}$, $0\leqq\eta\leqq1$, and $|\nabla\eta|\leqq\frac{C}{R}$, where C is a constant which is independent of R, and B_R denotes a geodesic ball with the fixed center x_0 and radius R. Thus

$$\begin{aligned}0&\leqq\int_{B_{2R}}(\eta^2\sqrt{v+\varepsilon})\,\Delta(\sqrt{v+\varepsilon})\,d\,\mathrm{vol}\\&=-\int_{B_{2R}}g^{\mu\nu}\frac{\partial\sqrt{v+\varepsilon}}{\partial x^\mu}\frac{\partial(\eta^2\sqrt{v+\varepsilon})}{\partial x^\nu}\,d\,\mathrm{vol}\\&=-\int_{B_{2R}}g^{\mu\nu}\frac{\partial\sqrt{v+\varepsilon}}{\partial x^\mu}\frac{\partial\sqrt{v+\varepsilon}}{\partial x^\nu}\cdot\eta^2\,d\,\mathrm{vol}\\&\quad-\int_{B_{2R}}2g^{\mu\nu}\cdot\eta\cdot\frac{\partial\sqrt{v+\varepsilon}}{\partial x^\mu}\cdot\frac{\partial\eta}{\partial x^\nu}\cdot\sqrt{v+\varepsilon}\,d\,\mathrm{vol}.\end{aligned}$$

Therefore

$$\int_{B_R} |\nabla\sqrt{v+\varepsilon}|^2 \eta^2 \, d\,\mathrm{vol} + \int_{B_{2R}-B_R} |\nabla\sqrt{v+\varepsilon}|^2 \eta^2 \, d\,\mathrm{vol}$$
$$= \int_{B_{2R}} |\nabla\sqrt{v+\varepsilon}|^2 \eta^2 \, d\,\mathrm{vol}$$
$$\leqq 2\sqrt{\int_{B_{2R}-B_R} \eta^2 \cdot |\nabla\sqrt{v+\varepsilon}|^2 \, d\,\mathrm{vol}} \cdot \sqrt{\int_{B_{2R}-B_R} (v+\varepsilon)|\nabla\eta|^2 \, d\,\mathrm{vol}}$$

Let

$$\mathsf{X} = \sqrt{\int_{B_{2R}-B_R} \eta^2 |\nabla\sqrt{v+\varepsilon}|^2 \, d\,\mathrm{vol}},$$

then the above inequality becomes

$$\mathsf{X}^2 - 2\sqrt{\int_{B_{2R}-B_R} (v+\varepsilon)|\nabla\eta|^2 \, d\,\mathrm{vol}} \cdot \mathsf{X} + \int_{B_R} |\nabla\sqrt{v+\varepsilon}|^2 \, d\,\mathrm{vol} \leqq 0.$$

Since X is real, the discriminant of this quadratic must be non-negative. Thus

$$\int_{B_R} |\nabla\sqrt{v+\varepsilon}|^2 \, d\,\mathrm{vol} \leqq \int_{B_{2R}-B_R} (v+\varepsilon)\cdot|\nabla\eta|^2 \, d\,\mathrm{vol} \leqq \frac{C^2}{R^2} \int_{B_{2R}-B_R} (v+\varepsilon) \, d\,\mathrm{vol}.$$

Set $B'_R = B_R \setminus \{x | v=0\}$, and let $\varepsilon \to 0$. We have

$$\int_{B'_R} \frac{|\nabla v|^2}{4v} d\,\mathrm{vol} \leqq \frac{C^2}{R^2} \int_{B_{2R}-B_R} v \, d\,\mathrm{vol} \leqq \frac{C^2}{R^2} \int_{B_{2R}} v \, d\,\mathrm{vol}.$$

Let $R \to \infty$, using the assumption (2) in the theorem, we obtain

$$\int_{M \setminus \{x|v=0\}} \frac{|\nabla v|^2}{4v} d\,\mathrm{vol} = 0.$$

Thus $|\nabla v| = 0$ on $M \setminus \{x | v=0\}$, and therefore also on M. Hence $v = \text{const.}$ on M. Then, substituting this into the inequality (21), we have $v=0$, i.e.,

$$f_{\mu\nu}^- = 0. \quad \text{Q.E.D.}$$

Remark. If we replace the self-dual condition of M in the Theorem by the anti-self-dual condition and f^- by f^+, then this field would be an anti-self-dual field.

Finally, we point out that the corollary is a direct consequence of the theorem and the remark above.

References

1. Atiyah, M.F., Hitchin, N.J., Singer, I.M.: Self duality in four dimensional Riemannian Geometry. Proc. Roy. Soc. London Ser. A **362**, 425–461 (1978)
2. Bourguignon, J.P., Lawson, H.B.: Stability and isolation phenomena for Yang-Mills fields. Comm. Math. Phys. **79**, 189–230 (1981)

3. Hildebrandt, S.: Liouville theorems for harmonic mappings and an approach to Bernstein theorem. Annals of Mathematics Studies. Princeton: Princeton University Press (to appear)
4. Min-Oo: An $\mathscr{L}_2$-isolation theorem for Yang-Mills fields. Preprint
5. Shen, C.L.: On the sourceless $SU(N)$ gauge field over the four-dimensional self-dual compact Riemannian manifold with positive scalar curvature. In: Proceedings of the Beijing Symposium in Differential Geometry and Partial Differential Equations (Beijing 1980) (to appear)
6. Yau, S.T.: Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry. India Univ. Math. J. **25**, 659–670 (1976)

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