Math. Z. 181, 55–70 (1982)

Mathematische Zeitschrift

A Generalization of the Matrix-Tree Theorem

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1. Definitions

Let $G=(V,E)$ be a multigraph of order n without loops. Put $V=\{1,2,\dots,n\}$. Let $\rho(x)$ denote the valency of the vertex x. $\mathscr{V}(G)=(v_{ij})$ be its *valency matrix*, with $v_{ij}=\rho(i)\,\delta_{ij}$, δ_{ij} being the Kronecker delta. Let $\mathscr{A}(G)=(a_{ij})$ denote the *adjacency matrix* of G where a_{ij} equals the multiplicity of the edge (i,j) of G. Since G has no loops, the matrix $\mathscr{A}$ has a zero diagonal. The *admittance matrix* $A(G)$ of G is defined by $A(G)=\mathscr{V}-\mathscr{A}$. Let B be an $m\times n$ matrix. Let $\alpha\subset\{1,\dots,m\}$, $\beta\subset\{1,\dots,n\}$. We shall use the notation introduced in [13] for identifying submatrices of B. $B[\alpha|\beta]$ is the submatrix of B consisting of rows α and columns β. $B(\alpha|\beta)$ is the complementary submatrix to $B[\alpha|\beta]$ obtained by deleting from B rows of α and columns of β. $B(\alpha|\beta]$ denotes the submatrix of B whose rows are precisely those complementary to α and whose columns are designated by β. Likewise $B[\alpha|\beta)$. The unit matrix of order n will be denoted by I_n. The matrix of order $m\times n$ all whose entries are 1 will be denoted by $J_{m,n}$. If $m=n$ we just write J_n instead of $J_{n,n}$.

The following definition will be needed in the sequel.

$$0^0=1.$$

The number of elements of a set S is denoted by $|S|$. We shall write x for the one element set $\{x\}$. The determinant of a matrix X will be denoted by $\det X$. We define $\det[\emptyset|\emptyset]=1$.

Let $\kappa(G)$ denote the complexity of the multigraph G.

2. Introduction

The *complexity* of a graph is defined as the number of its spanning trees. The first explicit result in this direction has been obtained by Cayley in 1889 who stated that the complexity of the complete graph of order n is n^{n-2} [6]. The first to give a satisfactory proof to Cayley's result seem to be Dziobek [8] in

0025-5874/82/0181/0055/$03.20

1917 and Prüfer [17] in 1918. With time this beautiful result has received many proofs and the interested reader is referred to a thorough survey article on this subject by Moon [14]. See also [5] and [2].

The theorem which is sometimes referred to as the *Matrix Tree Theorem*, also known as *Kirchhoff's Theorem* and the *Kirchhoff-Trent Theorem*, supplies a formula for the complexity of an arbitrary multigraph G.

Matrix Tree Theorem. *The complexity of a multigraph G of order n equals any principal minor of order $n-1$ of the admittance matrix of G.*

We shall refer to this theorem as MTT. The idea to this theorem appears already in the fundamental work of Kirchhoff in 1847 [12]. Although it has been rediscovered several times since, a clear proof based on the theory of determinants was given as late as 1954 by Trent [22]. An even shorter and more comprehensible proof was supplied by Hutschenreuther in 1967 [11].

Leaning on an idea of Hutschenreuther we shall give a slightly different proof of the MTT and then generalize the result which will reveal the precise meaning of every principal minor of the admittance matrix of a graph.

The difference in approach lies mainly in the fact that Hutschenreuther proves a lemma on certain special classes of determinants before he starts the induction whereas we endeavour not to take our sight off the multigraph during the course of proof. A minor difference is also that we drop a complete multiple edge whereas in [11] a single edge is dropped to use induction.

A paper on this subject was also written by Chaiken and Kleitman [24]. In [25] Chaiken skillfully treats the general minors of the admittance matrix and obtains an interpretation which is distinct from ours.

Moon states in [15] that the usefulness of the MTT is limited by the difficulty in practically evaluating determinants of large matrices. This being generally true, we still would like to show a number of special cases which may be solved by either the MTT or its generalization, including most cases appearing in [2].

Since the presence of loops does not affect complexity, there is no restriction in assuming the graph in question to be without loops.

Counting of trees in different ways has been extensively used to provide combinatorial identities. In Sect. 5 we shall add some new and some known combinatorial identities based on counting trees.

3. The Main Result

We shall first give a short proof of the MTT which is a variation of a proof by Hutschenreuther [11].

Let G be a multigraph of order n. Let the admittance matrix of the graph be $A=(a_{ij})$ and put $a_{ij}=-b_{ij}$ for $i\neq j$. If G is not connected, then $A(G)$ is reducible, its submatrices of order $n-1$ are singular, so that $\det A(1|1)=0=\kappa(G)$, so we may assume G connected.

For $n=2$ the theorem is clearly true, so we assume the theorem to be true for all graphs of order less than n. Since the theorem holds for disconnected graphs, we may assume the theorem to hold for graphs with less edges than G.

