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Global Pinching Theorems for Even Dimensional Minimal Submanifolds in the Unit Spheres

Lin, Jun-min and Xia, Chang-yu
Institut of Mathematics, Fudan University, Shanghai, People's Republic of China

In this paper, we prove the following result: Let M^{2n} be a minimal submanifold in the unit sphere with Euler characteristic no greater than two, then, if $\int_M S^n < c$, for some $c > 0$, M is totally geodesic. Here S is the square of the norm of second fundamental form. We also give a topological lower bound for $\int_M S^n$ in terms of Pontrjagin numbers.

§ 1. Introduction

Gauss-Bonnet formula establish a very powerful relation between the geometry and topology of a manifold. Lots of results have been obtained using Gauss-Bonnet formula as a key ingredient. But, this is done only within the intransic geometry of a manifold. Surely enough, it will play a key role in the theory of submanifolds. Here, we give an effort to obtain some consequences of Gauss-Bonnet formula.

Let $M^n \to S^{n+p}(1)$ be an oriented minimal immersion. Assume S is the norm of the second fundamental form of M, it is well known that the Simons' inequality can be used to obtain pinching theorems for S. For example, we have: If $S < n/(2 - 1/p)$, then, M is totally geodesic [6]. Further discussion in this direction have been carried out [3, 8]. All these discussion have pointwise condition for S. Hence, one may consider the global condition on S. By using eigenvalue estimates, Shen proved the following result:

Global Pinching Theorem [7]. Let $M^n \to S^{n+1}(1)$ be a minimal embedding with $\mathrm{Ric}_M \geqq 0$. Then, there exists an universal constant $c(n) > 0$, such that, if $\int_M S^{n/2} < c(n)$, M is totally geodesic.

In this paper, we will prove some global pinching theorems using topological information of the submanifold M instead of geometry condition like curvature pinching condition.

First, we have the following theorems for minimal surface or minimal hypersurface M^4 in the unit sphere.

Theorem A. *Let M^2 be an oriented compact minimal surface in the unit sphere S^n which is not totally geodesic. Then, there exists an universal constant $c>0$, such that:*

$$\int_M S \geqq g\pi g + c$$

where g is the genus of M.

Theorem B [5]. *Let M^4 be an oriented compact minimal hypersurface in $S^5(1)$. If M is not totally geodesic, then*

$$\int_M S^2 \geqq 64\pi^2(2-\chi)/3 + c,$$

for some universal constant $c>0$. Here χ is the Euler characteristic of M.

From Theorem A and B, we can get global pinching theorem for minimal submanifolds with Euler characteristic no greater than two in each case. We go further in this direction, and consider M^6 in the unit sphere. Then, we have following theorem:

Theorem C. *Let M^6 be an oriented compact minimal hypersurface in the unit sphere $S^7(1)$. If M is not totally geodesic, then:*

$$\int_M S^3 \geqq 2880\pi^3(2-\chi)/49 + c,$$

for some constant $c>0$. Here χ is the Euler characteristic of M.

If M^{2n} is an oriented compact minimal submanifold in the unit sphere, we can show that similar estimate holds.

Theorem D. *Let M^{2n} be an oriented compact minimal submanifold in $S^{2n+p}(1)$. If M^{2n} is not totally geodesic, then,*

$$\int_M S^n \geqq b(n,p)(2-\chi) + c(n,p),$$

for some constants $b(n,p)$, $c(n,p)>0$. Here χ is the Euler characteristic of M.

From this, one can easily show the following global pinching theorem.

Theorem E (Main Theorem). *Let M^{2n} be an oriented compact minimal submanifold in $S^{2n+p}(1)$. If the Euler characteristic of M is not greater than two, and $\int_M S^n < c(n,p)$, then, M is totally geodesic.*

Theorem A to D give lower bounds of $\int_M S^n$ in terms of Euler characteristic. Then, one may ask whether or not there is a similar bound in terms of other characteristic numbers. This is answered in the following theorem.

Theorem F. *Let M^{4k} be a compact minimal submanifold in $S^{4k+p}(1)$, and $P(M)$ is one of its Pontrjagin number. Then, there exists a constant $c(k, p, P) > 0$, such that:*

$$\int_M S^{2k} \geqq c(k, p, P)\, P(M).$$

This paper is arranged as follows: In § 2, we give facts that are used in our proof and a basic lemma (generalized Simons' inequality). In § 3, we review the proof of Theorem A and B, then, give the proof of Theorem C. In § 4, we prove Theorem D and our main theorem, Theorem E. In our last section, § 5, we prove Theorem F.

§ 2. Preliminaries

In this section, we recall some well known facts and prove a generalization of Simons' inequality which is needed in the proof of our theorems.

