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On Locally Indicable Groups

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1. Introduction

A group is said to be *locally indicable* if each of its non-trivial finitely generated subgroups has the infinite cyclic group as a homomorphic image. Such groups were studied by Higman [9] in connection with the zero-divisor and unit problems for group rings. More recently, they have arisen [2, 12] in the study of equations over groups.

In [12], I proved a Freiheitsatz for locally indicable groups. This has been proved independently by Brodskiĭ [2] and Short [22]. The present paper arises from an investigation of a question put to me by S.J. Pride – whether torsion-free 1-relator groups are locally indicable. The question was raised originally by Baumslag ([1], Problem 19) and an affirmative solution has recently been announced by Brodskiĭ [2].

As a consequence, the group algebra RG of a torsion-free 1-relator group G over an integral domain R has no non-trivial zero-divisors, and no non-trivial units (using Higman's results [9]). The first fact was also proved by Lewin and Lewin [16], by embedding RG in a division ring. The second appears to be new, as was pointed out to me by K.A. Brown.

A second consequence is that no 1-relator group has a non-trivial finitely generated perfect subgroup, which answers [1], Problem 7 and [10], Question 1.

In fact, using the Freiheitsatz, and the tower method described in [12], it is possible to prove the following general version of Brodskiĭ's theorem.

Theorem 4.2. *Let A and B be locally indicable groups, and let G be the quotient of $A*B$ by the normal closure of a cyclically reduced word r of length at least 2. Then the following are equivalent:*

(i) *G is locally indicable;*
(ii) *G is torsion-free;*
(iii) *r is not a proper power in $A*B$.*

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The group G in Theorem 4.2 will be called a 1-relator product of A and B. Thus the class of locally indicable groups is closed under the formation of 1-relator products in which the relator is not a proper power.

The methods used in the proof of Theorem 4.2 can also be used to study asphericity in 2-complexes with locally indicable fundamental groups.

Theorem 5.2. *Let L be a connected 2-complex with $H_2(L)=0$ and $\pi_1(L)$ locally indicable. Then L is aspherical.*

Theorem 5.3. *Let L be a 2-complex formed by attaching a 1-cell and a 2-cell to a 2-complex K, each component of which is aspherical with locally indicable fundamental group. Suppose the attaching map for the 2-cell of $L \backslash K$ is not homotopic in $K \cup L^{(1)}$ to a map into K, and does not represent a proper power in $\pi_1(K \cup L^{(1)})$. Then L is aspherical.*

These results may be applied to a class of group presentations which I call "reducible", and which includes, for example, all 1-relator and staggered presentations.

Corollary 4.5. *A group given by a reducible presentation with no proper powers is locally indicable.*

Corollary 5.4. *A reducible presentation with no proper powers is aspherical.*

These terms will be made precise in § 2, but an "aspherical" presentation is one whose geometric realisation is an aspherical 2-complex. There exist in the literature a variety of inter-related notions of asphericity for presentations, which have been extensively investigated by Chiswell, Collins and Huebschmann [4] (see also [3, 7, 17, 21]). Any staggered presentation (with or without proper powers) is Cohen-Lyndon aspherical in the sense of [4]. In contrast, the proper power condition in Corollary 5.4 is crucial: a group given by a reducible presentation need not have finite quasi-projective dimension in the sense of Howie and Schneebeli [13], so the presentation need not satisfy any of the asphericity conditions discussed in [4].

A large class of locally indicable groups arise as fundamental groups of suitable 3-manifolds. In particular all knot-groups and link-groups are locally indicable, which answers Question 3 of [10]. This fact seems to be understood by 3-dimensional topologists, but does not seem to be in the literature. I have included a proof in § 6, based on Scott's compact submanifold theorem. I am grateful to G.A. Swarup for the ideas behind this proof.

Short ([22, 23]) applies the Freiheitsatz for knot-groups to a problem of Lickorish ([15], Problem 1.1): a band sum of two knots can be the unknot only if each of the original knots is the unknot.

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2. Notation and Definitions

A group G is *locally indicable* if every finitely generated subgroup A of G (other than the identity) admits an epimorphism onto the infinite cyclic group.

Equivalent ways of saying this are that $H^1(A)\neq 0$ or that $H_1(A)$ $(=A^{ab}=A/[A,A])$ is finite. (All homology and cohomology is with integer coefficients unless stated otherwise.)

The main object of study of this paper is 1-relator products of locally indicable groups, that is groups of the form $G=(A*B)/N$, where A and B are locally indicable, and N is the normal closure of a single element r, not conjugate to an element of A or of B. The relator r may be represented (up to conjugacy) as a cyclically reduced word of length at least 2. We must therefore study cyclically reduced words in free products, or more generally in graphs of groups with trivial edge groups.

We use the notation of Dicks [5] for graphs of groups. Thus a graph of groups $(\mathcal{G},X)$ consists of a connected, oriented graph X, with vertex set $V=V(X)$ and edge set $E=E(X)$, together with groups $\mathcal{G}(v)$ $(v\in V)$ and $\mathcal{G}(e)$ $(e\in E)$, and monomorphisms $\iota_e\colon \mathcal{G}(e)\to\mathcal{G}(\iota e)$, $\tau_e\colon \mathcal{G}(e)\to\mathcal{G}(\tau e)$ $(e\in E)$, where ιe, τe denote the initial and terminal vertices respectively of e.

Following Higgins [8], we define the *fundamental groupoid* $\mathcal{F}(\mathcal{G},X)$ of $(\mathcal{G},X)$ to be the groupoid generated by the groups $\mathcal{G}(v)$ $(v\in V)$ and the edges $e\in E$, subject to the relations $e\cdot\tau_e(g)=\iota_e(g)\cdot e$ $(e\in E, g\in\mathcal{G}(e))$.