Without loss of generality we may assume vertices 1 and 2 adjacent in G. As in [11] we contract vertices 1, 2 to a single vertex v and consider the graph G' of order $n-1$. Each spanning tree of G' corresponds to b_{12} spanning trees of G each containing one of the multiple edges of $(1,2)$ and conversely. Thus the number of spanning trees of G containing a single edge of $(1,2)$ is $b_{12}\,\kappa(G')$. Now remove the multiple edge $(1,2)$ from G. The graph G'' so obtained has the same number of spanning trees as has G of the kind that do not use the edge $(1,2)$. We thus have

$$\kappa(G)=b_{12}\,\kappa(G')+\kappa(G''). \tag{1}$$

By the induction hypothesis we have

$$\kappa(G')=\det A(G')(v|v)=\det A(G)(\{1,2\}|\{1,2\}), \tag{2}$$

$$\kappa(G'')=\det A(G'')(1|1). \tag{3}$$

Now the matrices $A(G)(1|1)$ and $A(G'')(1|1)$ differ only in their upper left entry which is a_{22} and $a_{22}-b_{12}$ respectively. Therefore

$$\det A(G'')(1|1)+b_{12}\det A(G)(\{1,2\}|\{1,2\})=\det A(G)(1|1). \tag{4}$$

Combining (1), (2), (3) and (4) yields the theorem.

Now let G be a multigraph and let F be some subforest of G. How many spanning trees of G contain F? To answer this question we adjoin to F as trees those vertices of G which are not in F, thus expanding the forest F to a spanning forest F' of G. Therefore there is no restriction in posing our problem as follows. How many spanning trees of G contain the spanning forest F of G? (We intentionally replaced F' by F.) We now construct a new multigraph G' derived from G. Let $T_1, T_2, \ldots, T_k$ be the trees constituting the forest F. Let $G'=(V',E')$, $V'=\{T_i|i=1,2,\ldots,k\}$. The multiplicity of the edge (T_i, T_j) of E' is equal to the number of edges (counting multiplicities) having one vertex in T_i and one in T_j. Each spanning tree of G containing F corresponds to one spanning tree of G' and vice versa. This establishes a bijection from the spanning trees of G containing F to the spanning trees of G'. The number of spanning trees of G containing F is therefore equal to the complexity of G'. We have in fact shown the following theorem.

Theorem 1. *Let G be a graph and let F be some subforest of G. The number of spanning trees of G containing F is equal to the complexity of the graph obtained from G by contracting each tree of F to a single vertex and at the same time preserving outer adjacencies (including multiplicities).*

Various special cases are of quite general nature.

Theorem 2. *Let $G=(V,E)$ be a multigraph, A its admittance matrix and let T be some subtree of G on the vertex set α of V. Then the number of spanning trees of G containing T is given by*

$$\mu(T)=\det A(\alpha|\alpha).$$

Proof. Let G' be the graph derived from G by contracting T to a vertex. Consider $A'=A(G')$. Since T is a vertex of G', the number of spanning trees of

G containing T equals the complexity of G' which is given by $\kappa(G') = \det A'(T|T)$. But $A'(T|T) = A(\alpha|\alpha)$, so that $\mu(T) = \det A(\alpha|\alpha)$, which proves the theorem.

Theorem 2 is thus a generalization of the MTT, since in the latter α may be chosen as a one-element set.

Let $G=(V,E)$ be a multigraph and let U be a subset of V. Let $G(U)$ denote the subgraph of G induced by U. Let $\varphi(G(U))$ be some spanning forest of $G(U)$. Let $f(G,U)$ denote the number of spanning forests of G where each component of such a spanning forest contains precisely one component of $\varphi(G(U))$. By adding just enough edges to $G(U)$ to make it connected, each spanning forest of $G(U)$ becomes a spanning tree of $G(U)$ and each spanning forest of G' with the above mentioned restriction becomes a spanning tree of G', where G' is G strengthened by the additional edges. The problem now reduces to counting the number of spanning trees of G' containing the modified spanning tree of $G'(U)$ as a subtree. Once the counting has been completed, the auxiliary edges may be dropped. The conclusion is that the two problems have an identical solution. We thus have

Theorem 3. *Let $G=(V,E)$ be a multigraph and let U be a subset of V. We then have $f(G,U) = \det A(U|U)$.*

Theorem 3 gives rise to a converse theorem which interprets the meaning of a principal minor in general of the admittance matrix of some given multigraph. Let $G=(V,E)$ be a multigraph of order n. Let $S \subset V = \{1,2,\dots,n\}$. Let $\gamma(G)$ denote the number of components of G. We then have

Theorem 4. *Let A be the admittance matrix of some multigraph $G=(V,E)$ and let β be some arbitrary subset of V. Then $\det A[\beta|\beta]$ designates the number of spanning forests of G having $\gamma(G(V\setminus\beta))$ components, containing a spanning forest of $G(V\setminus\beta)$ as subforest.*

Proof. Consider some spanning forest of $G' = G(V\setminus\beta)$. Connect the forest by additional edges from G, so as to obtain as few components as possible, that is $\gamma(G(V\setminus\beta))$. Applying now Theorem 3 we get Theorem 4.

Corollary 1. *All the principal minors of an admittance matrix of a multigraph are nonnegative.*

Proof. Interpret the principal minors of an admittance matrix in the light of Theorem 4.

Let $R^{n,n}$ denote the set of all $n \times n$ matrices over the real field. Define $Z^{n,n} = \{A = (a_{ij}) | a_{ij} \leqq 0,\ i \neq j\}$.

