Math. Z. 178, 429–442 (1981)

Mathematische Zeitschrift


# Perturbation Classes of Semi-Fredholm Operators

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## 1. Introduction

Around 1956 Kato improved the classical perturbation theorems for densily defined, closed semi-Fredholm operators. In [14] he called a bounded, linear operator $T: X \to Y$ in Banach spaces $X$ and $Y$ *strictly singular* ($T \in \mathfrak{S}(X, Y)$) if the restriction of $T$ to any infinite-dimensional subspace of $X$ is not an isomorphism; and he showed that not only compact operators but also strictly singular operators $T \in \mathfrak{S}(X, Y)$ are *admissible $\Phi_+$-perturbations* ($T \in \mathfrak{F}_+(X, Y)$), i.e. $S+T: X \to Y$ is a $\Phi_+$-operator whenever $S: X \to Y$ is one. A class of operators in a certain sense dual to strictly singular operators was introduced by Pelczynski in [19]: he called a bounded linear operator $T: X \to Y$ *strictly cosingular* ($T \in \mathfrak{C}\mathfrak{S}(X, Y)$) if the composition $\Phi T$ with any infinite-dimensional quotient map $\Phi$ on $Y$ is not surjective. Later it was shown by Vladimirskii that all strictly cosingular operators are *admissible $\Phi_-$-perturbations* ($T \in \mathfrak{F}_-(X, Y)$), i.e. $S+T: X \to Y$ is a $\Phi_-$-operator whenever $S$ is one. The classes $\mathfrak{F}_+(X, Y)$ and $\mathfrak{F}_-(X, Y)$ where first studied in their own right by Gohberg, Markus and Feldman in [5] but the question of whether strictly singular operators are the maximal class of admissible $\Phi_+$-perturbations was left open (see [5], p. 74). It is also not known if all admissible $\Phi_-$-perturbations are strictly cosingular. These questions are not only of interest because a positive answer would provide a topological characterization of the perturbation classes $\mathfrak{F}_+$ and $\mathfrak{F}_-$, but also because $\mathfrak{F}_+$ and $\mathfrak{F}_-$ form an operator ideal as defined by Pietsch if and only if $\mathfrak{S}(X, Y) = \mathfrak{F}_+(X, Y)$ and $\mathfrak{C}\mathfrak{S}(X, Y) = \mathfrak{F}_-(X, Y)$ for all Banach spaces $X$ and $Y$.

There are some partial results in this direction. It follows already from [5], §5 that $\mathfrak{S}(X) = \mathfrak{F}_+(X)$ and $\mathfrak{C}(X) = \mathfrak{F}_-(X)$ for $X = l_p$, $1 \leq p < \infty$. The same is true for $X = L_p(\Omega, \mu)$ or $X = C[0, 1]$ according to [18] and [32]. In this paper we give a positive answer for a large class of Banach spaces, including most classical Banach spaces, and we reduce the general question to some long unsolved problems in Banach space theory. More precisely:

**Theorem A.** *If $X$ is weakly compactly generated, then $\mathfrak{F}_+(X) = \mathfrak{S}(X)$ and $\mathfrak{F}_-(X) = \mathfrak{C}\mathfrak{S}(X)$.*

0025-5874/81/0178/0429/$02.80

Recall that a Banach space $X$ is *weakly compactly generated (w.c.g.)* if the linear span of some weakly compact subset is dense in $X$. Hence all separable and all reflexive Banach spaces are w.c.g. as well as $L_1(\Omega, \mu)$ if $(\Omega, \mu)$ is $\sigma$-finite. (For a discussion of w.c.g. spaces, see [3], Chap. 5.)

Our proofs also work for some more general classes of Banach spaces as detailed in Sect. 3. They imply for example that $\mathfrak{F}_-(X)=\mathfrak{C}\mathfrak{S}(X)$ for all Banach spaces if every Banach space had a separable quotient space. Wether or not this is true still seems to be unknown. Dealing with non-endomorphisms leads to another open problem, namely whether each infinite dimensional Banach space $X$ contains, infinite dimensional subspaces $M$ and $N$ such that $M+N$ is closed in $X$ (and $M+N/N$, $M+N/M$ are infinite dimensional).

**Theorem B.** *Let $Y$ be weakly compactly generated.*

a) $\mathfrak{F}_+(X, Y)=\mathfrak{S}(X, Y)$ *for all Banach spaces $X$ if and only if $Y$ contains at least two infinite dimensional subspaces $M$ and $N$ such that $M \cap N=\{0\}$ and $M+N$ is closed in $Y$.*

b) $\mathfrak{F}_-(Y, X)=\mathfrak{C}\mathfrak{S}(Y, X)$ *for all Banach spaces $X$ if and only if $Y$ contains at least two infinite codimensional subspaces $M$ and $N$ such that $M+N=Y$.*

Theorem A and B are proved in Sect. 3. Section 2 includes the necessary properties of subspaces $M$ and $N$ with $M+N$ closed. We also use a theorem on quasi-complements by Lindenstrauß, Rosenthal and Johnson from [12].

Actually in [18] and [32] a stronger result than $\mathfrak{F}_+(X)=\mathfrak{S}(X)$ and $\mathfrak{F}_-(X)=\mathfrak{C}\mathfrak{S}(X)$ was proved for $X=L_p(\Omega, \mu)$ or $X=C[0, 1]$, namely that $\mathfrak{S}(X)$ and $\mathfrak{C}\mathfrak{S}(X)$ both equal the class of admissible Fredholm perturbations. In Sect. 4 we give some examples showing that this cannot be extended to all $C(K)$-spaces or to function spaces "close" to $L_p$. As a contrast to the "constructed" operators used so far we give in Sect. 5 a "natural" example of a non-compact admissible Fredholm perturbation: we show that for $1<p<2$ the Fouriertransform $\mathscr{F}: L_p(G, m) \to L_{p'}(\Gamma, n)$, $\frac{1}{p}+\frac{1}{p'}=1$, on a locally compact group $G$ is strictly singular and strictly cosingular. In particular, the range of $\mathscr{F}$ on $L_p$ does not contain an infinite dimensional subspace closed in $L_{p'}$.