The *fundamental group* $\pi(\mathcal{G},X)$ is then just a vertex group of $\mathcal{F}(\mathcal{G},X)$, or alternatively the quotient of $\mathcal{F}(\mathcal{G},X)$ by a maximal tree subgroupoid.

An element of $\mathcal{F}(\mathcal{G},X)$ can be expressed as a *word*

$$w=g_0\cdot e_1^{\varepsilon(1)}\cdot g_1\cdot\ldots\cdot e_n^{\varepsilon(n)}\cdot g_n$$

where $g_j\in\mathcal{G}(v_j)$ $(v_j\in V)$, $\varepsilon(j)=\pm 1$, $e_j\in E$ and $\iota(e_j)=v_{j-1}$, $\tau(e_j)=v_j$ (if $\varepsilon(j)=1$), or $\iota(e_j)=v_j$, $\tau(e_j)=v_{j-1}$ (if $\varepsilon(j)=-1$).

The integer n is the *length* of the word (w). A word (w) is *reduced* if it does not contain a sequence of the form $e\cdot\tau_e(h)\cdot e^{-1}$ or $e^{-1}\cdot\iota_e(h)\cdot e$ $(h\in\mathcal{G}(e))$. It is *closed* if $v_0=v_n$. A closed word (w) is *cyclically reduced* if it is reduced and if $e_n^{\varepsilon(n)}\cdot(g_n g_0)\cdot e_1^{\varepsilon(1)}$ is reduced.

Elements of $\mathcal{F}(\mathcal{G},X)$ have a normal form [8], which in general depends on choices of transversals for the images of the maps ι_e,τ_e. In the case where the edge groups $\mathcal{G}(e)$ are trivial, there is a unique choice of transversals, so a unique normal form, and the normal forms are precisely the reduced words. Hence every element of $\mathcal{F}(\mathcal{G},X)$ is uniquely defined by a reduced word. Also, every element in $\pi(\mathcal{G},X)$ is conjugate in $\mathcal{F}(\mathcal{G},X)$ to (an element represented by) a cyclically reduced closed word.

Let

$$w=g_0\cdot e_1^{\varepsilon(1)}\cdot g_1\cdot\ldots\cdot e_n^{\varepsilon(n)}\cdot g_n,$$
$$w'=h_0\cdot f_1^{\eta(1)}\cdot h_1\cdot\ldots\cdot f_m^{\eta(m)}\cdot h_m$$

be reduced words, such that $g_n, h_0\in\mathcal{G}(v)$ for the same vertex $v\in V$. Then one can form the product word

$$w\cdot w'=g_0\cdot e_1^{\varepsilon(1)}\cdot\ldots\cdot e_n^{\varepsilon(n)}\cdot(g_n h_0)\cdot f_1^{\eta(1)}\cdot\ldots\cdot f_m^{\eta(m)}\cdot h_m.$$

This is reduced if and only if $e_n^{\varepsilon(n)}\cdot(g_n h_0)\cdot f_1^{\eta(1)}$ is reduced.

Let w be a cyclically reduced closed word. If w_1, w_2 are reduced words such that $w_1 \cdot w_2$ is defined, and is a cyclically reduced conjugate of w, then we will call w_1 and w_2 *subwords* of w. If both w_1 and w_2 have positive length, then they are *proper subwords* of w.

A *2-complex* is a CW-complex K of dimension at most 2. Its 1-skeleton $K^{(1)}$ may be thought of as a graph, and the attaching maps for 2-cells may be thought of as closed paths in $K^{(1)}$, which we will always assume to be cyclically reduced.

An *elementary reduction* consists of a pair (L, K) of 2-complexes, with $K \subseteq L$, such that $L \backslash K$ has no 0-cells, exactly one 1-cell, and at most one 2-cell; and such that the attaching map for the 2-cell (if there is one) *strictly involves* the 1-cell, in the sense that it is not homotopic in $K \cup L^{(1)}$ to a map into K.

We will be concerned with the case when L is connected, so that K has either 1 or 2 components. We associate a graph of groups $(\mathscr{G}, X)$ to the elementary reduction (L, K) as follows. The graph X is obtained from $K \cup L^{(1)}$ by shrinking each component of K to a point. Thus X has a single edge e and one vertex v for each component K_v of K. Define $\mathscr{G}(e)=1$, and $\mathscr{G}(v)=\pi_1(K_v)$.

Clearly $\pi(\mathscr{G}, X) \cong \pi_1(K \cup L^{(1)})$. In particular, if there is no 2-cell in $L \backslash K$, then $\pi(\mathscr{G}, X) \cong \pi_1(L)$. If there is a 2-cell in $L \backslash K$, its attaching map "spells out" a closed word w in $\mathscr{F}(\mathscr{G}, X)$, and $\pi_1(L) \cong \pi(\mathscr{G}, X)/N$, where N is the normal closure of the element represented by w.

Note that, if K is connected, then w is not well-defined in general, but depends on a choice of maximal tree in $K^{(1)}$. In general also w is not cyclically reduced (although this is independent of a choice of tree), but is nevertheless conjugate to a cyclically reduced word of positive length. Replacing w by a conjugate does not alter the (simple) homotopy type of the pair (L, K), so there is no loss of generality in considering only the elementary reductions (L, K) for which the corresponding word w is cyclically reduced.

An important special case of an elementary reduction is an elementary collapse $L \searrow K$. An elementary reduction (L, K) is an elementary collapse precisely when (L, K) has a 2-cell, K is connected and w has length 1.

Recall [10] that a *combinatorial map* $f: L' \to L$ between CW-complexes is a cellular map which maps each cell of L' homeomorphically onto a cell of L (necessarily of the same dimension). Suppose (L', K'), (L, K) are elementary reductions, and $f: L' \to L$ is a combinatorial map. We will say f *carries* (L', K') *to* (L, K) if f maps each cell of $L' \backslash K'$ to a cell of $L \backslash K$. In particular if $L' \backslash K'$ has a 2-cell then so has $L \backslash K$. Note that f is *not* required to map K' into K.