Let M^n be an n-dimensional Riemannian manifold. As usual, we denote by R_{ijkl}, R_{ij} and ρ be its curvature tensor, Ricci curvature tensor and scalar curvature under orthonormal frame near a point, respectively. The Einstein tensor E of M is defined by:

$$E_{ij} = R_{ij} - \rho\, \delta_{ij}/n.$$

Let $M^n \to S^{n+p}(1)$ be a minimal immersion. Assume h^{α}_{ij} be the second fundamental form of M, where $i, j, k, l = 1, \ldots, n$, $\alpha = n+1, \ldots, n+p$. The minimal condition can be read as:

$$\sum_{i=1}^{n} h^{\alpha}_{ii} = 0, \qquad \alpha = n+1, \ldots, n+p.$$

The Gauss' equations can be written as:

$$R_{ijkl} = \delta_{ik}\, \delta_{jl} - \delta_{il}\, \delta_{jk} + h^{\alpha}_{ik}\, h^{\alpha}_{jl} - h^{\alpha}_{il}\, h^{\alpha}_{jk}.$$

We also have the lower bound for the volume of minimal submanifolds in the unit sphere:

Proposition [2]. *Let $M^n \to S^{n+p}(1)$ be a compact minimal submanifold. If M is not totally geodesic, then, there exists a constant $c(n) > 0$, such that:*

$$V(M) > (1 + c(n))\, V(S^n(1)).$$

Let M^{2n} be an oriented Riemannian manifold with Euler characteristic χ and curvature form $\Omega_{ij} = \frac{1}{2} \sum_{k,l} R_{ijkl}\, \omega^k\, \omega^l$, where ω^k is the orthonormal coframe of M. Then, we have Gauss-Bonnet formula.

Proposition [4].

$$2^{2n}\pi^n n!\,\chi=\int_M \varepsilon_{i_1 i_2 \ldots i_{2n}} \Omega_{i_1 i_2} \wedge \ldots \wedge \Omega_{i_{2n-1} i_{2n}}.$$

Lemma (Generalized Simons' inequality). *Let $M^n \to S^{n+p}(1)$ be a closed minimal submanifold, then, for any integer $t \geqq 0$, we have:*

$$\int_M S^{t+1}(S-n/(2-1/p))\,dv_M \geqq 0.$$

Proof. It follows from [6] that

$$\tfrac{1}{2}\Delta S \geqq S(n-(2-1/p)S).$$

For any $\varepsilon>0$, we have:

$$(\tfrac{1}{2}\Delta S)(S+\varepsilon)^t \geqq S(n-(2-1/p)S)(S+2)^t.$$

Since

$$\Delta(S+\varepsilon)^{t+1}=(t+1)\,t(S+\varepsilon)^{t-1}|\nabla S|^2+(t+1)(S+\varepsilon)^t \Delta S,$$

we have:

$$\begin{aligned}
&\int_M S(n-(2-1/p)S)(S+\varepsilon)^t\\
&\quad \leqq \tfrac{1}{2}\int_M (S+\varepsilon)^t \Delta S\\
&\quad \leqq \int_M \Delta(S+\varepsilon)^{t+1}/(2t+2)-t(t+1)(S+\varepsilon)^{t-1}|\nabla S|^2/2\\
&\quad \leqq 0.
\end{aligned}$$

The last inequality above follows from Green's formula and the assumption $t \geqq 0$. Thus, we obtain:

$$\int_M S(n-(2-1/p)S)(S+\varepsilon)^t \leqq 0.$$

Letting $\varepsilon \to 0$, we conclude the proof of the lemma.

§3. The Proves of Theorem A, B and C

In this section, we first give a proof of Theorem A, then, we prove Theorem C. For the proof of Theorem B, one may consult [5], since the proof is similar to the proof we give here.

Proof of Theorem A. From Gauss' equation, we have that for minimal surface in the unit sphere

$$S=2-2K,$$

where K is the Gauss curvature of the surface. Then, we have:

$$\int_M S = 2V(M) - 8\pi(1-g).$$

Combining with the lower bound of $V(M)$ in § 2, we have:

$$\int_M S \geqq 8\pi g + c.$$

Proof of Theorem C. From Gauss-Bonnet formula, we have:

$$2^6 \pi^3 3!\, \chi = \int_M \varepsilon_{i_1 \ldots i_6} \Omega_{i_1 i_2} \wedge \Omega_{i_3 i_4} \wedge \Omega_{i_5 i_6}.$$