*Notation.* $X$ and $Y$ always denote Banach spaces and $\mathfrak{B}(X, Y)$ stands for the space of all bounded linear operators $T: X \to Y$. For a closed subspace $M$ of $X$, $\Phi_M$ is the quotient map $\Phi_M: X \to X/M$, $x \to \hat{x}$. If $S: X \to Y$ is a closed operator we denote its domain by $\mathscr{D}(S)$ and the Banach spaces $X_s$ is $\mathscr{D}(S)$ with the graph norm $\|x\|_s=\|x\|+\|Sx\|$. Let $S: X \to Y$ be a densely defined, closed operator $S: X \to Y$ with a closed range. Then $S$ is a *$\Phi_+$-operator* ($S \in \Phi_+(X, Y)$) if $S$ has a finite dimensional kernel, and $S$ is a *$\Phi_-$-operator* ($S \in \Phi_-(X, Y)$) if its range has finite codimension in $Y$. $\Phi(X, Y)=\Phi_+(X, Y) \cap \Phi_-(X, Y)$ is the class of Fredholm-operators. Observe that $\Phi(X, Y) \neq \varnothing$ if $X$ is separable and $Y'$ is separable in the $w^*$-topology (indeed, by [7] there is an injective, compact operator $K: Y \to X$ and $K^{-1}$ defines a Fredholm-operator). If $\Phi(X, Y) \neq \varnothing$, the class $\mathfrak{F}(X, Y)$ of admissible Fredholm-perturbations consists of all operators $T \in \mathfrak{B}(X, Y)$ such that $S+T$ is a Fredholm operator whenever $S \in \Phi(X, Y)$. If $\Phi(X, Y)=\varnothing$ we put $\mathfrak{F}(X, Y)=\{T \in \mathfrak{B}(X, Y)$: $Id_x+ST \in \Phi(X)$ *for all* $S \in \mathfrak{B}(Y, X)\}$.

We defined above $\mathfrak{F}_+(X, Y)$ and $\mathfrak{F}_-(X, Y)$ for $\Phi_+(X, Y) \neq \emptyset$ and $\Phi_-(X, Y) \neq \emptyset$, respectively. If $\Phi_+(X, Y) = \emptyset$ we put $\mathfrak{F}_+(X, Y) = \mathfrak{S}(X, Y)$ and for $\Phi_-(X, Y) = \emptyset$ we set $\mathfrak{F}_-(X, Y) = \mathfrak{C}\mathfrak{S}(X, Y)$. The latter definitions differ from [5] but they are chosen to allow for the possibility that $\mathfrak{F}_+$, $\mathfrak{F}_-$ and $\mathfrak{F}$ define operator ideals. Sometimes (e.g. [27]) the symbols $\mathfrak{F}_+$, $\mathfrak{F}_-$, $\mathfrak{F}$ are used for the perturbation classes of *bounded* (Semi-) Fredholm operators. Except for $\mathfrak{F}(X)$ it seems to be unknown if the two kinds of perturbation classes are the same.

For basic properties of strictly singular and strictly cosingular operators we refer to [6, 15 and 22].

## 2. Preliminaries on Perpendicular Subspaces

Two subspaces $M$ and $N$ of a Banach space $X$ are called *perpendicular* (see [24], p. 20, or "pseudo-complemented" in [29]) in short $M \perp N$, if $M \cap N = \{0\}$ and $M+N$ is closed in $X$. In 2.1. we collect some known properties of perpendicular subspaces which we will use many times in the sequel.

2.1. **Proposition.** *For infinite dimensional subspaces $M$ and $N$ of a Banach space $X$ the following assertions are equivalent:*

a) *$M$ and $N$ are perpendicular*

b) *$i: M \oplus N \to X$, $i(x, y) = x + y$ is an isomorphism into $X$, a.e. there is a constant $C > 0$ such that $\|x+y\| \geqq C\|x\|$ for all $x \in M$, $y \in N$.*

c) $M^\perp + N^\perp = X'$

d) *$M \cap N = \{0\}$ and there is a $\varepsilon > 0$ such that no infinite dimensional subspace $\tilde{M}$ of $M$ has an embedding $J: \tilde{M} \to N$ with $\|Jx - x\| \leqq \varepsilon \|x\|$ for all $x \in \tilde{M}$.*

a)$\Leftrightarrow$b) follows from the open-mapping theorem; a proof of a)$\Leftrightarrow$d) using basic sequences is implicit in [23], Theorem 1, but 2.1. can also be shown in a way similar to 2.2. which deals with the "dual" property $M+N=X$.

2.2. **Proposition.** *For infinite codimensional subspaces $M$ and $N$ of a Banach space $X$ the following assertions are equivalent:*

a) $M+N=X$

b) *$\Phi: X \to X/M \oplus X/N$, $\Phi(x) = (\Phi_M x, \Phi_N x)$, is surjective.*

c) *$M^\perp$ is perpendicular to $N^\perp$ in $X'$.*

d) *$N^\perp \cap M^\perp = \{0\}$ and there is a $\varepsilon > 0$ such that no w*-closed, infinite dimensional subspace $\tilde{M}$ of $M^\perp$ has a w*-continuous embedding $J: \tilde{M} \to N^\perp$ with $\|Jx' - x'\| \leqq \varepsilon \|x'\|$ for all $x' \in \tilde{M}$.*

*Proof.* a)$\Leftrightarrow$b) This follows from $\operatorname{Im}\Phi = \{(\Phi_M x, \Phi_N y): x - y \in M+N\}$ which is easy to verify.

b)$\Rightarrow$c) We have $\Phi': N^\perp \oplus M^\perp \to X'$, $\Phi'(x', y') = x' + y' \cdot \Phi'$ is an isomorphism because $\Phi$ is surjective (see [22], C.I. Theorem 2.7).

c)$\Rightarrow$d) This is clear since there is a $C > 0$ such that $\|x' + y'\| \geqq C\|x'\|$ for all $x' \in M^\perp$, $y' \in N^\perp$.

c)⇒b) Since $(M+N)^{\perp}=M^{\perp}\cap N^{\perp}=\{0\}$, $M+N$ is dense in $X$ and $\operatorname{Im}\Phi=\{(\Phi_M x, \Phi_N y): x-y\in M+N\}$ is dense in $X/M\oplus X/N$. Assume that $\Phi$ is not surjective. Then $\Phi': M^{\perp}\oplus N^{\perp}\to X'$ is not a bounded $\Phi_+$-operator (see [22], C.I. Theorem 2.7'). By C.III. Lemma 6.2. and its proof there is for every $\varepsilon>0$ a $w^*$-closed subspace $L$ of $M^{\perp}\oplus N^{\perp}$ such that $\|\Phi'|_L\|\leqq\varepsilon$.

For all $x' M^{\perp}$, $y'\in N^{\perp}$ with $(x', y')\in L$ we have

$$2\|x'\|\geqq\|x'\|+\|y'\|-\|x'+y'\|=\|(x', y')\|-\|\Phi'(x', y')\|\geqq(1-\varepsilon)\,\|(x', y')\|$$

and similarly: $2\|y'\|\geqq(1-\varepsilon)\,\|(x', y')\|$.

If $P: M^{\perp}\oplus N^{\perp}\to M^{\perp}$ and $Q: M^{\perp}\oplus N^{\perp}\to N^{\perp}$ denote the natural projections it follows that $M_1=P(L)$ and $N_1=Q(L)$ are $w^*$-closed subspaces of $M^{\perp}$ and $N^{\perp}$ resp., and $J=-P_2(P_1|_{M_1})^{-1}: M_1\to N_1$ is a $w^*$-isomorphism with

$$\|Jx'-x'\|\leqq\varepsilon\|(x', -Jx')\|\leqq\frac{2\varepsilon}{1-\varepsilon}\|x'\|$$

because $(x', -Jx')\in L$. But this contradicts d).