An *M-matrix* is a matrix A of the form $A = sI - B$ with $s > 0$ and B nonnegative, and such that $s \geqq \sigma(B)$, the *spectral radius* of B. We then have

Theorem 5. *Every admittance matrix of a multigraph is a (singular) M-matrix.*

Proof. Let A be an admittance matrix of some multigraph of order n. Then $A \in Z^{n,n}$. By Theorem 4 all its principal minors are nonnegative and hence by a result of Fiedler and Ptak [9] (see also [3]) the matrix A is an M-matrix.

Corollary 2. *Let G be a multigraph of order n. Then for some labelling of its vertices the admittance matrix A of G is factorizable as $A = BB^T$, where B is triangular and singular.*

Proof. This follows from our Theorem 5 and from Theorem 2 in [20].

Corollary 3. *The admittance matrix A of a connected graph is factorizable as $A = BB^T$, where B is triangular and singular.*

Proof. Since G is connected, the admittance matrix is irreducible. Theorem 5 together with Corollary 1 in [20] now imply Corollary 3.

Let $T=(V,E)$ be a tree and let U_1 and U_2 be disjoint subsets of V such that the subgraphs induced by U_1 and U_2 are trees. Let $A=A(T)$ be the admittance matrix of T.

The *distance* between two subgraphs of a graph is the minimal number of edges needed to traverse from one subgraph to the other within the given graph. We now have

Theorem 6. *The distance between $T(U_1)$ and $T(U_2)$ in T is given by*

$$d(T(U_1), T(U_2)) = \det A(U_1 \cup U_2 \,|\, U_1 \cup U_2). \tag{5}$$

Proof. If the subgraph of T induced by $U_1 \cup U_2$ is connected, then, since $U_1 \cap U_2 = \emptyset$, we have $d(T(U_1), T(U_2)) = 1$ and the right hand side of (5) equals one by Theorem 2 and the fact that T is a tree. If the subgraph of T induced by $U_1 \cup U_2$ is not connected, introduce an auxiliary edge e not in T connecting $T(U_1)$ and $T(U_2)$. There is a unique shortest path in T between $T(U_1)$ and $T(U_2)$. Let this path be the sequence of edges $e_1, e_2, \ldots, e_t$ with $t = d(T(U_1), T(U_2))$. Replacing e_i by e we get a unique spanning tree of $(T \backslash e_i) \cup e$ for every i, $i=1,2,\ldots,t$. The t distinct spanning trees of $T \cup e$ are the only spanning trees of $T \cup e$ containing $T(U_1) \cup T(U_2)$. Their number is given by the corresponding minor of the admittance matrix of T modified by e. But e has one vertex in U_1 and one in U_2, the corresponding minor is not affected by the addition of e and hence the number of trees is given by the minor of the original matrix T. Since t is the distance in T between $T(U_1)$ and $T(U_2)$, the theorem is proved.

Corollary 4. *Let T be a tree, A its admittance matrix. For $i,j \in V(T)$, $i \neq j$, let $d(i,j)$ be the distance (number of edges) in T between i and j. Then*

$$\det A(\{i,j\} \,|\, \{i,j\}) = d(i,j).$$

Proof. Put $i = U_1$, $j = U_2$ and apply Theorem 6.

Corollary 5. *Let $T=(V,E)$ be a tree, A its admittance matrix. Then $\det A(\{i,j\} \,|\, \{i,j\}) = 1$ if and only if $(i,j) \in E$.*

4. Some Special Cases

We start with a few simple lemmas.

Lemma 1. *Let x be some complex number and let s be some positive integer. We then have*

$$\det(xI_s - J_s) = (x-s)x^{s-1}.$$

Proof. Subtract the first row from every other row and add all the columns to the first one as it is done in the special case of Cayley's Theorem in [10, p. 154]. We obtain an upper triangular matrix whose determinant is $(x-s)x^{s-1}$. This proves the lemma.

For the special case of the complexity of K_n put $x=n$, $s=n-1$ and the theorem of Cayley follows.

Lemma 2. *Let x, y, z, u, v, w be arbitrary complex numbers. Let*

$$A=(a_{ij})=\begin{pmatrix} xI_p+yJ_p & zJ_{p,s} \\ uJ_{s,p} & vI_s+wJ_s \end{pmatrix}$$

be a matrix of order n in block form. Then

$$\det A=[(x+py)(v+sw)-szpu]x^{p-1}v^{s-1}. \tag{6}$$

The *proof* is a standard procedure similar to the one described in Lemma 1. Subtract the first row of A from each of the following rows down to the p-th row inclusive. Subtract the last row of A from each of the preceding up to the first row inclusive of the two lower blocks. Now add to the first column the sum of the remaining $p-1$ columns of the left blocks and add to the last column the sum of the remining $s-1$ columns of the right blocks. The result now follows quite easily.

Corollary 6. *Let p, s be positive integers. Then we have*

$$\det\begin{pmatrix} nI_p & -J_{p,s} \\ -J_{s,p} & mI_s \end{pmatrix}=m^{s-1}n^{p-1}(mn-ps).$$

Proof. Put $x=n$, $v=m$, $y=w=0$, $z=u=-1$ in Lemma 2 and the result follows.

