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Nonnegativity of the Curvature Operator and Isotropy for Isometric Immersions

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1. Introduction

Positivity of curvature plays a crucial role in riemannian geometry. However, such positivity conditions can be expressed in different ways of varying strength. In the more qualitative parts of riemannian geometry, the sectional curvature K and its sign is prevailing. The reason is that K enters the fundamental formulas for the geodesic variation in a very direct manner. For other questions, the curvature operator ρ and its definiteness is more dominant. For example, the positive semidefiniteness $\rho \geqq 0$ is the only known universal condition ensuring the nonnegativity of the general Gauss-Bonnet integrand, while $K \geqq 0$ is merely sufficient for dimensions less than 6. Moreover, for noncompact complete manifolds the Gauss-Bonnet integral and the Euler characteristic, if existing at all, do in general not coincide, but their difference is in some instances manageable if the curvature operator is positive semidefinite (see Theorem (2.F) below).

It is well known that $\rho \geqq 0$ implies $K \geqq 0$, the converse being true only for dimensions less than 4. So, a natural question is: What conditions have to be added to conclude from $K \geqq 0$ that $\rho \geqq 0$? Some known results in the realm of intrinsic geometry are listed in Sect. 2.

Now, if a riemannian manifold M of dimension m is isometrically immersed in another riemannian manifold $\tilde{M}$ of dimension $\tilde{m}$, we may conjecture that, for small codimensions $p := \tilde{m} - m$, there are weaker conditions that in the abstract case to ensure the positive semidefiniteness of ρ. The purpose of the present paper is to initiate some answers in this direction. For hypersurfaces ($p = 1$) of euclidean space, $\rho \geqq 0$ is directly implied by $K \geqq 0$ since ρ equals the exterior square of the Weingarten map. For submanifolds of euclidean space with codimension $p = 2$, Weinstein [31] deduced $\rho > 0$ from $K > 0$. In Sect. 3 we generalize this result to the semidefinite case and to nonflat ambient spaces. Also, an application to the Gauss-Bonnet formula is given. However, there is no hope to go beyond $p = 2$ in this manner as is shown by a counterexample at the end of Sect. 3. For higher codimensions, an attack is then made in Sect. 4

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in order to establish conditions, additional to $K \geqq 0$, which ensure $\rho \geqq 0$. They are expressed in terms of the two extremal values L and l of $|B(x,x)|$ on the unit tangent sphere where B is the vector valued second fundamental form. The conditions are related to a sort of mixed pinching for K and B. We call the quantities L and l *isotropy bounds* because their coincidence is equivalent to the notion of isotropy, introduced by O'Neill [22]. In Sect. 5, we conclude with some characterizations of isotropic submanifolds of low codimension, thus extending results of [22] to the variable curvature case. In particular, for $m=2$ and $p=2$, we meet the Veronese surface once more.

2. Notations and Known Results

Let $(M, \langle\ ,\ \rangle)$ always be a riemannian manifold of class C^∞ with dimension $m \geqq 2$ which is supposed to be hausdorff, paracompact, and connected. Beside the sectional curvature K of M we consider its curvature operator ρ, i.e. the riemannian curvature tensor, operating as a symmetric linear mapping from tangent bivectors to themselves. The curvature operator is called *positive semidefinite*, denoted by $\rho \geqq 0$, if $\langle \rho(\mathbf{X}), \mathbf{X} \rangle \geqq 0$ for all bivectors $\mathbf{X}$ (decomposable *or not*), where induced scalar products and norms for tensors are denoted the same way as for vectors. Similarly, the *positive definiteness* of ρ, denoted by $\rho > 0$, is defined. For decomposable $\mathbf{X} = u \wedge v \neq 0$, the sectional curvature of the plane E spanned by u, v is related to ρ by

$$K(E) = K(u,v) = \frac{\langle \rho(u \wedge v), u \wedge v \rangle}{|u \wedge v|^2}. \tag{2.1}$$

So, $K \geqq 0$ means that $\langle \rho(\mathbf{X}), \mathbf{X} \rangle \geqq 0$ for all *decomposable* bivectors $\mathbf{X}$. We express the last property by saying: ρ is *positive semidefinite on all decomposable bivectors*. Here and in the sequel, letters like $u, v, x, y, \ldots$ denote vectors at a fixed point of M. As a principle, in all inequalities we include the equality sign, leaving it to the reader to pursue the strict case along the lines given here.

Between the quantities K, ρ and the general Gauss-Bonnet integrand $\mathscr{K}$, the following pointwise relations are known:

(2.A) $\rho \geqq 0 \Rightarrow K \geqq 0$ (Eq. (2.1)).

(2.B) $\rho \geqq 0 \Rightarrow \mathscr{K} \geqq 0$ (Avez, Johnson, Kostant, cf. also Kulkarni [19]).

(2.C) $K \geqq 0 \Rightarrow \rho \geqq 0$ if $m \leqq 3$, but *not* for $m \geqq 4$ (if $m \leqq 3$, all bivectors are decomposable, for a counterexample in case $m \geqq 4$ see (3.D) below).