It still seems to be unknown if every Banach space $X$ contains two perpendicular infinite dimensional subspaces or two infinite codimensional subspaces $M$ and $N$ with $M+N=X$ (see [29], Problem 3, [1], p. 101). But such subspaces do exist if the space of bounded operators defined on $X$ is rich enough, e.g.

2.3. **Corollary.** a) *If $T\in\mathfrak{B}(X, Y)$ is neither strictly singular nor a bounded $\Phi_+$-operator, then $X$ contains two infinite dimensional perpendicular subspaces.*

b) *If $T\in\mathfrak{B}(X, Y)$ is neither strictly cosingular nor a bounded $\Phi_-$-operator, then there are infinite codimensional subspaces $M$ and $N$ of $Y$ with $M+N=Y$.*

*Proof.* b) If $\Phi_M T: X\to Y/M$ is surjective, there is an $\alpha>0$ such that $\|T'x'\|\geqq\alpha\|x'\|$ for all $x'\in M^{\perp}$.

Since $T$ is not a $\Phi_-$-operator, $T': Y'\to X'$ is not a $\Phi_+$-operator ([22], C.I. 2.7') and by [22], C.II. 6.2., and its proof there is a $w^*$-closed infinite dimensional subspace $L$ of $Y'$ such that $\|T'x'\|\leqq(\alpha/2)\,\|x'\|$ for $x'\in L$. For $x'\in M^{\perp}$, $y'\in L$ we have

$$\|T\|\cdot\|x'+y'\|\geqq\|T'x'+T'y'\|\geqq\|T'x'\|-\|T'x'\|\geqq(\alpha/2)\,\|x'\|.$$

Hence $M$ and $L$ are perpendicular and from 2.2. we conclude $M+L^{\perp}=Y$.

a) can be shown in the same way.

If $X=Y$ the following variation of 2.3. gives some useful additional information.

2.4. **Lemma.** *Let $T\in\mathfrak{B}(X)$ be a Rieszoperator, e.a. $\lambda Id-T\in\Phi(X)$ for all $\lambda\in\mathbb{C}$, $\lambda\neq0$.*

a) *If $T$ is not strictly singular, there is an infinite dimensional subspace $M$ of $X$ such that $T|_M$ is an isomorphism and $M$ and $T(M)$ are perpendicular.*

b) *If $T$ is not strictly cosingular, there is an infinite codimensional subspace $M$ of $X$ such that $\Phi_M T$ is surjective and $M+T^{-1}(M)=X$.*

*Proof.* a) In [28] Schechter introduced the quantity $\tau(T)=\sup_L \inf\{\|Tx\|: x\in L;\ \|x\|=1\}$, where the supremum is taken over all infinite dimensional subspaces $L$ of $X$, and he showed that $\lim_n \tau(T^n)^{1/n}$ is the Fredholm radius of $T$. Hence in our case: $\tau(T^n)^{1/n}\to 0$. Now choose a subspace $N$ of $X$ and a $\alpha>0$ such that $\|Tx\|\geqq\alpha\|x\|$ for all $x\in N$.

$$\text{Then there exists a } n\in\mathbb{N} \text{ and a subspace } \tilde{M} \text{ of } T^n(N) \tag{$*$}$$

such that $\tau(T|_{\tilde{M}})<\alpha$.

Otherwise one could select inductively a sequence $(M_i)$ of subspaces such that $M_0=N$, $M_{i+1}\subset T(M_i)$, $\|Tx\|\geqq(\alpha/2)\|x\|$ for $x\in M_{i+1}$.

For $N_i=(T^i|_{N_{i-1}})^{-1}(M_i)$ we get $\tau(T^i)\geqq \inf_{x\in N_i,\ \|x\|=1} \|T^i x\|\geqq(\alpha/2)^i$ and this contradicts $\tau(T^n)^{1/n}\to 0$. Let $n_0$ be the smallest natural number for which there exists a $\tilde{M}$ as in $(*)$, and choose for a small enough $\varepsilon>0$ a subspace $M\subset T^{-1}(\tilde{M})$ such that

$$\|Tx\|\geqq(\alpha-\varepsilon)\|x\| \quad \text{for } x\in M,\ \tau(T|_{T(M)})<\alpha-2\varepsilon. \tag{$**$}$$

This implies that $T(M)\cap M$ is at most finite dimensional and so we can alter $M$ so that in addition to $(**)$ $T(M)\cap M=\{0\}$. Assume now that $M$ and $T(M)$ are not perpendicular. By 2.1. there is a subspace $L$ of $T(M)$ and an isomorphism $J\colon L\to M$ with $\|Jx-x\|\leqq(\varepsilon/2)\|T\|\cdot\|x\|$ for $x\in L$. By $(**)$ there is a $x_0\in L$, $\|x_0\|=1$, with $\|Tx_0\|<\alpha-2\varepsilon$.

For $y_0=(\|Jx_0\|)^{-1}Jx_0$ we obtain

$$\|Ty_0\|\leqq\|T\|\cdot\|x_0-y_0\|+\|Tx_0\|\leqq\varepsilon+\|Tx_0\|<\alpha-\varepsilon$$

which contradicts $(**)$. Consequently $M\perp T(M)$.

b) Since $T'$ is not strictly singular there is a $w^*$-closed subspace $M_1\subset X'$ and $\alpha>0$ such that $\|T'x'\|\geqq\alpha\|x'\|$ for $x'\in M_1$. Define $\tau(T')=\sup_L \inf\{\|T'x'\|: x'\in L, \|x'\|=1\}$ where the supremum is taken only over all $w^*$-closed subspaces of $X'$. Because of $\tau'(T')\leqq\tau(T')$ we still have $\tau'((T')^n)^{1/n}\to 0$ and working only with $w^*$-closed subspaces we find as in a) a $w^*$-closed subspace $M$ of $X'$ such that $T'|_M$ is an isomorphism and $M\perp T'(M)$. Then $\Phi_{M^\perp}T$ is surjective and $M^\perp+[T'(M)]^\perp=X$ by 2.2. Now observe that $[T'(M)]^\perp=T^{-1}(M^\perp)$.

We shall also need quasi-complements: Two subspaces $M$ and $N$ of $X$ are quasi-complemented if $M\cap N=\{0\}$ and $M+N$ is dense in $X$. The following existence theorems for quasi-complements are known. For a separable Banach space $X$ they are due to Mackey, Guarii and Kadec (see [8]) and part c) is a result of Johnson, Lindenstrauß and Rosenthal (see [12]).