Let $G=K_n$ and let T_t be a tree of order t. Applying Theorem 2 to this special case we find that α is a t-set. Applying Lemma 2 with $x=n$, $p=n-t$, $y=-1$, $s=0$ we get

$$\mu(T_t)=(x+py)x^{p-1}=tn^{n-t-1}.$$

We have thus proved

Corollary 7. *Let T be some fixed tree of order t in K_n. The number of spanning trees of K_n containing T is given by*

$$\mu(T)=tn^{n-t-1}.$$

For $t=1$ we get Cayley's result. For $t=2$ we get

Corollary 7′. *The number of spanning trees of K_n containing some fixed edge e is*

$$\mu(e)=2n^{n-3}.$$

Let $F(n,k)$ denote the number of forests on n labelled vertices consisting of k disjoint trees so that the first k vertices all belong to different trees. Already Cayley [6] established the equality

$$F(n,k)=kn^{n-k-1}. \tag{7}$$

This was also proved by Rényi [18]. Equality (7) follows immediately from Corollary 7 and the fact that the two problems have an identical solution.

Multiplying the result of Corollary 7 by t^{t-2} we get the following result.

Corollary 8. *Let α be a subset of t vertices of K_n. The number of those spanning trees of K_n whose subgraphs induced by α are connected (i.e. trees) is*

$$n^{n-2}\left(\frac{t}{n}\right)^{t-1}.$$

For $t=1$ and $t=n$ this is Cayley's result. For $t=2$ this is Corollary 7′.

Now let F be a spanning forest of K_n consisting of trees of two different orders q_1 and q_2. Let the number of trees of orders q_1 and q_2 equal r_1 and r_2 respectively. Then clearly $q_1r_1+q_2r_2=n$. Contract the trees to vertices and consider G'. The derived graph is of order r_1+r_2. Let the first r_1 vertices of G' be referred to the trees of the first order and the rest refer to the second order. The admittance matrix of G' is in block form

$$\begin{aligned} A' &= \begin{pmatrix} (q_1^2r_1+q_1q_2r_2)I_{r_1}-q_1^2J_{r_1} & -q_1q_2J_{r_1,r_2} \\ -q_1q_2J_{r_2,r_1} & (q_2^2r_2+q_1q_2r_1)I_{r_2}-q_2^2J_{r_2} \end{pmatrix} \\ &= \begin{pmatrix} q_1nI_{r_1}-q_1^2J_{r_1} & -q_1q_2J_{r_1,r_2} \\ -q_1q_2J_{r_2,r_1} & q_2nI_{r_2}-q_2^2J_{r_2} \end{pmatrix}. \end{aligned}$$

Applying Lemma 2 to $A'(1|1)$ with $x=q_1n$, $p=r_1-1$, $y=-q_1^2$, $v=q_2n$, $s=r_2$, $w=-q_2^2$, $z=-q_1q_2=u$, we get, after appropriate simplification,

$$\mu=q_1^{r_1}q_2^{r_2}n^{r_1+r_2-2}.$$

We have thus established the following

Theorem 7. *Let the complete graph K_n have a spanning forest F consisting of r_1 trees of order q_1 and r_2 trees of order q_2. Then the number of spanning trees of K_n containing F is*

$$q_1^{r_1}q_2^{r_2}n^{r_1+r_2-2}.$$

It should be noted that the trees of F which are of the same order need not be isomorphic.

Corollary 9. *Let F be a subforest of the complete graph K_n such that F consists of r trees of order q, then the number of spanning trees of K_n containing F is*

$$q^rn^{n-2-(q-1)r}.$$

Proof. Put $q_1=q$, $r_1=r$, $q_2=1$ in Theorem 7.

For $q=1$ this is again Cayley's result.

Corollary 10. *Let F be a subforest of K_n consisting of r copies of stars S_k. Then the number of spanning trees of K_n containing F is*

$$(k+1)^rn^{n-2-kr}.$$

Proof. Put $q=k+1$ in Corollary 9.

Corollary 11 (O'Neil [16]). *Let G be the complete graph of order n with k edges incident with the same vertex missing. Then*

$$\kappa(G)=n^{n-3}(1-1/n)^{k-1}(n-k-1). \tag{8}$$

Proof. Put $r=1$ in Corollary 10 and apply the inclusion exclusion principle. We get

$$\kappa(G)=n^{n-2}\sum_{i=0}^{k}(-1)^i(i+1)\binom{k}{i}n^{-i}.$$

Using well known combinatorial identities we easily obtain (8).

For $k=1$ the star degenerates into an edge and the forest becomes a matching. We thus have

Theorem 8. *Let H be a matching of r pairs of vertices of K_n. Then the number of spanning trees of K_n containing H is*

$$2^r n^{n-2-r}=(2/n)^r n^{n-2}.$$

Applying the inclusion-exclusion principle to Theorem 8 we obtain a result of Weinberg [23] on the *almost complete* graphs. (See also [4].)

Theorem 9 ([23]). *Let G be the graph obtained by removing r disjoint edges from K_n. Then*

$$\kappa(G)=n^{n-2}(1-2/n)^r.$$

Proof. We have

$$\kappa(G)=n^{n-2}\sum_{i=0}^{r}(-1)^i\binom{r}{i}(2/n)^i$$

$$=n^{n-2}\sum_{i=0}^{r}\binom{r}{i}(-2/n)^i=n^{n-2}(1-2/n)^r,$$

proving Theorem 9.