(2.D) $K \geqq 0 \Rightarrow \mathscr{K} \geqq 0$ if $m \leqq 5$, but *not* for $m \geqq 6$ (Chern-Milnor [7], Geroch [11]).

(2.E) $K_{\min} \geqq \dfrac{m'-2}{m'+1} K_{\max}$, $m' := 2 \cdot \left[\dfrac{m}{2}\right] \Rightarrow \rho \geqq 0$ (Bourguignon-Karcher [4]).

If nothing else is said, additional notions and conventions on signs of curvatures etc. are the same as in [29]. For m odd, $\mathscr{K} := 0$.

One further significant impact of $\rho \geqq 0$ is that on the Gauss-Bonnet *boundary* integrands and subsequently on the noncompact Gauss-Bonnet theory. The following theorem can be drawn rather immediately from former results of the second named author:

(2.F) **Theorem.** *Let M be a complete and oriented riemannian manifold. For $m \leqq 5$ let the sectional curvature K be nonnegative outside a compact subset A of M; for $m \geqq 6$ let the curvature operator ρ be positive semidefinite outside a compact subset A of M. Then a generalized version of Cohn-Vossen's inequality is valid for M, namely*

$$\int_M \mathscr{K}\, d\mu \leqq \chi(M), \tag{2.2}$$

where both quantities, the total curvature and the Euler characteristic, exist finitely. Moreover, $\mathscr{K} \geqq 0$ on $M \setminus A$.

For compact M, (2.2) with the equality sign is the generalized Gauss-Bonnet theorem of Allendoerfer-Weil [1] and Chern [6] (a recent version of Chern's intrinsic proof is to be found in [29]). In the noncompact case, an essential tool in proving Theorem (2.F) is the extended Gauss-Bonnet formula for compact locally convex subsets of M established in [27]. For intrinsic convexity in general, see [26, 28 and 30]. (A different approach to (2.2) which, instead of convex sets, uses convex functions has been proposed by Poor [24] and also appealed to in Greene-Wu [13, 14]. However, this reasoning causes difficulties because certain signs cannot be controlled.) Another main step is the exhaustion of a noncompact M by suitable compact totally convex subsets whose existence in case $A = \emptyset$ is part of the Cheeger-Gromoll structure theory [5]. The full proof of Theorem (2.F) in this case is carried out in [27]. For $A \neq \emptyset$ one can essentially argue the same way, using Greene-Wu's generalization of the Cheeger-Gromoll construction in [12]. The necessary homotopy equivalence of the exhausting totally convex subsets with M follows explicitly from results of Bangert [2] or, in the widest generality, [3].

Remark. For $m = 2$, the classical Cohn-Vossen inequality (2.2) holds true under much weaker assumptions, cf. [10].

3. Isometric Immersions of Codimension 2

We consider an isometric immersion of M into a second riemannian manifold $\tilde{M}$. All quantities related to $\tilde{M}$ will be marked by a tilde. Dimensions are sometimes indicated by writing M^m, $\tilde{M}^{\tilde{m}}$; the codimension is $p := \tilde{m} - m \geqq 1$.

First, let p be arbitrary. Then we have the Gauss equation

$$\langle \rho(u \wedge v), w \wedge z \rangle - \langle \tilde{\rho}(u \wedge v), w \wedge z \rangle = \langle \rho^B(u \wedge v), w \wedge z \rangle \tag{3.1}$$

where

$$\langle \rho^B(u \wedge v), w \wedge z \rangle := \langle B(u, w), B(v, z) \rangle - \langle B(u, z), B(v, w) \rangle. \tag{3.2}$$

Here, we look upon M as a submanifold of $\tilde{M}$. B is the second fundamental form, interpreted as a symmetric bilinear mapping of the tangent space of M into the normal space. Of course, ρ^B can be extended uniquely to a symmetric linear mapping from the set of all bivectors tangent to M to itself, and the definiteness notions introduced above for ρ apply to ρ^B as well. For every normal vector e, the corresponding scalar second fundamental form is $(u,v) \mapsto \langle B(u,v), e\rangle$.

Generalizing [31, Theorem 1], we prove the following purely algebraic

(3.A) **Proposition.** *At a fixed point of M, consider the following conditions on B:*

(a) *There is an orthonormal base $e_1, \ldots, e_p$ of the normal space such that the quadratic forms $u \mapsto \langle B(u,u), e_i\rangle$ are all positive semidefinite.*
(b) *ρ^B is positive semidefinite.*
(c) *ρ^B is positive semidefinite on all decomposable bivectors.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) *and, for $p=2$, also* (c) $\Rightarrow$ (a).