2.5. **Proposition.** a) *Let $M$ be a subspace of $X$ such that $X/M$ is separable and infinite dimensional. For every separable subspace $N$ of $M$ with* $\dim N=\infty$ *there is a quasi-complement $N_1$ of $M$ in $X$ such that $N\approx N_1$ and $X/N\approx X/N_1$.*

b) *Let $N$ be a separable, infinite dimensional subspace of $X$. For every $M \supset N$ with $X/M$ separable and infinite dimensional, there is a quasi-complement $M_1$ of $N$ in $X$ such that $M \approx M_1$ and $X/M \approx X/M_1$.*

c) *Let $M$ be a subspace of $X$ such that $M'$ is w*-separable and $X/M$ has a separable infinite dimensional quotient space. Then $M$ has a quasi-complement $N$ in $X$ such that $N^\perp$ is w*-separable.*

*Proof* (of a) and b)). Let $(y_n, y'_n)$ be a biorthogonal system in $X$ such that $\|y_n\| = 1$, $(\Phi_M y_n)$ is complete in $X/M$ and $y'_n \in M^\perp$ (see [17], IX.1 Theorem 7) and let $(x_n, x'_n)$ be a biorthogonal system in $X$ such that $\|x_n\| = 1$, $(x_n)$ is complete in $N$ and $(x'_n)$ is total over $N$. Define

$$J: X \to X, \quad Jx = x + \sum_{n=1}^{\infty} 2^{-n-1}(\|x'_n\|)^{-1} x'_n(x)\, y_n.$$

Then $\|Id - J\| \leq \frac{1}{2}$ and $J$ is an isomorphism onto $X$. Put $N_1 = J(N)$. Hence $J|_N: N \to N_1$ and $J\Phi_{N_1}: X/N \to X/N_1$ are isomorphism. $M + N_1$ is dense in $X$ since $M + N_1$ contains $M$ and $(y_n)$. If $x \in M \cap N_1$ then

$$x - J^{-1}x = \sum_{n=1}^{\infty} 2^{-n-1}(\|x'_n\|)^{-1} x'_n(J^{-1}x)\, y_n \in M$$

implies $x'_n(J^{-1}x) = 0$ for all $n$ (since $y'_n \in M^\perp$) and therefore $J^{-1}x = 0$ (since $(x'_n)$ is total over $N$). Hence $x = 0$. This proves a). In order to prove b) put $M_1 = J^{-1}(M)$ and recall that $J^{-1}$ is an isomorphism.

## 3. Proof of the Main Results

Theorem A will follow from 3.2., 3.7. and Theorem B from 3.1., 3.6. and 3.5.

3.1. **Theorem.** *Let $Y$ be weakly compactly generated. Then $\mathfrak{S}(X, Y) = \mathfrak{F}_+(X, Y)$ for all Banach spaces $X$ if and only if $Y$ contains at least two infinite dimensional subspaces $L_1, L_2$ with $L_1$ perpendicular to $L_2$.*

*Proof.* "$\Leftarrow$" Assume that $T \in \mathfrak{B}(X, Y)$ restricted to some separable infinite-dimensional subspace $M_0$ of $X$ is an isomorphism. Given some injective $\Phi_+$-operator $U: X \to Y$ we shall construct another $S \in \Phi_+(X, Y)$ such that $T - S$ is not a $\Phi_+$-operator.

By 2.1. there is an infinite dimensional subspace of $T(M_0)$ which is perpendicular either to $L_1$ or $L_2$. Indeed, if $T(M_0)$ is not perpendicular to $L_2$ and $\|x + y\| \geq C\|x\|$ for $x \in L_1$, $y \in L_2$ there is a subspace $M$ of $M_0$ and an isomorphism $J: T(M) \to L_2$ with $\|Jx - x\| \leq (C/2)\|x\|$ for $x \in T(M)$. This implies $T(M) \perp L_1$. Hence $T(M) \perp L$ with $L = L_1$ or $L = L_2$. Next we choose a projection $P: Y \to Y$ such that $P(Y)$ is separable, $T(M) + L \subset P(Y)$ and $P(\operatorname{Im} U) \subset \operatorname{Im} U$. This is possible according to Lemma 4 in Chap. 5, §1 of [3] since $Y$ is weakly compactly generated. If $N_1 = \operatorname{Ker} P \cap \operatorname{Im} U$ then $N_1 \perp T(M)$, $N_1 \perp L$ and $\operatorname{Im} U/N_1$ is separable.

Since $U^{-1}: N_1 \to X$ is continuous, $X/\overline{(U^{-1}(N_1)+M)}$ is separable and there exists a separable quasi-complement $K$ of $U^{-1}(N_1)+M$ in $X$ by 2.5.a. Finally, we choose a quasi-complement $N$ of $U(M) \cap N_1$ in $N_1$ using 2.5.c. Indeed, since $U^{-1}: N_1 \cap U(M) \to M$ is continuous and $M$ is separable, the Banach space $U(M) \cap N_1$ has $w^*$-separable dual and $N_1/(U(M) \cap N_1)$ has an infinite dimensional separable quotient space by [11], Cor. 1. Then $U^{-1}(N) \cap M = \{0\}$, $U^{-1}(N) \cap K = \{0\}$ but $U^{-1}(N)+M+K$ is dense in $X$. Pick some surjective $V \in \Phi(K, L)$ (see [7]) and define $S: M+\mathscr{D}(V)+U^{-1}(N) \to Y$ by $S|_M = T|_M$, $S|_{\mathscr{D}(V)} = V$ and $S|_{U^{-1}(N)} = U|_{U^{-1}(N)}$. Then $\mathscr{D}(S)$ is dense in $X$, $\operatorname{Im} S = T(U)+L+N$ is closed in $Y$ and $S$ is a closed operator because $S^{-1}: \operatorname{Im} S \to X$ is continuous. Hence $S \in \Phi_+(X, Y)$ but $S-T \notin \Phi_+(X, Y)$.

"$\Rightarrow$" Assume that $Y$ has no subspaces perpendicular to each other. Then $Y$ has to be separable because every non-separable, weakly compactly generated Banach space has a separable, complemented subspace (see [3], Chap. 5, §2, Theorem 3).

Put $X = Y \times Y$. Then $\Phi_+(X, Y) \neq \varnothing$ by [7], but for every $S \in \Phi_+(X, Y)$ the inclusion map $j: X_s \to X$ is strictly singular. Indeed, since a finite codimensional subspace of $X_s$ is isomorphic to the subspace $\operatorname{Im} S$ of $Y$, $X_s$ cannot contain perpendicular subspaces. Then $j$ is not a bounded $\Phi_+$-operator, because otherwise $X_s \approx X = Y \times Y$; and it follows from 2.3. that $j$ is strictly singular.

If $T \in \mathfrak{B}(X, Y)$, then $X_{S+T} \approx X_S$ and $jT \in \mathfrak{S}(X_{S+T}, Y)$. Therefore $j(S+T) = jS + jT$ is a bounded $\Phi_+$-operator and $S+T \in \Phi_+(X, Y)$. Hence $\mathfrak{B}(X, Y) = \mathfrak{F}_+(X, Y)$ but the operator $T: X \to Y$ defined by $T((y_1, y_2)) = y_1$ is certainly not strictly singular.