By a *tower* (of height h) we mean a map between connected 2-complexes which can be expressed as a composite $i_0 \circ p_1 \circ i_1 \circ \ldots \circ p_h \circ i_h$, where each p_r is an infinite cyclic covering, and each i_r is the inclusion of a finite, connected subcomplex. (This is an $(\mathfrak{F}, \mathfrak{C}_\infty)$-tower in the notation of [12], where more general towers are considered.) Clearly towers are combinatorial maps.

Lemma 2.1. *Let (L, K) be an elementary reduction, and $g: L' \to L$ a tower such that $g(L') \nsubseteq K$. Then there exists an elementary reduction (L', K') which is carried to (L, K) by g.*

Proof. Since $g(L') \nsubseteq K$, it is clear that the 1-cell of $L \backslash K$ lies in $g(L')$, and it follows that $(g(L'), K \cap g(L'))$ is an elementary reduction. Hence there is no loss of generality in assuming g to be surjective. Also, by an inductive argument on the height of g, it is sufficient to consider the height 1 case. Thus we may take g to be $p \circ i$, where $p: \tilde{L} \to L$ is an infinite cyclic covering, and $i: L' \hookrightarrow \tilde{L}$ is an inclusion.

Let c denote the generator of the group of covering transformations of p. Since L' is finite and $p \circ i$ is surjective, we may choose a 1-cell y of L' such that $p(y) \notin K$ and $c^t(y) \notin L'$ for any $t \geqq 1$. If y is involved in the attaching map for at most 1 2-cell (α, say) of L', we may define K' to be $L' \backslash \{y\}$ or $L' \backslash \{y, \alpha\}$, and the result holds.

Suppose then that y is involved in the attaching maps for 2 distinct 2-cells α_1 and α_2 of L'. Then $p(y)$, the 1-cell of $L \backslash K$ is involved in the attaching maps for $p(\alpha_1)$ and $p(\alpha_2)$, which must therefore be 2-cells in $L \backslash K$, and so $p(\alpha_1) = p(\alpha_2)$. Thus there is a covering transformation c^s $(s \neq 0)$ such that $c^s(\alpha_1) = \alpha_2$. Since y is involved in the attaching map for α_1, it follows that $c^s(y)$ is involved in that for α_2, so $c^s(y) \in L'$. Similarly $c^{-s}(y) \in L'$, which contradicts the choice of y.

Define a 2-complex L to be *reducible* if for every finite subcomplex L' either $L' \subset L^{(0)}$ or there exists an elementary reduction (L', K'). Recall that to any group presentation

$$\mathscr{P}: \langle x_1, x_2, \ldots | r_1, r_2, \ldots \rangle$$

is associated a connected 2-complex $K(\mathscr{P})$, with a single 0-cell, a 1-cell for each generator, and a 2-cell for each defining relator – attached along a path which "spells" the relator. Define a presentation $\mathscr{P}$ to be *reducible* if $K(\mathscr{P})$ is reducible. Equivalently, $\mathscr{P}$ is reducible if every finite subpresentation with at least one defining relator has the form

$$\langle x_1, \ldots, x_m, y | r_1, \ldots, r_n, s \rangle,$$

such that the r_i are words in the x_j, and s is not conjugate in $G * \langle y \rangle$ to an element of $G = \langle x_1, \ldots, x_m | r_1, \ldots, r_n \rangle$.

Suppose $\mathscr{P}'$ is the presentation obtained by omitting the defining relator r from the presentation $\mathscr{P}$. Say that r is a *proper power* of $\mathscr{P}$ if it is a proper power in the group G presented by $\mathscr{P}'$. That is there exist a word q in the generators and an integer $m \geqq 2$ such that r and q^m represent the same element of G. We will say that a presentation has *no proper powers* if none of its defining relators is a proper power.

Clearly all 1-relator presentations, and indeed all staggered presentations, are reducible. In fact reducible presentations may be thought of as being "staggered on one side", and so are a natural generalisation of staggered presentations. It turns out that reducible presentations with no proper powers share some of the "nice" properties of staggered presentations, while those with proper powers do not, in general.

A topological space X is *aspherical* if $\pi_n(X)=0$ for each $n\geqq 2$. It is well known that a 2-complex L is aspherical if and only if $\pi_2(L)=0$. A presentation $\mathscr{P}$ is *aspherical* if the corresponding 2-complex $K(\mathscr{P})$ is aspherical. Now any presentation which is aspherical in this sense has no proper powers. There are, however, various slightly different asphericity conditions, for presentations, which allow proper powers (see [4] for a full discussion). Staggered presentations (even those with proper powers) are aspherical under the strongest of these conditions, that of Cohen-Lyndon asphericity. We will show in § 5 that reducible presentations with no proper powers are aspherical in the topological sense, but those with proper powers need not be aspherical in any sense.

Indeed, any group G with a presentation satisfying any of the asphericity conditions of [4] must have quasi-projective dimension (qpd) at most 2 in the sense of [13]. That is, there exists an exact sequence

$$0\to Q\oplus P_2\to P_1\to P_0\to \mathbb{Z}\to 0$$

of $\mathbb{Z}G$-modules, with Q a permutation module, and each P_i projective. Thus the invariant qpd may be used to show that certain presentations are not aspherical in any sense. We give examples of groups of qpd more than 2, with reducible presentations.

3. Preliminary Results

The simplest case of Theorem 4.2 is when r has length 2, that is when G is the free product of A and B with an infinite cyclic subgroup amalgamated. This case is a result of Karrass and Solitar [14]. In particular, it follows that the class of locally indicable groups is closed under the process of "adjoining an n^{th} root" to a non-trivial element. We will make use of this fact in the proof of Proposition 3.3 below, so it is convenient to state the special case of Theorem 4.2 separately.