It should be noted that Theorem 9 as well as Corollary 11 are special cases of a result of Temperley [21].

Now let H be some fixed matching of K_n. What is the number of spanning trees of K_n for which H is a maximal matching? Put $V(G)\setminus V(H)=V'$. We note that in a tree that contains H as a maximal matching, the subgraph induced by V' is totally disconnected. Let H be a matching of order q so that $n=2q+r$. By putting $x=4q+2r=2n$, $y=-4$, $z=u=-2$, $v=2q$, $w=0$; $p=q-1$, $s=r$ in Lemma 2 we get

$$v(K_n, H_{\max})=2^{q+r-1}n^{q-1}q^{r-1}=2^{n-q-1}n^{q-1}q^{n-2q-1}.$$

We thus have

Theorem 10. *Let H be a matching of order q in K_n. Then there are $2^{n-q-1}n^{q-1}q^{n-2q-1}$ spanning trees in K_n containing H as a maximal matching.*

For $q=1$ we get 2^{n-2} such trees for $n\geqq 2$, which is quite obvious. For $q=2$ we get $4^{n-4}n$ such trees for $n\geqq 4$.

Treating the complete bipartite graph $K_{m,n}$ in a similar manner, we obtain the following auxiliary admittance matrix A' of order $m+n-q$ where q is the order of the matching.

$$A' = \begin{pmatrix} (m+n)I_q - 2qJ_{q,q} & -J_{q,m-q} & -J_{q,n-q} \\ -J_{m-q,q} & nI_{m-q} & -J_{m-q,n-q} \\ -J_{n-q,q} & -J_{n-q,m-q} & mI_{n-q} \end{pmatrix}.$$

Taking the minor of order $m+n-q-1$ and using the same principle as before, we get

$$v(K_{m,n}, q) = \det A'(1|1) = (m+n)^{q-1} n^{m-q-1} m^{n-q-1}(m+n-q). \tag{9}$$

For $q=0$ we get

$$v(K_{m,n}, 0) = \kappa(K_{m,n}) = m^{n-1} n^{m-1} \quad [19].$$

For $m=n$ we get

$$v(K_{n,n}, q) = 2^{q-1} n^{2n-q-3}(2n-q). \tag{10}$$

For $m=n=q$ we get

$$v(K_{n,n}, n) = 2^{n-1} n^{n-2}. \tag{11}$$

Again using the inclusion-exclusion principle, we get, for q missing edges from $2q$ vertices in $K_{m,n}$:

$$\sum_{r=0}^{q} (-1)^r \binom{q}{r} (m+n)^{r-1} m^{n-r-1} n^{m-r-1}(m+n-r).$$

Summing up and simplifying this yields

$$m^{n-q-1} n^{m-q-1}(mn-m-n)^{q-1}(mn-m-n+q).$$

We thus have

Theorem 11. *The complexity of the graph $G_{m,n,q}$ which is the complete bipartite graph $K_{m,n}$ with q edges missing from $2q$ vertices is*

$$\kappa(G_{m,n,q}) = m^{n-2} n^{m-2}(1-1/m-1/n)^{q-1}(mn-m-n+q).$$

For $m=n$ Theorem 11 yields

$$\kappa(G_{n,n,q}) = n^{2n-q-3}(n-2)^{q-1}(n^2-2n+q). \tag{12}$$

For $m=n=q$ we get

$$\kappa(G_{n,n,n}) = n^{n-2}(n-1)(n-2)^{n-1}. \tag{13}$$

It should be noted that for $n=1,2$ the graph $G_{n,n,n}$ is not connected.

The number of spanning trees containing a given matching H of order q of $K_{m,n}$ as a maximal matching is obtained by putting in Lemma 2, $x=m+n=N$, $y=-2$, $z=u=-1$, $v=q$, $w=0$, $p=q-1$, $s=N-q$. We get

Theorem 12. *Let H be a matching of order q in $K_{m,n}$. The number of spanning trees in $K_{m,n}$ containing H as a maximal matching is given by*

$$v(K_{m,n}, H) = (N+q-q^2)N^{q-2}q^{N-q-1}.$$

In [15] (see also [2]) the following problem is solved. Let $K_n=(X,E)$ be a simple complete graph of order n. Let X be partitioned into disjoint sets X_i so that $X=\bigcup_{i=1}^{p} X_i$ and let T_i be a tree of order n_i on X_i. How many spanning trees of K_n that contain every T_i as a subgraph are there? Contracting each tree to a vertex we obtain a multigraph G' of order p. The admittance matrix $A'=(a_{ij})$ will be as follows. $a_{ij}=n_i(n\delta_{ij}-n_j)$. ($\delta_{ij}$ is the Kronecker delta.) Now consider $\det A'(1|1)$. By adding all the rows to the first row of $A'(1|1)$ we can extract n_1 from the first row and n_k from column $k-1, k=2,\ldots,p$, of $A'(1|1)$. In the determinant that remains subtract the first column from every other, so we get a lower triangular matrix of order $p-1$ whose determinant is clearly n^{p-2}. It follows that the number of trees containing X_i is

$$n^{p-2}\prod_{i=1}^{p} n_i. \tag{14}$$

A slight generalization of Moon's result (14) is obtained if we do not require that the X_i partition X but merely that they be disjoint. Consider $X' =X\setminus\bigcup_{i=1}^{p} X_i$. Regarding the set X' as consisting of $|X'|=n-\sum_{i=1}^{p} n_i$ separate trees of order 1, we apply (14) to the modified partition and we get