3.2. **Corollary.** *Assume that every separable subspace of $X$ is contained in a weakly compactly generated and complemented subspace of $X$. Then $\mathfrak{F}_+(X) = \mathfrak{S}(X)$.*

*Proof.* Assume there were an operator $T \in \mathfrak{F}_+(X) - \mathfrak{S}(X)$. Then, by 2.4., we could pick a separable subspace $M$ such that $T|_M$ is an isomorphism and $M \perp T(M)$. By assumption there is a w.c.g. subspace $X_1$ of $X$ containing $M + T(M)$ and a subspace $X_2$ of $X$ such that $X = X_1 \oplus X_2$. If $S_1 \in \Phi_+(X_1, X_1)$ with $S_1|_M = T|_M$, we define $S: \mathscr{D}(S_1)+X_2 \to X$ by $S|_{\mathscr{D}(S_1)} = S_1$, $S|_{X_2} = Id_{X_2}$. Then $S \in \Phi_+(X)$ but $S-T \notin \Phi_+(X)$ which contradicts $T \in \mathfrak{F}_+(X)$. The existence of $S_1$ follows from the proof of 3.1. or by the following more direct argument: Choose a sequence $x_i' \in X_1'$ with $x_i'|_M = 0$ which norms $T(M)$, i.e. $\sup_i |x_i'(x)| \geqq C\|x\|$ for some $C>0$ and all $x \in T(M)$. If $N = [x_i]^\perp \subset X_1$, then $M \subset N$ and $X_1/N$ is separable because $X_1/N$ is weakly compactly generated and $(X_1/N)| = [x_i]^{\perp\perp}$ is $w^*$-separable (this follows e.g. from [3], p. 163, Theorem 3). By 2.5.a there is a quasicomplement $N_1$ of $M$ in $X_1$ and an isomorphism $J: N_1 \to N$. Define $S_1: M+N_1 \to X_1$ by $S|_M = T|_M$, $S|_{N_1} = J$.

3.3. *Remark.* The assumption of 3.2. is fulfilled for example by Banach lattices with an order continuous norm $\Big($see [26], II.5; a sequence $(x_i) \subset X$ is contained in the band generated by $x = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i|}{\|x_i\|}$, which is w.c.g. by 5.10$\Big)$.

Some important Banach spaces, like $l_\infty$, which are not covered by 3.2. can be treated by the following variation:

3.4. **Corollary.** *If $X \approx X \oplus X$ has at least one separable infinite dimensional quotient space, then $\mathfrak{F}_+(X) = \mathfrak{S}(X)$.*

*Proof.* Let $P_1$ and $P_2 = Id - P_1$ be projections in $X$ with $P_1(X)$ and $P_2(X)$ isomorphic to $X$. If $T \in \mathfrak{B}(X)$ is not strictly singular, then there are $i, j \in \{1, 2\}$ such that $P_i T P_j$ is not strictly singular. Pick a separable $M \subset P_j(X)$ such that $P_i T P_j|_M$ is an isomorphism. Then $T(M)$ is perpendicular to $(Id - P_i)(X)$. Since $X/M$ has a complemented subspace isomorphic to $X$, namely $(I - P_j)(X)$, it follows that $X/M$ has a separable quotient space. Choose a quasi-complement $N$ of $M$ in $X$ (by 2.5.a) and an isomorphic embedding $J: N \to (I - P_i)(X)$. Then $S: M + N \to X$ defined by $S|_N = J$, $S|_M = T|_M$ belongs to $\Phi_+(X)$, but $T - S \notin \Phi(X)$.

3.5. *Remark.* It still seems to be an open question if every Banach space has a separable quotient space. But it is known that $X$ has a separable quotient in each of the following cases:

a) $X$ is weakly compactly generated or just a subspace of a quotient of a weakly compactly generated space: (see [23], Chap. 5, §2, Theorem 3 and [11], Corollary 1).

b) $X$ contains a subspace isomorphic to $l_1$ or $c_0$: (in the first case $X$ has a quotient isomorphic to $l_2$ by a result in [20]; in the second case it follows that $X'$ has a subspace isomorphic $l_1$ and Corollary 1 of [9] can be applied).

c) $X$ is a Banach lattice: (if $X$ does not contain $c_0$ or $l_1$ then $X$ is reflexive, see [26], II. Theorem 5.16).

d) $X$ is a quotient of a $C(K)$-space: (by [19], Theorem 1, $X$ is either reflexive or contains a subspace isomorphic to $c_0$).

a) and d) show that the assumption of Theorem 3.6 below is shared by all subspaces of weakly compactly generated spaces and all $C(K)$-spaces.

3.6. **Theorem.** *Assume that every infinite dimensional quotient space of the Banach space $X$ has an infinite dimensional separable quotient space. Then we have $\mathfrak{F}_-(X, Y) = \mathfrak{C}\mathfrak{S}(X, Y)$ for all Banach spaces $Y$ if and only if $X$ contains at least one pair of infinite codimensional closed subspaces $L_1$, $L_2$ with $X = L_1 + L_2$.*

*Proof.* "$\Leftarrow$" Given a $T \in \mathfrak{B}(X, Y)$ which is not strictly cosingular and some surjective $U \in \Phi_-(X, Y)$ we construct another $S \in \Phi_-(X, Y)$ so that $S - T \notin \Phi_-(X, Y)$. By assumption there is a subspace $M_0$ of $Y$ such that $\Phi_{M_0} T$ is surjective, and $Y/M_0 \approx X/T^{-1}(M_0)$ is separable and infinite dimensional. Next we choose an infinite codimensional subspace $\tilde{M}$ of $X$ containing $T^{-1}(M_0)$ such that $\tilde{M} + L_1 = X$ or $\tilde{M} + L_2 = X$: as in 3.1. it follows from 2.2. that $T^{-1}(M_0)^\perp \subset X'$ contains a $w^*$-closed subspace $\tilde{M}^\perp$ which is either perpendicular to $L_1^\perp$ or $L_1^\perp$ since $L_1^\perp \perp L_2^\perp$. Assume $\tilde{M} + L = X$ with $L = L_1$ or $L = L_2$ and put $M = T(\tilde{M})$. By assumption we can alter $L$ so that in addition $X/L$ is separable. Therefore we can find separable subspaces $N_1$, $N_2$, $N_3$ of $X$ such that $L+N_1=X$, $\tilde{M}+N_2=X$, $U(N_3)+M=Y$ (e.g. see [30], Lemma 2) and $U(N_3)\cap M$ is infinite dimensional. If $N_0=\operatorname{Ker} U$ and $N=\left[\bigcup_{i=0}^{3} N_i\right]$ then $U(N)$ is closed in $Y$ and $U(N)'$ is $w^*$-separable since $\Phi_{N_0}(N)$ is closed and separable in $X/N_0$ and $U$ induces a continuous injective operator $\hat{U}^{-1}: Y\to X/N_0$. The latter operator induces a continuous, injective operator $Y/U(N)\to X/N$ with dense range, and its inverse $U_1$ belongs to $\Phi_-(X/N, Y/U(N))$. Since $U(N)'$ is $w^*$-separable, the dual of $U(N)\cap M$ is $w^*$-separable and $Y/(U(N)\cap M)$ has a separable quotient space, namely $Y/M$. Therefore, by 2.5.c, $U(N)\cap M$ has a quasi-complement $K$ in $Y$ such that $(Y/K)'\approx K^\perp$ is $w^*$-separable. Pick some $V\in\Phi_-(X/L, Y/K)$ (see [7]) and denote by $S_0$ the composition operator given by