Proposition 3.1 ([14], Theorem 9). *Let G be an amalgamated free product $G=A\underset{C}{*}B$, where A and B are locally indicable, and C is infinite cyclic. Then G is locally indicable.*

Corollary 3.2. *Let a be a non-trivial element of the locally indicable group A, and let n be a positive integer. Then the group $G=\langle A, t\,|\,t^n=a\rangle$ is locally indicable.*

Proof. Put $B=\langle t\rangle$ and $C=\langle t^n\rangle=\langle a\rangle$ in Proposition 3.1.

The following technical result generalises a result of Weinbaum [24], that any proper subword of the defining relator represents a non-trivial element in a one-relator group.

Proposition 3.3. *Let $(\mathscr{G}, X)$ be a graph of groups with trivial edge groups and locally indicable vertex groups. Let $w_1, \ldots, w_n$ $(n\geqq 2)$ be reduced closed words in $(\mathscr{G}, X)$, not all of length zero, such that the product $w=w_1\ldots w_n$ is defined and is cyclically reduced. Let N denote the normal closure of w in $\pi(\mathscr{G}, X)$. If $w_1N=\ldots=w_nN$, then either*

(i) $w_1 = \ldots = w_n \notin N$; *or*

(ii) *all but one of the* w_i *are the empty word.*

Proof. The proof is by induction on the length λ of w. If $\lambda=1$, then w has the form $ge^{\pm 1}$ or $e^{\pm 1}g$, where $g \in \mathcal{G}(v)$, $v=\iota(e)=\tau(e)$. Let Y be the subgraph $X \setminus \{e\}$ of X, and denote the restriction of $(\mathcal{G}, X)$ to Y by $(\mathcal{G}, Y)$. Then the composite $\pi(\mathcal{G}, Y) \to \pi(\mathcal{G}, X) \to \pi(\mathcal{G}, X)/N$ is an isomorphism. Now $\pi(\mathcal{G}, Y)$ is a free product of the vertex groups $\mathcal{G}(v)$ $(v \in V(x))$ and the free group $\pi_1(X)$, all of which are locally indicable. It is an immediate consequence of the Kuroš subgroup theorem that a free product of locally indicable groups is locally indicable. Thus $\pi(\mathcal{G}, X)/N$ is locally indicable, so torsion-free. Since $(w_i N)^n = wN = N$, it follows that $w_i \in N$ for all i. But all but one of the w_i are words in $(\mathcal{G}, Y)$, and are thus equal to the empty word. Hence (ii) holds.

Suppose then that $\lambda > 1$ and that the result holds for words of length less than λ. Next suppose that w involves two distinct edges. Let f be one of those edges, let X_0 be the smallest subgraph containing f, and let X' be the graph obtained from X by shrinking X_0 to a vertex 0. Define a graph of groups $(\mathcal{G}', X')$ by $\mathcal{G}'(0) = \pi(\mathcal{G}, X_0)$, $\mathcal{G}'(v) = \mathcal{G}(v)$ for $v \in V(X) \setminus V(X_0)$ and $\mathcal{G}'(e) = 1$ for $e \in E(X')$. Then $\pi(\mathcal{G}', X') \cong \pi(\mathcal{G}, x)$, and the words w_i determine reduced closed words w_i' in $(\mathcal{G}', X')$, not all of length 0. Furthermore the product $w' = w_1' \cdot \ldots \cdot w_n'$ is defined, and is cyclically reduced of length $\lambda' < \lambda$. The result holds for w' by inductive hypothesis, and thus holds also for w.

Hence we may assume that precisely one edge, e say, occurs in w. Let X_1 denote the smallest subgraph containing e, let $(\mathcal{G}, X_1)$ denote the restriction of $(\mathcal{G}, X)$ to X_1, and let N_1 be the normal closure of w in $\pi(\mathcal{G}, X_1)$. Then $\pi(\mathcal{G}, X_1)/N_1$ is a free factor of $\pi(\mathcal{G}, X)/N$, so it is sufficient to prove the proposition for w as a word in $(\mathcal{G}, X_1)$. We may therefore assume that $X = X_1$, that is X consists only of the edge e and its initial and terminal vertices (which may coincide). By cyclically permuting the w_i, conjugating all the w_i by some fixed element of some vertex group, and possibly changing the orientation of e, we may assume that w_1 (and hence also w) begins $1 \cdot e \cdot \ldots$. If e has non-zero exponent sum in w, then X has a single vertex, v say, and w has a subword of the form e.g. e or $e^{-1} \cdot g^{-1} \cdot e^{-1}$ for some $g \in \mathcal{G}(v)$. There is an automorphism θ of $\pi(\mathcal{G}, X)$ defined by $\theta = \mathrm{id}$ on $\mathcal{G}(v)$ and $\theta(e) = eg^{-1}$. Then $\theta(w) = \theta(w_1) \cdot \ldots \cdot \theta(w_n)$ is a cyclically reduced word of length λ, beginning $1 \cdot e \cdot \ldots$, and containing a subword of the form $e \cdot 1 \cdot e$ or $e^{-1} \cdot 1 \cdot e^{-1}$. Clearly the result for w will follow from that for $\theta(w)$, so we may also assume that either e has exponent sum zero in w, or w has a subword of the form $e \cdot 1 \cdot e$ or $e^{-1} \cdot 1 \cdot e^{-1}$. Finally, we may assume that the vertex groups of $(\mathcal{G}, X)$ are generated by those elements which occur in some w_i.