But

$$\begin{aligned} &\varepsilon_{i_1 \ldots i_6} \Omega_{i_1 i_2} \wedge \Omega_{i_3 i_4} \wedge \Omega_{i_5 i_6} \\ &= \tfrac{1}{8} \varepsilon_{i_1 \ldots i_6} R_{i_1 i_2 j_1 j_2} R_{i_3 i_4 j_3 j_4} R_{i_5 i_6 j_5 j_6} \omega^{j_1} \wedge \ldots \wedge \omega^{j_6} \\ &= 6!\left(1 + \tfrac{1}{5} \sum_{i<j} \lambda_i \lambda_j + \tfrac{1}{5} \sum_{i<j<k<l} \lambda_i \lambda_j \lambda_k \lambda_l + \prod_{i=1}^{6} \lambda_i\right) dv_M, \end{aligned}$$

here, λ_i are the principal curvature of M. And, we also have used Gauss' equation:

$$R_{ijkl} = (1 + \lambda_k \lambda_l)(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}).$$

Since M is a minimal submanifold, we have

$$\sum_i \lambda_i = 0.$$

Hence, we have:

$$\sum_{j<k} \lambda_j \lambda_k = -\tfrac{1}{2} S.$$

As is well known, $\sum_{i<j<k<l} \lambda_i \lambda_j \lambda_k \lambda_l$ can be expressed by linear combination of $(\sum \lambda_i^2)^2$ and $(\sum \lambda_i^4)$ (Notice that we have $\sum \lambda_i = 0$). Hence, we can assume:

$$\sum_{i<j<k<l} \lambda_i \lambda_j \lambda_k \lambda_l = a S^2 + b \sum \lambda_i^4.$$

Let $\lambda_1 = \ldots = \lambda_5 = 1$, $\lambda_6 = -5$. Then,

$$-45 = 900a + 630b.$$

Let $\lambda_1 = \ldots = \lambda_3 = 1$, $\lambda_4 = \ldots = \lambda_6 = -1$. We get

$$3 = 36a + 6b.$$

Hence, $a = \frac{1}{8}$, $b = -\frac{1}{4}$. That is

$$\sum_{i<j<k<l} \lambda_i \lambda_j \lambda_k \lambda_l = \tfrac{1}{8}(S^2 - 2 \sum \lambda_i^4).$$

Hence, we have the following formula:

$$\int_M (-720\lambda_1\ldots\lambda_6 + 36(\sum \lambda_i^4) - 18S^2 + 72S) = 6!\,V(M) - 2^6\pi^3\,3!\,\chi.$$

Lemma. $-\lambda_1\ldots\lambda_6 \leqq S^3/216$.

Proof. We use the Lagrange multiple method to calculate the maximum of the function $f = -\mu_1\ldots\mu_6$ under the constraints $\sum\mu_i = 0$ and $\sum\mu_i^2 = S$. Let $\bar{f} = -\mu_1\ldots\mu_6 + \lambda(S - \sum\mu_i^2) + \mu\sum\mu_i$. Then, at the maximum point of f, we have:

$$\begin{gathered} -\mu_2\ldots\mu_6 - 2\lambda\mu_1 + \mu = 0 \\ \vdots \\ -\mu_1\ldots\mu_5 - 2\lambda\mu_6 + \mu = 0. \end{gathered}$$

Thus $-\mu_1\ldots\mu_6 = \lambda S/6$. We can assume $\mu_1\ldots\mu_6 \neq 0$. Hence, μ_i, $i = 1, \ldots, 6$ are the roots of the following equation:

$$6\lambda\chi^2 - 3\mu\chi - \lambda S = 0.$$

It follows from direct calculation that;

$$\max(-\mu_1\ldots\mu_6) = S^3/216,$$

which is reached by:

$$\mu_1 = \mu_2 = \mu_3 = -\mu_4 = -\mu_5 = -\mu_6 = \sqrt{S/6}.$$

Lemma. $\sum\lambda_i^4 \leqq 7S^2/10$.

Proof. We use the Lagrange multiple method to calculate the maximum of $\sum\mu_i^4$ under the constraints $\sum\mu_i = 0$ and $\sum\mu_i^2 = S$. Let $f = \sum\mu_i^4 + \lambda(S - \sum\mu_i^2) + \mu\sum\mu_i$. Then, at the maximal point of $\sum\mu_i^4$, we have:

$$4\mu_i^3 - 2\lambda\mu_i + \mu = 0.$$

It follows from direct calculation that:

$$\max(\sum\mu_i^4) = 7S^2/10,$$

which is reached by:

$$\mu_1 = \mu_2 = \ldots = \mu_5 = \sqrt{S/30}, \qquad \mu_6 = -\sqrt{5S/6}.$$

From the lemmas, we have:

$$\int_M 10S^3/3 + 36S^2/5 + 72S \geqq 6!\,V(M) - 2^6\pi^3\,3!\,.$$

From the generalized Simons' inequalities, we have:

$$\int_M 72S \leqq 2\int_M S^3,$$

$$\int_M 36S^2/5 \leqq \int_M 6S^3/5.$$

Hence, we have (using the lower bound of $V(M)$):

$$\int_M S^3 \geqq 2880\pi^3(2-\chi)/49 + c$$

for some constant $c > 0$. Here χ is the Euler characteristic of M.