$$X \xrightarrow{\Phi} X/\tilde{M}\oplus X/N\oplus X/L \xrightarrow{\widehat{\Phi_M T}\oplus U_1\oplus V} Y/M\oplus Y/U(N)\oplus Y/K$$

where $\Phi(x)=(\Phi_{\tilde{M}}x, \Phi_N x, \Phi_L x)$ and $\widehat{\Phi_M T}$ is induced by $\Phi_M T: X\to Y/M$. Since $\Phi$ is surjective by 2.2. it follows that $S_0$ is a $\Phi_-$-operator with $\mathscr{D}(S_0) =\Phi^{-1}(X/\tilde{M}\oplus\mathscr{D}(U_1)\oplus\mathscr{D}(V))$. The map $j$: $Y\to Y/M\oplus Y/U(N)\oplus Y/K$ given by $j(x)=(\Phi_M x, \Phi_{U(N)}x, \Phi_K x)$ is injective and has dense range because $Y/M\oplus YU(N)=Y/(M\cap U(N))$ and $K$ and $M\cap U(N)$ are quasi-complements. Therefore we can identify $Y$ with a dense subspace of $Y/M\oplus Y/U(N)\oplus Y/K$. Now it is easy to check that the operator $S=S_0|_{\mathscr{D}}$ where $\mathscr{D}=S_0^{-1}(Y)$, belongs to $\Phi_-(X, Y)$ and that $\Phi_M(T-S)=0$.

"$\Rightarrow$" Assume there are no infinite codimensional subspaces $L_1, L_2$ of $X$ with $X=L_1+L_2$. Then $X$ has to be separable because if $X$ is non-separable we can choose a separable quotient space $X/M$ and a separable subspace $N$ with $M+N=X$ (see [30], Lemma 2) where $N$ has to have infinite codimension.

Put $Y=X\oplus X$. Then $\Phi_-(X, Y)\neq\varnothing$ (see [7]). If $S\in\Phi_-(X, Y)$, then the inclusion map $j$: $X_S\to X$ is not a bounded $\Phi_-$-operator because otherwise $X/\operatorname{Ker} S\approx X_S/\operatorname{Ker} S\approx \operatorname{Im} S\approx X_1\oplus X_2$ with finite-codimensional subspaces $X_1$, $X_2$ of $X$, and this contradicts our assumption. In view of 2.3. $j$ has to be strictly cosingular. For every $T\in\mathfrak{B}(X, Y)$ we have $X_S\approx X_{S+T}$ and $jT$: $X_S\to Y$ is strictly cosingular. Then $j(T+S)=jT+jS$: $X_S\to Y$ is a bounded $\Phi_-$-operator and $T+S\in\Phi_-(X, Y)$. So we showed $\mathfrak{B}(X, Y)=\mathfrak{F}_-(X, Y)$. On the other hand, the operator $T$: $X\to Y$, $Tx=(x, 0)$, is not strictly cosingular.

3.7. **Corollary.** *Assume that every infinite dimensional quotient space of the Banach space $X$ has an infinite dimensional separable quotient space. Then $\mathfrak{F}_-(X)=\mathfrak{C}\mathfrak{S}(X)$.*

*Proof.* This is a consequence of 3.6. and 2.3. or of the following more direct argument: If there were a $T\in\mathfrak{F}_-(X)-\mathfrak{C}\mathfrak{S}(X)$ we could choose by 2.4. a subspace $M$ of $X$ such that $\Phi_M T$ is surjective, $T^{-1}(M)+M=X$, and $X/M$ is infinite-dimensional and separable. Pick a separable subspace $N$ of $M$ such that $T^{-1}(M)+N=X$ (see [30], Lemma 2). By 2.5.b there is a quasi-complement $N_1$ of $M$ in $X$ such that there is an isomorphism $V$ of $X/N$ onto $X/N_1$.

As in 3.6. we find a $S \in \Phi_-(X)$ with $S-T \notin \Phi_-(X)$ using the composition

$$X \to X/T^{-1}(M) \oplus X/N \xrightarrow{\widehat{\Phi_M T} \oplus V} X/M \oplus X/N_1 \supset X.$$

This contradicts $T \in \mathfrak{F}_-(X)$.

## 4. Some Counterexamples Concerning Fredholm Perturbations

It is known already that there are Banach spaces $X$ with $\mathfrak{F}(X) \neq \mathfrak{S}(X)$ or $\mathfrak{F}(X) \neq \mathfrak{C}\mathfrak{S}(X)$ (see [5]) but it will follow from Proposition 4.1 that this can even occur for Banach spaces with a very nice Banach space structure, like $C(K)$-spaces, Orlicz-sequence spaces or rearrangement invariant function spaces 'close' to $L_p$-spaces.

4.1. **Proposition.** *Assume that $X$ has two subspaces $M$, $N$ such that*

a) *both are isomorphic to $c_0$ or the same $l_p$ for some $1 \leqq p < \infty$,*

b) *$M$ is complemented in $X$*

c) *$N$ contains no infinite dimensional subspace complemented in $X$.*

*Then* $\mathfrak{F}(X) \neq \mathfrak{S}(X)$.

*Proof.* If $P: X \to M$ is a projection and $J: M \to N$ an isomorphism onto $N$, then $T = JP \notin \mathfrak{S}(X)$. Assume that $T \notin \mathfrak{F}(X)$. By [27], Theorem 27, there is a bounded $S \in \Phi(X)$ such that $(S-T)|_L = 0$ for some infinite-dimensional subspace $L$ of $X$. If $X_1$ is a complement of $\operatorname{Ker} S$ in $X$ and $X_2 = \operatorname{Im} S$ then $X_1$ and $X_2$ have finite codimension in $X$ and we may assume that $M \subset X_1$, $L \subset X_1$ and $N \subset X_2$. Furthermore, $S|_L = JP|_L$ implies that $P|_L$ is an isomorphism. By [15], 2.a.2., there is an infinite dimensional projection $Q: M \to M$ with $\operatorname{Im} Q \subset P(L)$. Then $U = (P|_L)^{-1} QP$: $X_1 \to X_1$ is a projection onto a subspace of $L$ and $V = SU(S|_{X_1})^{-1}$: $X_2 \to X_2$ is a projection into $S(L) = T(L) \subset M$. This, contradicts c). Hence $T \in \mathfrak{F}(X)$.