Next suppose that some vertex group of $(\mathcal{G}, X)$, say $\mathcal{G}(v)$, vanishes. Then v is the only vertex of X (otherwise no closed reduced word in $(\mathcal{G}, X)$ has positive length). Thus $\pi(\mathcal{G}, X)$ is infinite cyclic, generated by e. Since the w_i and w are all reduced, we have $w_i = e^{\lambda(i)}$ and $w = e^{\lambda}$, where $0 \leqq \lambda(i)$ for all i, $1 \leqq \lambda(1)$, and $\lambda(1) + \ldots + \lambda(n) = \lambda$. Also $\pi(\mathcal{G}, X)/N \cong \mathbb{Z}/\lambda\mathbb{Z}$, so $\lambda(1) \equiv \ldots \equiv \lambda(n) \pmod{\lambda}$. Hence either $\lambda(1) = \ldots = \lambda(n) = \lambda/n \not\equiv 0 \pmod{\lambda}$, or $\lambda(1) = \lambda$ and $\lambda(2) = \ldots = \lambda(n) = 0$, corresponding to the possibilities (i) and (ii) in the statement.

We may therefore assume that no vertex group vanishes. Since the vertex groups are finitely generated and locally indicable, it follows that $H^1(\pi(\mathscr{G},X))$ has rank at least 2, so we may choose an epimorphism $\phi: \pi(\mathscr{G},X) \twoheadrightarrow \mathbb{Z}$ with $\phi(w)=0$. In particular, if $V(X)=\{v\}$ and e has exponent sum 0 in w, then ϕ may be defined by $\phi(\mathscr{G}(v))=0$ and $\phi(e)=1$. We make the convention that this choice of ϕ is used whenever possible. Clearly $\phi(w_i)=0$ for all i, since $\phi(w_i)^n = \phi(w)=0$.

For each vertex v of X, the image of $\mathscr{G}(v)$ under ϕ is a subgroup of $\mathbb{Z}$, so has the form $k\mathbb{Z}$ for some $k \geqq 0$. If $k \geqq 2$, choose $g \in \mathscr{G}(v)$ such that $\phi(g)=k$ and let $\overline{\mathscr{G}}(v)$ denote the group obtained by adjoining a k^{th} root to g. Then $\overline{\mathscr{G}}(v)$ is locally indicable, by Corollary 3.2, and the map ϕ on $\mathscr{G}(v)$ extends uniquely to an epimorphism $\phi: \overline{\mathscr{G}}(v) \twoheadrightarrow \mathbb{Z}$. Replacing each $\mathscr{G}(v)$ by $\overline{\mathscr{G}}(v)$ whenever $k \geqq 2$ gives a new graph of groups $(\overline{\mathscr{G}},X)$, where $\overline{\mathscr{G}}(v)=\mathscr{G}(v)$ if $k=0$ or $k=1$. Let $\bar{N}$ denote the normal closure of w in $\pi(\overline{\mathscr{G}},X)$. It follows from the Freiheitsatz [12] that

$$\pi(\overline{\mathscr{G}},X)/\bar{N} \cong \overline{\mathscr{G}}(u) \underset{\mathscr{G}(u)}{*} (\pi(\mathscr{G},X)/N) \underset{\mathscr{G}(v)}{*} \overline{\mathscr{G}}(v) \quad \text{if } V(X)=\{u,v\},$$

or

$$\pi(\overline{\mathscr{G}},X)/\bar{N} \cong \overline{\mathscr{G}}(v) \underset{\mathscr{G}(v)}{*} \pi(\mathscr{G},X)/N \qquad \text{if } V(x)=\{v\}.$$

Hence it is sufficient to prove the result for w as a word in $(\overline{\mathscr{G}},X)$. Note that the vertex groups of $(\overline{\mathscr{G}},X)$ are not in general generated by those elements which appear in the w_i. However, unless $\phi(\overline{\mathscr{G}}(v))=0$ for all $v \in V(X)$, some w_i contains an element g of some vertex group such that $\phi(g) \neq 0$. That is, either w_i begins or ends with such a g, or has a subword $e^{\mp 1} \cdot g \cdot e^{\mp 1}$. It follows that w also contains such a g. For otherwise $0=\phi(w)=\sigma \cdot \phi(e)$, where σ is the exponent sum of e in w. By our choice of ϕ, we must have $\sigma \neq 0$, so $\phi(e)=0$. Since the only elements with non-zero ϕ-image which can occur in w_j are the first and last, we have $w_j = g_j \cdot e^{\mp 1} \cdot \ldots \cdot e^{\mp 1} \cdot h_j$ with $\phi(g_j) = -\phi(h_j)$. In particular $\phi(g_i) \neq 0$. Since $h_{j-1} g_j$ is a vertex group element appearing in w, we also have $\phi(h_{j-1} g_j)=0$, and so $\phi(g_j) \neq 0$ for all j, contradicting the assumption that w_1 begins $1 \cdot e \ldots$. Thus either $\phi(\overline{\mathscr{G}}(v))=0$ for all v, or w ends $\ldots \cdot e^{\mp 1} \cdot g$ or has a subword $e^{\mp 1} \cdot g \cdot e^{\mp 1}$, with $\phi(g) \neq 0$.

Now let H denote the kernel of $\phi: \pi(\overline{\mathscr{G}},X) \to \mathbb{Z}$. By Bass-Serre Theory [5, 20], $G=\pi(\overline{\mathscr{G}},X)$ acts on a tree T, regularly on the edges of T, with quotient X. Also $H=\pi(\mathscr{H},\tilde{X})$, where $\tilde{X}$ is the quotient of T by the action of H, and $(\mathscr{H},\tilde{X})$ is a graph of groups with trivial edge groups, and vertex groups which are isomorphic to subgroups of those of $(\overline{\mathscr{G}},X)$. The regular action of G on $E(T)$ induces a regular action of $\mathbb{Z} \cong G/H$ on $E(\tilde{X})$, which may be used to label the edges of $\tilde{X}: \ldots, e_{-1}, e_0, e_1, \ldots$. There are 4 possibilities for the isomorphism type of $(\mathscr{H},\tilde{X})$ depending on X and ϕ:

A) $V(X)=\{u,v\}$, $\phi(\overline{\mathscr{G}}(u))=\mathbb{Z}=\phi(\overline{\mathscr{G}}(v))$. Then $V(\tilde{X})=\{u,v\}$, $\mathscr{H}(u)=H \cap \overline{\mathscr{G}}(u)$, $\mathscr{H}(v)=H \cap \overline{\mathscr{G}}(v)$, and each e_i connects u to v.