Proof. The first two implications are proven verbally the same way as in the strict positive case. The geometric core of the third implication will be discussed in a more general setting in Sect. 4. At this point, we give a formal, elementary proof of (c)$\Rightarrow$(a): Denote by Z the set of all vectors of the form $B(u,u)$ in the normal plane. By assumption

$$\langle \rho^B(u \wedge v), u \wedge v\rangle = \langle B(u,u), B(v,v)\rangle - |B(u,v)|^2 \geqq 0,$$

thus

$$\langle z, z'\rangle \geqq 0 \qquad \text{for all } z, z' \in Z. \tag{3.3}$$

Let Z_1 be the set of all vectors $x = z/|z|$ with $z \in Z \setminus \{0\}$. The closure $\bar{Z}_1$ is compact and has the property analogous to (3.3). Choose $e_1, y_0 \in \bar{Z}_1$ such that

$$\langle x, y\rangle \geqq \langle e_1, y_0\rangle \geqq 0 \qquad \text{for all } x, y \in \bar{Z}_1. \tag{3.4}$$

If e_2 is the unit vector in the normal plane orthogonal to e_1 with

$$y_0 = \mu_1 e_1 + \mu_2 e_2, \qquad \mu_2 \geqq 0,$$

then $\langle e_1, y_0\rangle = \mu_1 \geqq 0$ and $\mu_1^2 + \mu_2^2 = 1$. Applying (3.4) to $y = e_1$, resp. $y = y_0$, gives $\langle x, e_1\rangle \geqq 0$, resp.

$$\mu_2 \langle x, e_2\rangle \geqq \mu_1(1 - \langle x, e_1\rangle) \geqq 0.$$

In case $\mu_2 > 0$ this implies $\langle x, e_2\rangle \geqq 0$. If $\mu_2 = 0$, we have $\mu_1 = 1$, and it follows $\langle x, e_1\rangle = 1$, so $x = e_1$. In each case $\langle x, e_i\rangle \geqq 0$ for $x \in \bar{Z}_1$, and this implies $\langle z, e_i\rangle \geqq 0$ for $z \in Z$.

By combining the last result for $p=2$ with the Gauss equation (3.1), (3.2), we see immediately that the condition $K(E) \geqq \tilde{K}(E)$ for all tangent planes of M implies $\langle \rho(\mathbf{X}), \mathbf{X}\rangle \geqq \langle \tilde{\rho}(\mathbf{X}), \mathbf{X}\rangle$ for all bivectors of M over the fixed point. Thus we obtain

(3.B) **Theorem.** *Let the manifold M be isometrically immersed with codimension 2 in the riemannian manifold $\tilde{M}$. Then the following pointwise statement holds* *true: If the curvature operator $\tilde{\rho}$ of $\tilde{M}$ is positive semidefinite and if $K(E) \geqq \tilde{K}(E)$ for all tangent planes E of M, then the curvature operator ρ of M is positive semidefinite.*

Like all conditions sufficient for $\rho \geqq 0$ this leads directly to a corresponding Gauss-Bonnet result. A particular case arises when $\tilde{M}$ is a space form of nonnegative curvature since then $\tilde{\rho} \geqq 0$, automatically. We display the following significant case which generalizes a result of [27] and leave it to the reader to formulate corresponding Gauss-Bonnet corollaries in the other cases (especially those of Sect. 4).

(3.C) **Corollary.** *Let the complete oriented manifold M be isometrically immersed with codimension 2 in a flat space form $\tilde{M}$ (e.g. $\mathbb{R}^{m+2}$). If the sectional curvature K of M is nonnegative over all points outside a compact subset A of M, then*

$$\int_M \mathscr{K}\, d\mu \leqq \chi(M),$$

including the finite existence of both quantities. Moreover, $\mathscr{K} \geqq 0$ in $M \setminus A$.

Proof. Combine (2.F) and (3.B).

For $p \geqq 3$, it is no longer true that the sign of K alone determines the sign of ρ.

(3.D) *Example. There is a 4-dimensional submanifold M of $\mathbb{R}^7$ with $0 \in M$ such that all sectional curvatures of M over 0 are positive but the curvature operator ρ of M at 0 is indefinite.*

With respect to (2.C) and (3.B), the dimensions 4 and 7 are minimal for such an example.

Proof. Let a_1, a_2, a_3, a_4 resp. e_1, e_2, e_3 denote orthonormal bases for the tangent resp. normal space in a fixed point of our candidate M. We choose three scalar bilinear forms as candidates for the three second fundamental forms of e_1, e_2, e_3 according to the matrix representations:

$$B_1 = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \alpha_4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \beta_1 & & & \\ & \beta_2 & & \\ & & \beta_3 & \\ & & & \beta_4 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & & & a \\ & & a & \\ & a & & \\ a & & & 0 \end{pmatrix}. \tag{3.5}$$

The coordinates for bivectors over our point will be denoted as follows:

$$\mathbf{X} = p_1 \cdot a_1 \wedge a_2 + p_2 \cdot a_3 \wedge a_4 + p_3 \cdot a_1 \wedge a_3 \\ + p_4 \cdot a_2 \wedge a_4 + p_5 \cdot a_1 \wedge a_4 + p_6 \cdot a_2 \wedge a_3. \tag{3.6}$$