Theorem 13. *Let $T_1, T_2, \ldots, T_p$ be disjoint trees in K_n. Then the number of spanning trees of K_n containing every T_i, $i=1,2,\ldots,p$ is given by*

$$v(T_1, T_2, \ldots, T_p)=n^{n-2-\sum_{i=1}^{p}(n_i-1)}\prod_{i=1}^{p} n_i=n^{n-2}\prod_{i=1}^{p} n_i n^{-(n_i-1)}.$$

We shall now generalize some previous results.

Lemma 3. *Let k be some positive integer and let a_i, y_i be arbitrary complex numbers for $i=1,2,\ldots,k$ and such that $y_i\neq 0$. Let $B=(b_{ij})$ be a $k\times k$ matrix with the following entries. $b_{11}=y_1+a_1$, $b_{i1}=-y_1$ and $b_{ii}=y_i$ for $i>1$; for $j>1$ $b_{1j} =a_j$; for i, j and for $j>1$, $i\neq j$, $b_{ij}=0$. Then*

$$\det B=\left(1+\sum_{i=1}^{k}(a_i/y_i)\right)\prod_{i=1}^{k} y_i.$$

Proof. By induction on k. For $k=1$ the lemma is true. Let the lemma hold for matrices of order less than k. Expand the determinant by the last column. We get, using the induction hypothesis,

$$\begin{aligned}\det B&=y_k\det B(k|k)+(-1)^k a_k(-1)^{k-1}(-y_1)\prod_{i=2}^{k-1} y_i\\ &=y_k\left(1+\sum_{i=1}^{k-1}(a_i/y_i)\right)\prod_{i=1}^{k-1} y_i+a_k\prod_{i=1}^{k-1} y_i\\ &=\left(1+\sum_{i=1}^{k-1}(a_i/y_i)\right)\prod_{i=1}^{k} y_i+(a_k/y_k)\prod_{i=1}^{k} y_i=\left(1+\sum_{i=1}^{k}(a_i/y_i)\right)\prod_{i=1}^{k} y_i,\end{aligned}$$

proving the lemma.

Now consider a matrix in block form with the blocks B_{ij} being of order $s_i \times s_j$. We then have

Lemma 4. *Let $B=(B_{ij})$ be a square matrix in block form with*

$$B_{ii}=x_i I_{s_i}, \quad B_{ij}=a_j J_{s_i,s_j} \quad \text{for } i \neq j \quad \text{where } x_i \text{ and } a_j$$

are complex numbers such that $x_i \neq s_i a_i$. Then

$$\det B = \left(1+\sum_{i=1}^{k} \frac{s_i a_i}{x_i - s_i a_i}\right) \prod_{i=1}^{k} (x_i - s_i a_i) x_i^{s_i-1}. \tag{15}$$

Proof. For some fixed i consider the submatrix of B consisting of the blocks B_{ij}. Subtract the first row of the submatrix from all the other rows of this submatrix. After having completed the subtraction of the rows over all the submatrices of this kind, consider the submatrix of B consisting of the blocks B_{ij} for some fixed j. Add all the columns of this submatrix to the first column of the submatrix. Having done this for all j we arrive at a matrix which, except for the first rows of the horizontal submatrices, has zeros everywhere except on the diagonal. We may therefore extract the factor $\prod_{i=1}^{k} x_i^{s_i-1}$ and are left with a $k \times k$ submatrix $C=(c_{ij})$ whose entries are $c_{ii}=x_i$, $c_{ij}=s_j a_j$ for $i \neq j$. Subtracting the first row of C from every other row we arrive at a matrix which satisfies the conditions of Lemma 3. Applying Lemma 3 to the matrix C we get (15) which proves the lemma.

Corollary 12. *Let $B=(B_{ij})$ be a square matrix in block form with $B_{ii}=x_i I_{s_i}$, $B_{ij}=-J_{s_i,s_j}$ for $i \neq j$, where the x_i are complex numbers such that $x_i \neq -s_i$. Then*

$$\det B = \left(1-\sum_{i=1}^{k} \frac{s_i}{x_i + s_i}\right) \prod_{i=1}^{k} x_i^{s_i-1}(x_i+s_i).$$

Proof. Put $a_i=-1$ for every i, $1 \leqq i \leqq k$ and apply Lemma 4.