B) $V(X)=\{u,v\}$, $\phi(\overline{\mathscr{G}}(u))=\mathbb{Z}$, $\phi(\overline{\mathscr{G}}(v))=0$. Then $V(\tilde{X})=\{u; \ldots, v_{-1}, v_0, v_1, \ldots\}$, $\mathscr{H}(u)=H \cap \overline{\mathscr{G}}(u)$, $\mathscr{H}(v_i) \cong \overline{\mathscr{G}}(v)$, and each e_i connects u to v_i.

C) $V(X)=\{v\}$, $\phi(\overline{\mathscr{G}}(v))=\mathbb{Z}$. Then $V(\tilde{X})=\{v\}$ and $\mathscr{H}(v)=H \cap \overline{\mathscr{G}}(v)$.

D) $V(X)=\{v\}$, $\phi(\overline{\mathscr{G}}(v))=0$. Then $V(\tilde{X})=\{\ldots, v_{-1}, v_0, v_1, \ldots\}$, $\mathscr{H}(v_i) \cong \overline{\mathscr{G}}(v)$, and each e_i connects v_i to v_{i+1}.

The elements of G represented by the w_i all belong to H, and so can be represented by reduced closed words $\tilde{w}_i$ in $(\mathcal{H}, \tilde{X})$, such that $\tilde{w}_1$ begins $1 \cdot \tilde{e}_0 \cdot \ldots$, and such that the product $\tilde{w} = \tilde{w}_1 \cdot \ldots \cdot \tilde{w}_n$ is defined. Since w is cyclically reduced, so is $\tilde{w}$. Also $\tilde{w}$ has length λ.

Let s denote the least index such that e_s occurs in $\tilde{w}$, and let t denote the greatest such index. Let X_0 denote the smallest subgraph of $\tilde{X}$ containing the edges e_i $(s \leqq i \leqq t)$. Let X_1 be obtained from X_0 by omitting e_t (together with v_t in case B or v_{t+1} in case D), and let X_2 be obtained from X_0 by omitting e_s (together with v_s in cases B and D). Define $G_i = \pi(\mathcal{H}, X_i)$ $(i = 0, 1, 2)$, where $(\mathcal{H}, X_i)$ is the restriction of $(\mathcal{H}, \tilde{X})$, and let N_0 be the normal closure of $\tilde{w}$ in G_0. It is clear that X_1 and X_2 are connected, except possibly if $s = t$ in case A, when $X_1 = X_2$ consists of 2 vertices and no edges. However, we will see below that this possibility cannot arise: $\tilde{w}$ involves more than one edge in case A.

Now $\bar{G} = \pi(\bar{\mathcal{G}}, X)$ can be expressed as an HNN-extension of G_0, with associated subgroups G_1 and G_2, via the canonical embedding $\pi(\mathcal{H}, X_0) \to \pi(\mathcal{H}, \tilde{X}) \to \pi(\bar{\mathcal{G}}, X)$. In cases A and B, the stable letter may be taken to be an element p of $\bar{\mathcal{G}}(u)$ such that $\phi(p)$ is a generator of $\mathbb{Z}$. In case C it may be taken to be a similar element of $\bar{\mathcal{G}}(v)$, and in case D it may be taken to be the element e.

Also, the maps $G_i \to G_0 \to G_0/N_0$ $(i = 1, 2)$ are injective, by the Freiheitsatz [12], and hence $\bar{G}/\bar{N}$ can be expressed as an *HNN*-extension of G_0/N_0, also with associated subgroups G_1 and G_2. It follows that it is enough to prove the result for $\tilde{w}$, regarded as a word in $(\mathcal{H}, X_0)$.

If $s < t$, that is if $\tilde{w}$ involves more than one edge, then the result holds for $\tilde{w}$, and so for w. In particular this is true in cases A and B, for then w has a (cyclic) subword of the form $e \cdot g \cdot e^{-1}$ or $e^{-1} \cdot g \cdot e$ for some vertex group element g with $\phi(g) = k \neq 0$, and the corresponding cyclic subword of $\tilde{w}$ has the form $e_i \cdot h \cdot e_{i+k}^{-1}$ or $e_i^{-1} \cdot h \cdot e_{i+k}$ for some integer i and some element h of some vertex group of $(\mathcal{H}, X_0)$.

We also have $s < t$ in case C, for then $\phi(\bar{\mathcal{G}}(v)) \neq 0$, so e has non-zero exponent sum in w, by choice of ϕ. Then w has subwords

(a) $e \cdot 1 \cdot e$ or $e^{-1} \cdot 1 \cdot e^{-1}$.

(b) $e^{\pm 1} \cdot g \cdot e^{\pm 1}$ with $\phi(g) = k \neq 0$.

If $\phi(e) = l \neq 0$, then corresponding to (a) is a subword of $\tilde{w}$ of the form $e_i \cdot 1 \cdot e_{i+l}$ or $e_i^{-1} \cdot 1 \cdot e_{i-l}^{-1}$ for some i. If $\phi(e) = 0$, then corresponding to (b) is a subword of $\tilde{w}$ of the form $e_i^{\pm 1} \cdot h \cdot e_{i+k}^{\pm 1}$ for some i and some h.

Thus the only case in which $\tilde{w}$ can involve only one edge is case D, in which case X_0 is a graph with 1 edge and 2 vertices. Repeating the whole argument with w replaced by $\tilde{w}$ and $(\mathcal{G}, X)$ replaced by $(\mathcal{H}, X_0)$, the result holds for $\tilde{w}$ because we arrive in case A or B.