4.2. *Examples.* a) Let $X = C(K)$, where $K$ is the disjoint union of the one-point-compactification $\tilde{\mathbb{N}}$ of $\mathbb{N}$ and the Stone-Cech-compactification $\beta\mathbb{N}$ of $\mathbb{N}$. Then $\mathfrak{F}(X) \neq \mathfrak{S}(X)$.

b) Let $X = L_p(0, \infty) + L_q(0, \infty)$ with the norm $\|f\| = \inf\{\|h\|_p + \|g\|_q : f = h + g\}$. If $1 < p < q < 2$ then $\mathfrak{F}(X) \neq S(X)$.

c) Let $X = L_p(0, \infty) \cap L_q(0, \infty)$ with the norm $\|f\| = \max(\|f\|_p, \|f\|_q)$. If $2 < q < p < \infty$ then $\mathfrak{F}(X) \neq \mathfrak{C}\mathfrak{S}(X)$.

d) Let $X = L(p, q)$ be the Lorentz-space on $[0, 1]$ with the norm $\|f\| = \left(\frac{q}{p} \int_0^1 [t^{1/p} f^*(t)]^q \frac{dt}{t}\right)^{1/q}$ where $f^*$ is the decreasing rearrangement of $f$ (see [10] or [16], p. 120).

Then $\mathfrak{F}(X) \neq \mathfrak{S}(X)$ for $1 < p < q < 2$ and $\mathfrak{F}(X) \neq \mathfrak{C}\mathfrak{S}(X)$ for $2 < q < p < \infty$.

e) Let $X = U_{c,d}$ be an Orlicz sequence space as in [15], Theorem 4.b.12 with $\frac{1}{d} + \frac{1}{c} = 1$. Then $\mathfrak{F}(X) \neq \mathfrak{S}(X)$ and $\mathfrak{F}(X) \neq \mathfrak{C}\mathfrak{S}(X)$.

*Proof.* a) Since $C(K) \approx C(\tilde{\mathbb{N}}) \oplus C(\beta\mathbb{N}) \approx c_0 \oplus l_\infty$, and there is no projection in $l_\infty$ onto a subspace isomorphic to a subspace of $c_0$ (see e.g. [15], Theorem 2.a.7), one can apply 4.1.

b) $X$ has a complemented subspace $M$ isomorphic to $l_q$, e.g. take the span of $\chi_{[n^2,(n+1)^2]}$, $n \in \mathbb{N}$ (c.f. Lemma 1 in [13]). On the other hand, since $p<q$ we have $X|_{[0,1]} \approx L_p[0,1]$ and $X|_{[0,1]}$ contains a subspace $N$ isomorphic to $l_q$ but without complemented subspaces (by [13], Remark 2 and Corollary 3).

c) follows from b) by duality since $X' = L_{p'}(0,\infty) + L_{q'}(0,\infty)$ with $\frac{1}{q}+\frac{1}{q'}=1$, $\frac{1}{p}+\frac{1}{p'}=1$ and $T \in \mathfrak{F}(X)$ iff $T' \in \mathfrak{F}(X')$, $T \in \mathscr{S}(X)$ iff $T' \in \mathfrak{S}(X')$ (by [22], C.I. 2.6', 2.7', 6.9).

d) $X$ contains a complemented subspace $M$ isomorphic to $l_q$ by [4], Theorem 5.1. It is easy to calculate the Boyd indices of $X$ (see [16], p. 130): $p_x = q_x = p$.

Therefore, if $p<r<q$, we have a continuous inclusion map $j\colon L_r[0,1] \hookrightarrow X$ (by [16], Proposition 2.b.3.).

Since $r<q<2$, $L_r[0,1]$ contains a subspace $N_1$ isomorphic to $l_q$ ([13], Remark 2) such that the inclusion $N_1 \subset L_r[0,1] \hookrightarrow X \subset L_1[0,1]$ is an isomorphism into $L_1$ (combine Theorem 1 and 2 of [13]).

Then $N = j(N_1)$ is a subspace of $X$ isomorphic to $l_q$ and there is no infinite dimensional projection $P\colon X \to N$ because otherwise there is a projection $Q = (j|_{N_1})^{-1} P_j\colon L_r \to N_1$ which contradicts [13], Corollary 3. Hence 4.1 settles the case $1<p<q<2$. As in c) the case $2<q<p<\infty$ follows by duality since $L(p,q)' = L(p',q')$ with $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$ (see [10] 2.7).

e) By 4.b.12 of [15] $U_{c,d}$ contains a complemented $M$ isomorphic to $l_p$ for $c<p<d$, as well as an Orlicz space $X$ that contains a subspace $N$ isomorphic to $l_p$ but no complemented subspace isomorphic to $l_p$ ([15], Example 4.c.6).

Hence $\mathfrak{F}(X) \neq \mathfrak{S}(X)$ by 4.1. Since $U'_{c,d} \approx U_{c,d}$ (see the remark following 3.b.12 in [15]) $\mathfrak{F}(X) \neq \mathfrak{C}(X)$ follows by duality.

## 5. The Fourier Transform as an Admissible Fredholm Perturbation

Let $(G,m)$ be a locally compact group with its Haar-measure $m$. Denote by $(\Gamma,n)$ the dual group of $G$ with its Haar measure $n$ adjusted in such a way that the inversion formula holds. $\hat{f}(\gamma) = \int_G f(x)(-x,\gamma)\,dm(x)$, $\gamma \in \Gamma$ is the Fourier transform of $f \in L_1(G)$ (see [25], Sect. 1.2.). For $1 \leqq p \leqq 2$ the map $f \to \hat{f}$ extends to bounded operators $\mathscr{F}\colon L_p(G,m) \to L_{p'}(\Gamma,n)$ with $\frac{1}{p}+\frac{1}{p'}=1$. For $p=2$ we get an isometry (see [25], Theorem 2.6.1). For $1<p<2$, $\mathscr{F}$ is certainly not compact but we have

5.1. **Theorem.** *For $1<p<2$ the Fourier transform $\mathscr{F}\colon L_p(G,m) \to L_{p'}(\Gamma,n)$ is strictly singular and strictly cosingular.*

*Proof.* Assume that there is an infinite dimensional, separable subspace $M$ of $L_p(G,m)$ and a constant $C>0$ such that $\|\mathscr{F}f\| \geqq C\|f\|$ for $f\in M$. Choose a sequence $(K_n)$ of compact subsets of $G$ with

$$K_n \subset K_{n+1} \subset \ldots \quad \text{and} \quad \|(1-\chi_{K_n})f\| \xrightarrow[n\to\infty]{} 0$$

for all $f\in M$, and a sequence $(L_n)$ of compact subsets of $\Gamma$ such that $L_n \subset L_{n+1} \subset \ldots$ and $\|(1-\chi_{L_n})g\| \to 0$ for all $g\in\mathscr{F}(\chi_{K_n}M)$, $n\in\mathbb{N}$. Put $\chi_n=\chi_{K_n}$ and $\rho_n=\chi_{L_n}$. First we show that $\mathscr{F}\cdot\chi_n|_M$ is strictly singular for all $n\in\mathbb{N}$. For every normalized sequence $f_i\in\chi_n M$ weakly converging to zero it follows from the compactness of $\rho_m\mathscr{F}\chi_n$ ($\rho_m\mathscr{F}\chi_n$ is an integral operator with uniformly bounded kernel on $L_m \times K_n$) that $\|\rho_m\mathscr{F}\chi_n f_i\| \xrightarrow{i\to\infty} 0$ for all $m$. Therefore, we can find inductively a subsequence $m_k\in\mathbb{N}$ and a subsequence $g_k$ of $(\mathscr{F}\chi_n f_i)_{i\in\mathbb{N}}$ such that