Consider the star S_k in the complete bipartite graph $K_{m,n}$ with root in the m-part, so that $k \leqq n$. We obtain $\mu(S_k)$ by setting $x_1=n$, $s_1=m-1$, $x_2=m$, $s_2=n-k$ in Corollary 12. Put $m+n=N$. We then get

$$\begin{aligned}\mu(S_k) &= (1-(m-1)/(N-1)-(n-k)/(N-k))\, n^{m-2}(N-1)\, m^{n-k-1}(N-k)\\ &= [(m-1)k+n]\, m^{n-k-1}\, n^{m-2}.\end{aligned}$$

Applying the inclusion-exclusion principle and using well known combinatorial identities we get

Corollary 13. *Let G be the complete bipartite graph $K_{m,n}$ with k edges incident with one vertex from the m-set missing. Then*

$$\kappa(G) = m^{n-1} n^{m-1} (1-1/m)^{k-1} (1-1/m-k/n+k/mn).$$

Theorem 14. *Let $G=K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph. Let T be a subtree of G containing b_i vertices of the i-th part. Then the number of spanning* *trees of G containing T is given by*

$$\mu(T)=\left(1-\sum_{i=1}^{k}\frac{n_i-b_i}{n-b_i}\right)\prod_{i=1}^{k}(n-b_i)(n-n_i)^{n_i-b_i-1}. \tag{16}$$

Proof. The admittance matrix of $K_{n_1,n_2,\ldots,n_k}$ satisfies the conditions of Corollary 12. Applying Theorem 2 we are left with a principal submatrix also satisfying the conditions of Corollary 12. Putting $x_i=n-n_i$, $s_i=n_i-b_i$, $a_i=-1$, we get (16).

Here are some special cases.

Corollary 14 ([I]). *The complexity of the complete k-partite graph* $K_{n_1,n_2,\ldots,n_k}$ is *given by*

$$\kappa(K_{n_1,n_2,\ldots,n_k})=n^{k-2}\prod_{i=1}^{k}(n-n_i)^{n_i-1}$$

where $n=\sum_{i=1}^{k}n_i$.

Proof. Put $b_1=1$, $b_i=0$ for $i>1$ and substitute in (16).

Corollary 15. *Let e be an edge in the complete k-partite graph* $K_{n_1,n_2,\ldots,n_k}$ *with vertices in parts r and s. Then the number of spanning trees containing e is*

$$(n-1)[(n-n_r)^{-1}+(n-n_s)^{-1}]\,n^{k-3}\prod_{i=1}^{k}(n-n_i)^{n_i-1},$$

where $n=\sum_{i=1}^{k}n_i$.

Proof. Put $b_r=b_s=1$, $b_i=0$ for the rest in (16) and the result follows.

For $k=2$ we get

Corollary 16. *Let T be a tree of a complete bipartite graph* K_{n_1,n_2} *with* b_1 *vertices in one part and* b_2 *vertices in the other. Then the number of spanning trees containing T is*

$$\mu(T)=(n_1b_2+n_2b_1-b_1b_2)n_1^{n_2-b_2-1}\,n_2^{n_1-b_1-1}.$$

Corollary 16′ ([19]). *The complexity of the complete bipartite graph* K_{n_1,n_2} *is*

$$\kappa(K_{n_1,n_2})=n_1^{n_2-1}\,n_2^{n_1-1}.$$

Proof. Put $b_1=1$, $b_2=0$ in Corollary 16.

Consider the complete graph K_n. Let l be a positive integer less than n. Using the terminology of Clarke [7] a *tree of type l* is a tree in which precisely l edges end at a specified vertex. Let N_l denote the number of trees of type l. In [7] Cayley's result is based on the following theorem.

Theorem 15 ([7]). *The number of trees of type l in* K_n *is given by the formula*

$$N_l=\binom{n-2}{l-1}(n-1)^{n-l-1}.$$

We shall show that Theorem 15 is an immediate corollary of Theorem 2 and Lemma 1. Let T_l be the number of trees containing some fixed l-star. By Theorem 2 the minor considered is taken from a submatrix of order $n-l-1$, having $n-2$ on the main diagonal, since the set of vertices complementary to the l-star are precisely those which are not adjacent to the root of the star. Lemma 1 now prescribes, with $x=n-1$, $s=n-l-1$,

$$T_l = l(n-1)^{n-l-2}. \tag{17}$$

We get N_l by multiplying T_l by $\binom{n-1}{l}$, so that

$$N_l = \binom{n-1}{l} T_l = \binom{n-2}{l-1}(n-1)^{n-l-1}, \tag{18}$$

proving Theorem 15.

The lemma in [7] stating that $(n-1)N_{l-1}=(n-1)(l-1)N_l$ follows immediately from (18).

5. Some Combinatorial Identities

By counting trees in different ways it is possible to obtain combinatorial identities which are of interest in themselves. We shall provide but a few samples of the host of identities that may be derived. We first show the following identity.

Theorem 16. *Let r and n be integers such that $n \geqq r \geqq 2$. Then the following combinatorial identity holds.*

$$\sum_{i=0}^{n-r} \binom{n-r}{i} (i+1)^{i-1}(n-1-i)^{n-1-r-i} = \frac{r}{r-1} n^{n-1-r}. \tag{19}$$

Proof. Choose a fixed subtree T of K_n of order r and count the number of those spanning trees of K_n which contain T as a subtree. This is done by fixing some endvertex v of T. Let $V \setminus T = V_1$. Whenever a subset V_2 of V_1 forms a tree with v, the complement of V_2 in V_1 forms a tree with $T \setminus v$ and the two together form a spanning tree in K_n containing T. Counting all the possible ways to obtain such trees and using Corollary 7 leads us to

$$\begin{aligned}(r-1)(n-1)^{n-1-r} &+ \binom{n-r}{1} 2^0 (r-1)(n-2)^{n-2-r} + \dots \\ &+ \binom{n-r}{n-r-1}(n-r)^{n-r-2}(r-1)r^0 + (n-r+1)^{n-r-1} \\ &= (r-1)\sum_{i=1}^{n-r}\binom{n-r}{i}(i+1)^{i-1}(n-1-i)^{n-1-r-i}.\end{aligned}$$

Using Corollary 7 once again for the whole of T we get (19). This proves the theorem.