This completes the proof of Proposition 3.3.

Corollary 3.4. *Let $(\mathcal{G}, X)$ be a graph of groups as in Proposition* 3.3. *Let N be the normal closure in $\pi(\mathcal{G}, X)$ of a cyclically reduced closed word w in $(\mathcal{G}, X)$. Then no proper closed subword of w represents an element of N.*

Proof. Suppose $\bar{w} = w_1 \cdot w_2$ is a cyclically reduced conjugate of w, and w_1 represents an element of N. Then $w_2 N = (w_1 N)^{-1} (\bar{w} N) = N$, so $w_2 N = w_1 N$. By

Proposition 3.3, one of w_1, w_2 is the empty word, so w_1 is not a proper closed subword of w.

Corollary 3.5. *Let $(\mathcal{G}, X)$ be a graph of groups as in Proposition 3.3, and let N be the normal closure in $\pi(\mathcal{G}, X)$ of $w=u^m$, where u is a cyclically reduced closed word of positive length. Then uN is an element of order precisely m in $\pi(\mathcal{G}, X)/N$.*

Proof. Clearly $(uN)^m=wN=N$, so uN has order at most m. But for $k<m$, u^k is a proper closed subword of w, so $u^k \notin N$ by Corollary 3.4.

Corollary 3.6. *Let (L, K) be an elementary reduction, such that each component of K has locally indicable fundamental group. Suppose that $L \backslash K$ has a 2-cell, attached along a cyclically reduced closed path P in $L^{(1)}$, and let w denote the closed word defined by P in the graph of groups corresponding to (L, K) (see § 2). Suppose also that w is cyclically reduced and not a proper power. Let (L', K') be an elementary reduction with a 2-cell in $L' \backslash K'$, and let $g: L' \to L$ be a tower carrying (L', K') to (L, K), such that $g_*(\pi_1(K''))=1$ in $\pi_1(L)$ for some component K'' of K'. Then $K'=K''$ and (L', K') is an elementary collapse.*

Proof. Let $\tilde{e}$ denote the 1-cell of $L' \backslash K'$, and let e denote both the 1-cell $g(\tilde{e})$ of $L \backslash K$ and the unique edge of the graph X. The orientation of e as an edge of X induces an orientation of $\tilde{e}$. Since g is a tower, the 2-cell of $L' \backslash K'$ is attached along a cyclically reduced closed path $\tilde{P}$ in $L'^{(1)}$ such that $g(\tilde{P})=P$.

Suppose both $\tilde{e}$ and $\tilde{e}^{-1}$ occur in the path $\tilde{P}$. Then there exist adjacent occurrences of $\tilde{e}$ and $\tilde{e}^{-1}$ separated by a reduced closed path $\tilde{Q}$ in $K''^{(1)}$. In particular this holds if K' is disconnected. Thus $\tilde{P}$ has a subpath $\tilde{e} \cdot \tilde{Q} \cdot \tilde{e}^{-1}$ or $\tilde{e}^{-1} \cdot \tilde{Q} \cdot \tilde{e}$, so P has a subpath $e \cdot Q \cdot e^{-1}$ or $e^{-1} \cdot Q \cdot e$, where $Q=g(\tilde{Q})$. If e occurs in Q, then Q represents a proper subword w_1 of w, which lies in the normal closure of w in $\pi(\mathcal{G}, X)=\pi_1(K \cup L^{(1)})$, since Q is nullhomotopic in L. This contradicts Corollary 3.4. If e does not occur in Q, then Q is a path in $K^{(1)}$ which is nullhomotopic in L. By the Freiheitsatz [12], Q is nullhomotopic in K, so w has a subword $e \cdot 1 \cdot e^{-1}$ or $e^{-1} \cdot 1 \cdot e$. This contradicts the assumption that w is cyclically reduced.

Hence $\tilde{e}$ and $\tilde{e}^{-1}$ cannot both occur in $\tilde{P}$. Without loss of generality we may assume only $\tilde{e}$ occurs. In particular, $K'=K''$ is connected, and $\tilde{P} = \tilde{e} \cdot \tilde{Q}_1 \cdot \tilde{e} \cdot \ldots \cdot \tilde{e} \cdot \tilde{Q}_n$ up to cyclic permutation, where $\tilde{Q}_1, \ldots, \tilde{Q}_n$ are reduced paths in K''. Let $Q_i=g(\tilde{Q}_i)$ $(i=1, \ldots, n)$ and let w_i be the reduced closed word in $(\mathcal{G}, X)$ spelt out by the path $e \cdot Q_i$. Since $Q_i^{-1} \cdot Q_j=g(\tilde{Q}_i^{-1} \cdot \tilde{Q}_j)$ is the image in L of a closed path in K'', it is nullhomotopic in L, so $w_1 N=\ldots=w_n N$, where N is the normal closure of w in $\pi(\mathcal{G}, X)$. If $n \geqq 2$, we can apply Proposition 3.3 to get $w_1=\ldots=w_n$, contradicting the hypothesis that w is not a proper power. Hence $n=1$, and (L', K') is an elementary collapse, as claimed.

4. One-Relator Products

In this section we prove the main result, Theorem 4.2. First it is necessary to prove the following Lemma. The proof uses arguments similar to those of [11].

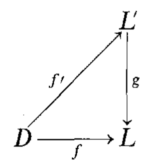
Lemma 4.1. *Let L be a connected 2-complex such that $\pi_1(L)$ is not locally indicable. Then there exists a tower $g: L' \to L$ such that*

(a) $H^1(L')=0$; *and*
(b) $g_*(\pi_1(L')) \neq 1$ *in* $\pi_1(L)$.