The curvature operator ρ can be computed from (3.2) to be represented by the matrix

$$\rho = \operatorname{diag}\left(\begin{pmatrix} \gamma_1 & -a^2 \\ -a^2 & \gamma_2 \end{pmatrix}, \begin{pmatrix} \gamma_3 & -a^2 \\ -a^2 & \gamma_4 \end{pmatrix}, \begin{pmatrix} \gamma_5 - a^2 & 0 \\ 0 & \gamma_6 - a^2 \end{pmatrix} \right). \tag{3.7}$$

Here, the right hand side stands for the (6×6)-matrix composed of the three displayed (2×2)-blocks along the main diagonal, where the following abbreviations have been used (same order as in (3.6)):

$$\gamma_1 := \alpha_1\alpha_2 + \beta_1\beta_2, \qquad \gamma_2 := \alpha_3\alpha_4 + \beta_3\beta_4, \dots$$

ρ becomes indefinite if e.g.

$$\gamma_1\gamma_2 - a^4 < 0. \tag{3.8}$$

Now, the decomposability condition for bivectors $\mathbf{X}$ over a 4-dimensional vector space is of scalar nature, namely the Plücker relation:

$$G(\mathbf{X},\mathbf{X}) := [\mathbf{X}\wedge\mathbf{X}] = 2p_1p_2 - 2p_3p_4 + 2p_5p_6 = 0. \tag{3.9}$$

Introducing the linear mapping γ associated with the symmetric bilinear form G on bivectors, this fact furnishes the following implication:

$$\text{there exists } t\in\mathbb{R} \text{ with } \rho + t\gamma > 0 \Rightarrow \langle\rho(\mathbf{X}),\mathbf{X}\rangle > 0 \text{ for all decomposable } \mathbf{X}\neq 0. \tag{3.10}$$

This conclusion is important because the positive definiteness of $\rho+t\gamma$, stated in the assumption, is easier to verify than the positivity of K, stated in the assertion. Since γ has by (3.9) the matrix representation

$$\gamma = \operatorname{diag}\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right),$$

the positive definiteness of $\rho+t\gamma$ is by (3.7) equivalent to:

$$\begin{aligned} &\gamma_1>0, \quad \gamma_2>0, \quad \gamma_3>0, \quad \gamma_4>0\\ &\gamma_5-a^2>0, \quad \gamma_6-a^2>0\\ &\gamma_1\gamma_2-(a^2-t)^2>0, \quad \gamma_3\gamma_4-(a^2+t)^2>0\\ &(\gamma_5-a^2)(\gamma_6-a^2)-t^2>0. \end{aligned} \tag{3.11}$$

The set of the ten inequalities (3.8) and (3.11) for the ten entries α_i, β_i, a, t, has solutions, a particular one being: $\alpha_1=\alpha_2=\alpha_4=1$, $\alpha_3=5$, $\beta_1=\beta_2=1$, $\beta_3=-1$, $\beta_4=4$, $a=\sqrt{2}$, $t=1$. Now, an M with $0\in M$ can be realized as the graph of the map $(x_1,x_2,x_3,x_4)\mapsto(f_1,f_2,f_3)(x_1,x_2,x_3,x_4)$ from $\mathbb{R}^4$ to $\mathbb{R}^3$ where f_1,f_2,f_3 are the quadratic polynomials corresponding to (3.5). For the numerical choice just made, we obtain the explicit example:

$$\begin{aligned} x_5 &= x_1^2+x_2^2+5x_3^2+x_4^2\\ x_6 &= x_1^2+x_2^2-x_3^2+4x_4^2\\ x_7 &= \sqrt{8}(x_1x_4+x_2x_3). \end{aligned}$$

4. Isometric Immersions of Higher Codimension

If M is immersed in $\tilde{M}$ with codimension $p \geqq 3$, there cannot exist a result as smooth as (3.B). Using again Proposition (3.A), we exhibit here some additional requirements allowing the conclusion from $K \geqq 0$ to $\rho \geqq 0$.

Obviously, the geometric idea behind the third implication of Proposition (3.A) is a statement on the unit circle $\mathbb{S}^1$, namely that a subset $Z_1 \subseteqq \mathbb{S}^1$ of geodesic diameter $\leqq \pi/2$ finds room in a geodesic interval on $\mathbb{S}^1$ of length $\pi/2$. For subsets of higher dimensional spheres there is a corresponding inequality of Jung's type, established by Santalo, which will be used for the following

(4.A) **Proposition.** *Let Z be a subset of a euclidean vector space W of dimension $p \geqq 2$ with the property*

$$\langle x, y\rangle \geqq \frac{p-2}{p-1} \cdot |x| \cdot |y| \qquad \text{for all } x, y \in Z. \tag{4.1}$$

Then there exists an orthonormal base $e_1, \ldots, e_p$ of W with $\langle x, e_i\rangle \geqq 0$ for all $x \in Z$ and $1 \leqq i \leqq p$.