For $r=2$ we obtain the special case

$$\sum_{i=0}^{n-2} \binom{n-2}{i} (i+1)^{i-1}(n-1-i)^{n-3-i} = 2n^{n-3}. \tag{20}$$

Another idea of counting trees based on Corollary 7 is the following. Let r, m, n be fixed positive integers such that $r<m<n$. Put $m-r=s$. Choose two disjoint trees T_r and T_s of orders r and s respectively connected by an edge e. The tree T_m is a union of two disjoint subtrees T_r' and T_s' such that $T_r' \supset T_r$ and $T_s' \supset T_s$ and T_r' and T_s' are joined by e. By counting the number of all possible trees T_r' and applying Corollary 7 in the process of counting as well as at the final stage, we arrive at the following identity.

$$r(m-r) \sum_{i=0}^{n-m} \binom{n-m}{i} (r+i)^{i-1}(n-r-i)^{n-m-1-i} = mn^{n-m-1}. \tag{21}$$

It may be of interest to note that the left hand side of (21) is independent of r.

For $r=2$ and $m=4$ we get the special case

$$\sum_{i=0}^{n-4} \binom{n-4}{i} (2+i)^{i-1}(n-2-i)^{n-5-i} = n^{n-5}. \tag{22}$$

Putting $r=4$ in (19) we get

$$\sum_{i=0}^{n-4} \binom{n-4}{i} (i+1)^{i-1}(n-1-i)^{n-5-i} = (4/3)n^{n-5} \tag{23}$$

which is a different identity from (22).

We shall conclude with several examples derived from bipartite graphs. Consider the complete bipartite graph $K_{m,n} = (V_m \cup V_n, V_m \times V_n)$, $|V_m| = m$, $|V_n| = n$. Let $T_{r,s}$ be a tree on $r+s$ vertices in $K_{m,n}$ with r vertices in V_m and s vertices in V_n. Let $v \in T_{r,s}$ be some fixed vertex in V_m. Choose a tree, consisting of i vertices from $V_m \setminus T_{r,s} \cup v$ and j vertices from $V_n \setminus T_{r,s}$, count the trees on the complementary vertex set. Performing this construction over all possible choices of i vertices from $V_m \setminus T_{r,s}$ and j vertices from $V_n \setminus T_{r,s}$ and summing up for all admissible values of i and j we obtain the following identity based on the number of spanning trees of $K_{m,n}$ containing $T_{r,s}$ as a subtree.

Theorem 17. *Let m, n be arbitrary positive integers and let r and s be positive integers such that $r \leqq m$, $s \leqq n$. Then the following identity holds.*

$$\begin{aligned} &\sum_{j=0}^{n-s} \sum_{i=1}^{m-r+1} \binom{m-r}{i-1} \binom{n-s}{j} i^{j-1} j^{i-1} ((m-i)s + (n-j-s)(r-1)) \\ &\cdot (m-i)^{n-j-s-1}(n-j)^{m-i-r} = (ms+nr-sr)m^{n-s-1}n^{m-r-1}. \end{aligned} \tag{24}$$

For $r=1$ this yields

Corollary 17. *Let m and n be arbitrary positive integers and let s be a positive integer $\leqq n$. Then we have*

$$s\sum_{j=0}^{n-s}\sum_{i=1}^{m}\binom{m-1}{i-1}\binom{n-s}{j} i^{j-1} j^{i-1}(m-i)^{n-j-s}(n-j)^{m-i-1}$$
$$=(ms+n-s)m^{n-s-1}n^{m-2}. \tag{25}$$

It may be pointed out that for $m=1$ both sides of the equality (25) yield 1 if we bear in mind our convention $0^0=1$.

Another identity is based on the following idea. Consider the complete bipartite graph $K_{m,n}=(V_1\cup V_2,\ V_1\times V_2)$ which is not a star. Choose two vertices v_1, v_2 of V_1 and consider them fixed. Now count the spanning forests of $K_{m,n}$ consisting of precisely two trees such that v_1 and v_2 are never in the same component. The number of such spanning forests is equal to the number of spanning trees of $K_{m,n}$ augmented by the edge $e=(v_1, v_2)$ and containing e. We then come to the following identity.

Theorem 18. *Let m and n be arbitrary positive integers with $m\geqq 2$. Then we have*

$$\sum_{j=0}^{n}\sum_{i=1}^{m-1}\binom{m-2}{i-1}\binom{n}{j} i^{j-1} j^{i-1}(m-i)^{n-j-1}(n-j)^{m-i-1}$$
$$=2n^{m-2}m^{n-1}. \tag{26}$$

Proof. The left hand side of (26) is based on Corollary 16′ on trees of complete bipartite graphs. The right hand side is derived from Theorem 2. This completes the proof.

Setting $m=2$ in (26) we obtain the well known identity:

$$\sum_{j=0}^{n}\binom{n}{j}=2^n. \tag{27}$$

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Received April 24, 1981; received in final form March 22, 1982