$$\|(\rho_{m_{k+1}}-\rho_{m_k})g_k - g_k\| \leqq C/2^{k+1}.$$

The functions $(\rho_{m_{k+1}}-\rho_{m_k})g_k$, having pairwise disjoint support span a subspace isomorphic to $l_{p'}$ (compare [13], Lemma 1) and then $[g_k]\approx l_{p'}$ by a standard stability result (see [15], 1.a.9). But $L_p$ for $p\neq 2$ has no subspace isomorphic to $l_{p'}$ and, consequently, $\mathscr{F}|_{[f_n]}$ cannot be an isomorphism. Hence all operators $\mathscr{F}\chi_n|_M$ are strictly singular and by [6], III.2.1. there is a sequence of infinite dimensional subspaces $M\supset M_1\supset\ldots\supset M_n$ such that $\|\mathscr{F}\chi_n|_{M_n}\|\to 0$ for $n\to\infty$. For $\varepsilon_k=\frac{1}{8}C^2\cdot 2^{-k}$ we choose inductively $n_k\in\mathbb{N}$ and a sequence $f_k\in M$, $\|f_n\|=1$ with

i) $f_k\in M_{n_k}$ and $\|\mathscr{F}\chi_{n_k}(f_k)\|\leqq\varepsilon_k$

ii) $\int f_k\cdot g_j^{p-1}\,dm=0$ for $j<k$ where $g_j=(\chi_{n_{j+1}}-\chi_{n_j})f_j$ (so that $g_j^{p-1}\in L_{p'}(G)$)

iii) $\|(1-\chi_{n_{k+1}})f_j\|\leqq\varepsilon_{k+1}$ for $j\leqq k$.

Then $Pf=\sum_{j=1}^{\infty}(\|g_j\|^{-p}\int f g^{p-1}\,dm)g_j$ is a projection of norm 1 onto the subspace $[g_j]$ isomorphic to $l_p$ (compare [13], Lemma 1). We have

$$\begin{aligned}\|Pf_k-g_k\|_p &\leqq \sum_{j>k}\|g_j\|_p^{1-p}\left|\int f_k g_j^{p-1}\,dm\right| \\ &\leqq \sum_{j>k}\|f_k(1-\chi_{n_j})\|_p \leqq \sum_{j>k}\varepsilon_j\leqq\varepsilon_k,\end{aligned}$$

and

$$\|g_k\|\geqq\|\mathscr{F}g_k\|\geqq\|\mathscr{F}f_k\|-\|\mathscr{F}\chi_{n_k}f_k\|-\|(1-\chi_{n_{k+1}})f_k\|\geqq C-2\varepsilon_k\geqq(3/4)\,C.$$

More generally, for all $\alpha_1\ldots\alpha_n\in\mathbb{C}$ we get

$$\begin{aligned}\left\|\sum\alpha_k\mathscr{F}g_k\right\| &\geqq \left\|\sum\alpha_k\mathscr{F}f_k\right\|-\sum|\alpha_k|\,\|\mathscr{F}\chi_{n_k}f_k\|-\|\mathscr{F}\|\sum_k|\alpha_k|\cdot\|(1-\chi_{n_{k+1}})f_k\| \\ &\geqq C\left\|\sum\alpha_k f_k\right\|-\left(\sum|\alpha_k|^p\right)^{1/p}2\cdot\left(\sum\varepsilon_k^{p'}\right)^{1/p'} \\ &\geqq C\cdot\left\|P\left(\sum\alpha_k f_k\right)\right\|-(2C^2/8)\left(\sum|\alpha_k|^p\right)^{1/p} \\ &\geqq C\cdot\left\|\sum\alpha_k g_k\right\|-\sum|\alpha_k|\,\|Pf_k-g_k\|-(2\cdot C^2/8)\left(\sum|\alpha_k|^p\right)^{1/p} \\ &\geqq C\cdot\left\|\sum\alpha_k g_k\right\|-(3C^2/8)\left[\frac{4}{3C}\left(\sum|\alpha_k|^p\cdot\|g_k\|^p\right)^{1/p}\right] \\ &\geqq C\cdot\left\|\sum\alpha_k g_k\right\|-C/2\left\|\sum\alpha_k g_k\right\|.\end{aligned}$$

Hence $\mathscr{F}|_{[gk]}$ is an isomorphism and this leads to the impossible conclusion that $l_p$ is isomorphic to a subspace of $L_{p'}(\Gamma, n)$. Hence $\mathscr{F}$ is strictly singular. By the Pontryagin Duality Theorem ([25], Theorem 1.7.2) we have that $\mathscr{F}'f = \overline{\mathscr{G}(\bar{f})}$ where $\mathscr{G}: L_p(\Gamma) \to L_{p'}(G)$ is the Fourier transform. Therefore $\mathscr{F}$ is strictly cosingular since $\mathscr{G}$ is strictly singular (see [22], C.II.6.9).

5.2. **Corollary.** *For $1<p<2$ the range of the Fourier transform $\mathscr{F}: L_p(G) \to L_{p'}(\Gamma)$ does not contain an infinite dimensional subspace closed in $L_{p'}$.*

*Proof.* This follows from 5.1. and the closed graph theorem because $\mathscr{F}$ is injective.

5.3. *Remark.* It was shown in [2] that $\mathscr{F}: L_1(G) \to C_0(\Gamma)$ is strictly singular if and only if $G$ is compact. One may add that $\mathscr{F}: L_1(G) \to C_0(\Gamma)$ is never strictly cosingular if $G$ is infinite. Indeed, if $\Gamma$ is not discrete then $\Gamma$ contains an infinite, compact Helson set $E \subset \Gamma$ ([25], 5.6.6), and if $\Gamma$ is discrete, there exists an infinite Sidon set $E \subset \Gamma$ ([25], 5.7.5). In any case, if $\Phi: C_0(\Gamma) \to C_0(E)$ denotes the quotient map $g \to g|_E$, it follows that $\Phi\mathscr{F}$ is surjective (cf. [25], 5.6.2 and [25], 5.7.3e).

The same argument also shows that $\mathscr{F}: M(G) \to C(E)$ is neither strictly cosingular, ([25], 5.6.2 and 5.7.3d) nor strictly singular: (observe that $C(E)$ contains a subspace isomorphic to $l_1$ and if $\mu_n \in M(G)$ are such that $\Phi\mathscr{F}(\mu_n)$ is equivalent to the unit vector basis of $l_1$, then $\Phi\mathscr{F}|_{[\mu_n]}$ is necessarily an isomorphism).

In particular, if $G$ is compact, the Fourier transform $\mathscr{F}: L_1(G) \to C_0(\Gamma)$ provides a natural example for a strictly singular operator such that the adjoint map is neither strictly singular nor strictly cosingular. For earlier examples of this kind, see [5] and [19].

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Received May 6, 1981