Proof. As is clear from the proof of (3.A), we may assume $p \geqq 3$ and consider Z a subset of the unit sphere $\mathbb{S}^{p-1}$ of W. The assumption (4.1) then means for the geodesic diameter d of Z:

$$d \leqq \arccos \frac{p-2}{p-1} < \frac{\pi}{2}. \tag{4.2}$$

If δ denotes the geodesic diameter of the smallest geodesic ball Δ on $\mathbb{S}^{p-1}$ containing Z then Santalo's inequalities [25, Theorem 1] imply

$$d \leqq \delta \leqq 2 \cdot \arccos \sqrt{\frac{(p-1)\cos d + 1}{p}}.$$

Observe the case 2^0 there is excluded since it can only happen when $d \geqq \pi/2$. With (4.2), this gives:

$$d \leqq \delta \leqq 2 \cdot \arccos \sqrt{\frac{p-1}{p}}.$$

Now, choose an orthonormal base $e_1, \ldots, e_p$ of W such that the center of Δ is represented by

$$a = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} e_j.$$

Then, for all $x \in Z$:

$$\langle x, a\rangle \geqq \sqrt{\frac{p-1}{p}}.$$

With

$$x = \sum_{k=1}^{p} x_k e_k$$

this means

$$\sum_{k=1}^{p} x_k \geqq \sqrt{p-1}. \tag{4.3}$$

If we had $x_j < 0$ for a specific j then the Cauchy-Schwarz inequality would imply

$$\sum_k x_k < \sum_{k \neq j} x_k \leqq \sqrt{\sum_{k \neq j} x_k^2} \cdot \sqrt{p-1} < \sqrt{p-1},$$

which is a contradiction to (4.3).

Remark. The constant $(p-2)/(p-1)$ in (4.1) has been selected minimally with respect to the idea of the proof. We do not know whether this is really the smallest possible choice for $p \geqq 3$.

Among the real invariants of the second fundamental form B, the simplest ones are the pointwise extrema

$$L := \max_{|u|=1} |B(u,u)|, \qquad l := \min_{|u|=1} |B(u,u)|. \tag{4.4}$$

We call L and l the *isotropy bounds* (the reason being explained in Sect. 5). There is no direct relation of L and l to the "length" of B dealt with by Chern-Do Carmo-Kobayashi in [9]. However, L is also a norm on the vector space of our bilinear mappings B and, in a sense, L and l are related to the maximal value of sectional curvature if $\tilde{M}$ is flat.

For the rest of this section, we only consider a flat ambient space $\tilde{M}$, hence $\tilde{\rho} = 0$.

(4.B) **Theorem.** *Let the manifold M be isometrically immersed with codimension p in the flat riemannian manifold $\tilde{M}$. Then the following pointwise statements hold true: If p is arbitrary then*

$$K_{\max} \leqq L^2. \tag{4.5}$$

If $p \leqq m-2$ (or $p \leqq m-1$ and $l > 0$) then

$$l^2 \leqq K_{\max}. \tag{4.6}$$

Proof. (4.5) follows immediately from the Gauss equation for orthonormal u, v:

$$K(u,v) = \langle B(u,u), B(v,v) \rangle - |B(u,v)|^2. \tag{4.7}$$

The proof of (4.6) uses essentially an argument due to Otsuki [23] (cf. also [20]): Denote by $\mathbb{S}^{m-1}$ the unit sphere in our fixed tangent space of M and let $u_0 \in \mathbb{S}^{m-1}$ be a vector at which the function $f(u) := |B(u,u)|^2$ attains its minimum on $\mathbb{S}^{m-1}$, so $|B(u_0, u_0)| = l$. The usual necessary conditions in first and second order for the minimum of f at u_0 read:

$$\langle B(u_0, u_0), B(u_0, u) \rangle = 0 \quad \text{for all } u \in \mathbb{S}^{m-1} \quad \text{with} \quad u \perp u_0, \tag{4.8}$$

$$2|B(u_0, u)|^2 + \langle B(u_0, u_0), B(u,u) \rangle - l^2 \geqq 0 \quad \text{for all } u \in \mathbb{S}^{m-1} \quad \text{with} \quad u \perp u_0. \tag{4.9}$$

The first condition is equivalent to

$$\langle B(u_0, u_0), B(u_0, u)\rangle = l^2 \cdot \langle u_0, u\rangle \quad \text{for all } u \in \mathbb{S}^{m-1}. \tag{4.8'}$$

The idea is to take an $u_1 \in \mathbb{S}^{m-1}$, orthogonal to u_0, *and* satisfying $B(u_0, u_1)=0$. For such an u_1, the Gauss equation (4.7) and (4.9) imply $K(u_0, u_1)=\langle B(u_0, u_0), B(u_1, u_1)\rangle \geqq l^2$, hence the assertion. Now, the existence of u_1 is clear by simple dimension arguments: If $p \leqq m-2$ then the kernel of the linear map $u \mapsto (B(u_0, u), \langle u_0, u\rangle)$ is nontrivial. If $p \leqq m-1$ and $l>0$, take $u_1 \in \mathbb{S}^{m-1}$ in the kernel of $u \mapsto B(u_0, u)$, which is nontrivial, and observe (4.8').