§4. The Proof of the Main Theorem

In this section, we prove Theorem D, our main theorem, Theorem E, is a consequence of Theorem D.

Proof of Theorem D. From Gauss-Bonnet formula, we have

$$2^{2n}\pi^n n!\,\chi = \int_M \varepsilon_{i_1\ldots i_{2n}} \Omega_{i_1 i_2} \wedge \ldots \wedge \Omega_{i_{2n-1} i_{2n}},$$

where $\Omega_{ij} = \frac{1}{2} R_{ijkl}\,\omega^k\,\omega^l$ is the curvature form. Since $M^{2n} \to S^{2n+p}(1)$ is a immersion, we know:

$$R_{ijkl} = \delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk} + h^\alpha_{ik}\,h^\alpha_{jl} - h^\alpha_{il}\,h^\alpha_{jk}$$

by Gauss' equation. Hence

$$\varepsilon_{i_1\ldots i_{2n}} \Omega_{i_1 i_2} \wedge \ldots \wedge \Omega_{i_{2n-1} i_{2n}} = ((2n)! + p(h))\,dv_M$$

where $p(h)$ is a fixed polynomial of h^α_{ij} which satisfies:

(i) $p(h)$ is of degree no greater than $2n$,

(ii) $p(h)$ is even polynomial of h,

(iii) $p(h)$ has no constant term.

Let $S^{2n} \to S^{2n+p}(1)$ be a totally geodesic embedding, we have $h=0$ in the above equality. Hence, we have:

$$(2n)!\; V(S^{2n}) = 2^{2n}\pi^n n!\,2.$$

Using the lower bound of $V(M)$ in §2, we have:

$$V(M) > V(S^{2n}(1)) + c(n)$$

if M is not totally geodesic. Hence

$$-\int_M p(h)\,dv_M \geqq c_1(n)(2-\chi) + c_2(n).$$

Since $p(h)$ is a fixed polynomial satisfying condition (i)–(iii), we have:

$$-p(h) \leqq c_3 S^n + c_4 S,$$

for some constants $c_3(n, p)$ and $c_4(n, p)$. By generalized Simons' inequality, we have:

$$\int_M S^n \geqq b(n, p)(2-\chi) + c(n, p).$$

Proof of Theorem E. This follows directly from Theorem D and the assumption $\chi \leqq 2$.

§ 5. Topological Lower Bound of $\int_M S^{\dim(M)/2}$

It is well know that the only totally geodesic n-dimensional submanifold of $S^{n+p}(1)$ is just the great circle. Notice that $S^n(1)$ has all its Pontrjagin number zero and $S^{n/2}=0$. Using this and the topological lower bound of Yang-Mills functional [1] as phototype, we can get a lower bound of $\int_M S^{\dim(M)/2}$ in terms of Pontrjagin number. This is just the content of Theorem F.

Proof of the Theorem F. We may assume $P(M) \geqq 0$, otherwise we use $-P$ instead of P. By Chern-Weil theory of characteristic class (4), we know $P(M)$ is the integral of some fixed curvature expressions (depend only on P). By Gauss' equation

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk};$$

we know:

$$P(M) = \int_M c(P)\, dv + \int_M f_P(h)\, dv.$$

Here, $c(P)$ is a constant depend only on the class P, $f_P(h)$ is a polynomial satisfies:

(i) $f_P(h)$ is a polynomial depending only on P,

(ii) $f_P(h)$ is of degree $2k$,

(iii) $f_P(h)$ is even with respect to h,

(iv) $f_P(h)$ has no constant term.

Now, let $M = S^{4k}$ be the totally geodesic embedding, then, we have $0 = c(P)\, V(S^{4k})$. Hence $c(P)=0$. From conditions (i)–(iv), we have;

$$f_P(h) \leqq c_1 S + c_2 S^{2k}$$

for some constant $c_i(k, p, P) > 0$. By generalized Simons' inequality, we have:

$$\int_M S^{2k} \geqq c_3 \int_M S.$$

Hence,

$$\int_M S^{2k} \geqq c(k, p, P)\,|P(M)|.$$

Remark. The above inequality become equality when M is totally geodesic submanifold. But, if one want to get the best constant in the above inequality, one must use some other method. This is the main difference between the topological lower bound of Yang-Mills functional in [1].

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