We now show:

(4.C) **Theorem.** *Let the manifold M be isometrically immersed with codimension p in the flat space form $\tilde{M}$ where $m \geqq 4$ and $p \geqq 3$. Then the following pointwise statement holds true: If*

$$K_{\min} \geqq \sqrt{\frac{p-3/2}{p-1}} \cdot L^2 \tag{4.10}$$

then the curvature operator ρ of M is positive semidefinite.

Proof. For each orthogonal pair u, v over our point, we have by the Gauss equation (4.7):

$$\frac{K_{\min}}{L^2} \leqq \frac{\langle B(u,u), B(v,v)\rangle}{|B(u,u)| \cdot |B(v,v)|}. \tag{4.11}$$

Here, it is assumed that all denominators are $\neq 0$. In order to gain an analogous relation for arbitrary tangent vectors x, y, choose z orthogonally to x, y and set

$$\varphi := \sphericalangle(B(x,x), B(y,y))$$
$$\varphi_1 := \sphericalangle(B(x,x), B(z,z))$$
$$\varphi_2 := \sphericalangle(B(y,y), B(z,z))$$

where, for the present, $B(x,x) \neq 0$, $B(y,y) \neq 0$, $B(z,z) \neq 0$. By (4.11) we have $\cos\varphi_\nu \geqq K_{\min}/L^2$, and the triangle inequality for angles yields:

$$\varphi \leqq \varphi_1 + \varphi_2 \leqq 2 \cdot \arccos \frac{K_{\min}}{L^2},$$

hence

$$\frac{\langle B(x,x), B(y,y)\rangle}{|B(x,x)| \cdot |B(y,y)|} = \cos\varphi \geqq 2 \cdot \frac{K_{\min}^2}{L^4} - 1.$$

Using (4.10) we deduce

$$\langle B(x,x), B(y,y)\rangle \geqq \frac{p-2}{p-1} \cdot |B(x,x)| \cdot |B(y,y)|. \tag{4.12}$$

This is trivially correct if $L=0$ or $B(x,x)=0$ or $B(y,y)=0$, the only case left being: $L \neq 0$, $B(x,x) \neq 0$, $B(y,y) \neq 0$, but $B(z,z)=0$ for all $z \neq 0$ orthogonal to x

and y. However, this last case does not exist since, for these z, we had by the Gauss equation (4.7):

$$0 \leqq |x \wedge z|^2 \cdot K(x,z) = \langle B(x,x), B(z,z)\rangle - |B(x,z)|^2 \leqq 0,$$

so $K_{\min}=0$, hence by (4.10): $L=0$.

From (4.12) and Propositions (3.A) and (4.A) follows $\rho \geqq 0$.

With respect to (4.5), the assumption (4.10) implies

$$K_{\min} \geqq \sqrt{\frac{p-3/2}{p-1}} \cdot K_{\max}.$$

Obviously, this pinching condition can be weaker than that of (2.E). We do not know whether such a pinching *alone* ensures $\rho \geqq 0$. However, from (4.B) and (4.C) we can draw the following *mixed* pinching condition:

(4.D) **Corollary.** *Let the manifold M be isometrically immersed with codimension p in the flat space form $\tilde{M}$ where $3 \leqq p \leqq m-1$. Then the following statement holds true pointwise: If $K \geqq 0$ and*

$$l^2 \cdot K_{\min} \geqq \sqrt{\frac{p-3/2}{p-1}} \cdot L^2 \cdot K_{\max} \tag{4.13}$$

then the curvature operator ρ of M is positive semidefinite.

Proof. In case $l=0$ the assertion is trivially clear since $K=0$. So we may assume $l>0$ and $K_{\max}>0$. But then (4.13) and (4.6) imply (4.10).

5. Isotropy

With regard to the "extrinsic" pinching discussed before, but also independently, it becomes important to elucidate the case where the isotropy bounds L and l coincide. We are able to do this in certain instances for which the codimension is not too big. The ambient manifold $\tilde{M}$ can again be arbitrary (not necessarily flat).

Following a notion of O'Neill [22], we call the immersion *λ-isotropic* at a fixed point if, for a real constant λ, we have

$$|B(u,u)| = \lambda \cdot |u|^2$$

for all vectors u over the given point. Of course, this is equivalent to $L=l=\lambda$. If there is a function $\varphi\colon M \to \mathbb{R}$ such that the immersion is $\varphi(q)$-isotropic at every $q \in M$, then the immersion will be called *φ-isotropic* or *isotropic* for short (necessarily, φ^2 is smooth). In case $\varphi=\lambda=$const., the immersion is said to be *λ-isotropic.*

Totally umbilic immersions are characterized by the following particular form of the second fundamental form: $B(u,v)=\langle u,v\rangle \cdot H$, where H is the mean curvature normal vector. So, totally umbilic immersions are isotropic with $\varphi = |H|$, and they are λ-isotropic if $|H|$ is constant, where $\lambda = |H|$ (this is certainly so if $\tilde{M}$ has constant curvature). In [22] O'Neill studied λ-isotropic immersions of space forms into space forms and particularly showed that such immersions are totally umbilic if $p < \frac{1}{2}m(m+1) - 1$ and not necessarily umbilic otherwise. We will prove that isotropic immersions are totally umbilic *in general* if $p < m/2$. The following Proposition contains the purely algebraic core. For simplicity, we deal with the 1-isotropic case:

(5.A) **Proposition.** *Let V and W be euclidean vector spaces of dimension $m \geqq 2$ and $p \geqq 1$ resp. and assume $B: V \times V \to W$ to be symmetric, bilinear and 1-isotropic, i.e.*

$$|B(u,u)| = |u|^2 \quad \text{for all } u \in V. \tag{5.1}$$

If $p < m/2$ then there is a unit vector $N \in W$ such that

$$B(x,y) = \langle x, y \rangle \cdot N \quad \text{for all } x, y \in V. \tag{5.2}$$

Proof. By polarizing (5.1) we get for all x, y, z, $w \in V$:

$$\begin{aligned} &\langle B(x,y), B(z,w)\rangle + \langle B(x,z), B(w,y)\rangle + \langle B(x,w), B(y,z)\rangle \\ &\quad = \langle x,y\rangle \cdot \langle z,w\rangle + \langle x,z\rangle \cdot \langle w,y\rangle + \langle x,w\rangle \cdot \langle y,z\rangle. \end{aligned} \tag{5.3}$$

In fact, both sides of (5.3) define a 4-linear form on V which is symmetric in any two of its arguments, and (5.1) says that both 4-linear forms coincide on the diagonal of $V \times V \times V \times V$. Specializing (5.3), we obtain the following identities in $x, y \in V$:

$$2|B(x,y)|^2 + \langle B(x,x), B(y,y)\rangle = 2\langle x,y\rangle^2 + |x|^2 \cdot |y|^2 \tag{5.4}$$

$$|B(x,x) - B(y,y)|^2 = (|x|^2 - |y|^2)^2 + 4|B(x,y)|^2 - 4\langle x,y\rangle^2 \tag{5.5}$$

$$\langle B(x,y), B(y,y)\rangle = \langle x,y\rangle \cdot \langle y,y\rangle. \tag{5.6}$$

For given unit vectors $x, y \in V$, define linear maps A_x and A_y from V to W by $A_x(z) := B(x,z)$ and $A_y(z) := B(y,z)$. Since $p < m/2$, we have $\dim(\ker A_x) > m/2$ and $\dim(\ker A_y) > m/2$. Hence, there must exist a unit vector $z \in V$ such that $B(x,z) = B(y,z) = 0$. Now (5.6) implies $\langle x,z\rangle = \langle y,z\rangle = 0$, and from (5.5) we get

$$|B(x,x) - B(z,z)|^2 = |B(y,y) - B(z,z)|^2 = 0.$$

Thus, $B(x,x) = B(y,y) =: N$ for any two unit vectors x, y. This implies (5.2).

Remark. Suppose that B is 1-isotropic and for some ON-base $e_1, \ldots, e_p$ of W the bilinear forms $B_i(x,y) := \langle B(x,y), e_i\rangle$, $i \in \{1, \ldots, p\}$ are simultaneously diagonalizable. (This is certainly the case if $p = 2$ and $m \geqq 3$, cf. [15].) Then, by (5.6), any diagonalizing base of V is automatically orthogonal, and it is not hard to see by means of (5.5) that the assertion of Proposition (5.A) holds true in this case, too.

From Proposition (5.A) and this remark we obtain:

(5.B) **Theorem.** *Let the manifold M be isometrically and isotropically immersed with codimension p in the riemannian manifold $\tilde{M}$. If $m \geqq 3$ and $p < \max\{m/2, 3\}$ then the immersion is totally umbilic. If, in addition, $\tilde{M}$ has constant curvature $\tilde{K}$ then the immersion is λ-isotropic and M has constant curvature $K = \tilde{K} + \lambda^2$ where $\lambda = |H|$.*

The last part follows again from the Gauss equation (3.1), (3.2), using Schur's lemma.

For $m=2$ and $p \geqq 2$, the assertion of (5.B) is false in general. In fact, *all minimal immersions of a differentiable 2-sphere into the round unit sphere $\mathbb{S}^4$ are isotropic.* This can be read off the results of Chern [8] where all such minimal immersions have been determined in principle, almost all of them having nonconstant curvature and, therefore, nonconstant φ; see (5.8) below. The essential point here is that for any minimal immersion $M^2 \to \mathbb{S}^4$ the isotropy in our sense is equivalent to the isotropy of the complex vector $V := B(e_1, e_1) + iB(e_1, e_2)$, introduced in [8], e_1, e_2 being any tangent ON-base of M. This is true pointwise and can be verified by a simple calculation. In particular, the *Veronese surface* of $\mathbb{S}^4$ is λ-isotropic with $\lambda = 1/\sqrt{3}$ (and $K = 1/3$). We prove the following converse:

(5.C) **Theorem.** *Let M^2 be a connected surface and φ a real function on it without zeros. Then, any φ-isotropic immersion of M^2 in a 4-dimensional riemannian manifold $\tilde{M}^4$ is either totally umbilic or minimal. Denoting by $\tilde{K}_t$ the sectional curvature of $\tilde{M}^4$ along the tangent planes of M^2, we have*

$$K - \tilde{K}_t = \varphi^2 \qquad \textit{(umbilic case)} \tag{5.7}$$

$$K - \tilde{K}_t = -2\varphi^2 \qquad \textit{(minimal case).} \tag{5.8}$$

If $\tilde{M}^4$ has constant curvature $\tilde{K}$, then any λ-isotropic immersion of M^2 in $\tilde{M}^4$ is in case $\tilde{K} \leqq 0$ totally umbilic and in case $\tilde{K} > 0$ either totally umbilic or an immersion in a Veronese surface of $\tilde{M}^4$.

Observe that the notion of a Veronese surface is by [9] locally of an intrinsic character. Related questions for $m \geqq 3$ or for constant curvature of M and $\tilde{M}$ are discussed in [16, 21, 22].

Proof. For the given φ-isotropic immersion, consider the square S of the length of B as defined in [9]. If X_1, X_2 is any ON-frame field on M^2, one calculates $S - 2\varphi^2 = 2|B(X_1, X_2)|^2$. So, $|B(X_1, X_2)|$ is invariant under rotation of X_1, X_2, and it is easily seen that $B(X_1, X_2) = 0$ characterizes the form $B(X, Y) = \langle X, Y\rangle \cdot H$ of B (pointwise).

Suppose that, at some point, there is an ON-basis e_1, e_2 with $B(e_1, e_2) \neq 0$. We extend this basis locally to a tangent ON-frame field X_1, X_2 such that $B(X_1, X_2) \neq 0$ and choose a normal ON-frame field X_3, X_4 with $B(X_1, X_2) = |B(X_1, X_2)| \cdot X_3$. By (5.6), $B(X_1, X_1)$ and $B(X_2, X_2)$ are orthogonal to $B(X_1, X_2)$, hence proportional (and of same length φ). By (5.5), they cannot agree, so $B(X_1, X_1) = -\varphi \cdot X_4$, $B(X_2, X_2) = \varphi \cdot X_4$, say. This shows the minimality. By (5.4), $|B(X_1, X_2)| = \varphi$, and this implies that the subset of M^2 where $B(e_1, e_2) \neq 0$ is possible for some ON-basis e_1, e_2 is not only open but also closed. For X_1, X_2, the Gauss-equation specializes to (5.8). That, in the umbilic case, (5.7) holds true is also seen easily.

Now assume $\varphi=\lambda=\text{const.}>0$ and $\tilde{M}^4$ of constant curvature $\tilde{K}$, and consider the case that the immersion is not totally umbilic. With the adapted frame fields just constructed one can enter the structure equations (in the form of [9]). The essential relation is the restriction of $d\tilde{\omega}=\tilde{\omega}\wedge\tilde{\omega}+\Phi$ onto M^2 where $\tilde{\omega}$ resp. Φ is the connection resp. curvature matrix in $\tilde{M}^4$ (superscripts counting columns, subscripts rows). Denoting the restrictions of $\tilde{\omega}$, Φ by ω, Ω, the second fundamental form part of ω is by the above choice of the frame fields: $\omega_1^3=\lambda\omega^2$, $\omega_2^3=\lambda\omega^1$, $\omega_1^4=-\lambda\omega^1$, $\omega_2^4=\lambda\omega^2$ where ω^1, ω^2 is dual to X_1, X_2, and Ω has entries $\Omega_2^1=-\Omega_1^2=\tilde{K}\omega^1\wedge\omega^2$, and zeros otherwise. Then $d\omega=\omega\wedge\omega+\Omega$ is expressed by

$$d\omega_2^1=(\tilde{K}-2\lambda^2)\omega^1\wedge\omega^2, \qquad d\omega_4^3=2\lambda^2\omega^1\wedge\omega^2 \tag{5.9}$$

$$d\omega^1=-\omega^2\wedge(\omega_2^1-\omega_4^3), \qquad d\omega^2=\omega^1\wedge(\omega_2^1-\omega_4^3). \tag{5.10}$$

Moreover

$$d\omega^1=\omega^2\wedge\omega_2^1, \qquad d\omega^2=-\omega^1\wedge\omega_2^1. \tag{5.11}$$

Eliminating $d\omega^1$, $d\omega^2$ from (5.10), (5.11) gives $\omega_4^3=2\omega_2^1$ and with this, (5.9) yields $\tilde{K}-2\lambda^2=\lambda^2$, hence $\tilde{K}=3\lambda^2>0$. So, for $\tilde{K}\leqq 0$, this case cannot occur. For $\tilde{K}>0$ we may assume $\tilde{K}=1$ and then obtain $\lambda=1/\sqrt{3}$ and $S=4\lambda^2=4/3$. Thus, by [9], we arrive at the Veronese surface here. (This case could also have been handled directly, using (5.8) and [8].)

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