Proof. Since $\pi_1(L)$ is not locally indicable, there is a finitely generated subgroup $G \neq 1$ of $\pi_1(L)$ such that G^{ab} is finite, of order m say. Since the inclusion-induced map $\eta: \pi_1(L^{(1)}) \to \pi_1(L)$ is surjective, there exists a finite subset $\{b_1, \ldots, b_n\}$ of $\pi_1(L^{(1)})$ such that $\{\eta(b_1), \ldots, \eta(b_n)\}$ generates G. Let H denote the subgroup of $\pi_1(L^{(1)})$ generated by $\{b_1, \ldots, b_n\}$.

For each i, we have $\eta(b_i^{-m}) \in [G, G] = \eta([H, H])$, so there exists $w_i \in [H, H]$ such that $\eta(b_i^{-m}) = \eta(w_i)$, that is $\eta(b_i^m w_i) = 1$. Now there exists a planar, 1-connected 2-complex D_i, and a combinatorial map $f_i: D_i \to L$, such that f_i maps the boundary path ∂D_i of D_i in the plane to a path in $L^{(1)}$ representing $b_i^m w_i$. (Indeed D_i may be taken as a wedge of complexes, each consisting of a disc with subdivided boundary, connected to the basepoint by a subdivided arc.) In particular ∂D_i is a composite $P_{i1} \cdot \ldots \cdot P_{im} \cdot Q_i$ of paths such that each $f_i(P_{ij})$ is a closed path representing b_i, and $f_i(Q_i)$ is a closed path representing w_i.

Let D denote the wedge $\bigvee_i D_i$ of the various D_i, and let $f = \bigvee_i f_i: D \to L$ be the induced map. Let



be a maximal tower-lifting of f [12, Lemma 3.1]. Now D is simply connected, so $H^1(D)=0$ and hence $H^1(L')=0$ (otherwise f' lifts over some infinite cyclic cover of L', contradicting maximality). Hence (a) holds.

To prove (b), suppose $\operatorname{Im} f \subseteq K \subseteq L$, and $\tilde{f}: D \to \tilde{L}$ is a lift of f over an infinite cyclic cover $p: \tilde{L} \to K$. Then $\tilde{f}(P_{ij})$ is a closed path in $\tilde{L}^{(1)}$, representing an element $\tilde{b}_i$ of $\pi_1(\tilde{L}^{(1)})$ (which depends only on i, not on j). Also each $\tilde{f}(Q_i)$ is a closed path representing some product of commutators of the $\tilde{b}_i$. Repeating this argument, we see that $f'(P_{ij})$ is a closed path in $L'^{(1)}$, representing an element b_i' (depending only on i). Clearly $g_*(b_i') = b_i$, so $1 \neq G \subseteq g_*(\pi_1(L')) \subseteq \pi_1(L)$.

Theorem 4.2. *Let A and B be locally indicable groups, and let G be the quotient of $A * B$ by the normal closure of a cyclically reduced word r of length at least 2. Then the following are equivalent:*

(i) *G is locally indicable;*
(ii) *G is torsion free;*
(iii) *r is not a proper power in $A * B$.*

Proof. That (i)$\Rightarrow$(ii) is immediate from the definition of locally indicable, and that (ii)$\Rightarrow$(iii) is Corollary 3.5, so it remains to prove (iii)$\Rightarrow$(i). Suppose then that r is not a proper power, and G is not locally indicable.

Choose presentations $\mathscr{P}_A, \mathscr{P}_B$ for A, B respectively, and let K be the disjoint union $K(\mathscr{P}_A) \cup K(\mathscr{P}_B)$. Attach a 1-cell e to K, connecting the two 0-cells of K. Then $\pi_1(K \cup \{e\}) \cong A * B \cong \pi(\mathscr{G}, X)$, where X is the tree with one edge and $(\mathscr{G}, X)$ has vertex groups A, B, and trivial edge group. The cyclically reduced word r of $A * B$ corresponds to a cyclically reduced closed word w in $(\mathscr{G}, X)$, and w can be represented by a cyclically reduced closed path P in $K^{(1)} \cup \{e\}$. Form a 2-complex L by attaching a 2-cell to $K \cup \{e\}$ along P. Then (L, K) is an elementary reduction, and $\pi_1(L) \cong G$.

Since G is not locally indicable, Lemma 4.1 applies, and there is a tower $g: L' \to L$ such that

(a) $H^1(L') = 0$; and

(b) $g_*(\pi_1(L')) \neq 1$ in $\pi_1(L)$.

Since L' is finite, there is no loss of generality in assuming it to be minimal with respect to (a) and (b), that is no proper connected subcomplex also satisfies these properties. Since each component of K has locally indicable fundamental group, it follows from (a) and (b) that $g(L') \nsubseteq K$. Hence, by Lemma 2.1, there is an elementary reduction (L', K') which is carried to (L, K) by g.

Suppose first that $L' \backslash K'$ has no 2-cell. Then by (a) it follows that K' is disconnected, say $K' = K'_1 \cup K'_2$, and so $H^1(K'_i) = 0$ for each $i = 1, 2$. By minimality of L', we must have $g_*(\pi_1(K_i)) = 1$ in $\pi_1(L)$, otherwise K'_i also satisfies (a) and (b). But $\pi_1(L') = \pi_1(K'_1) * \pi_1(K'_2)$, so $g_*(\pi_1(L')) = 1$, contradicting (b).

Now suppose $L' \backslash K'$ has a 2-cell. Then at least one component K'' of K' has $H^1(K'') = 0$. By minimality of L' we must have $g_*(\pi_1(K'')) = 1$ in $\pi_1(L)$. Hence by Corollary 3.6 (since w is not a proper power) it follows that $K' = K''$ is connected, and (L', K') is an elementary collapse. In particular, $\pi_1(K') \to \pi_1(L')$ is an isomorphism, so $g_*(\pi_1(L')) = g_*(\pi_1(K')) = 1$, again contradicting (b).

In either case, we have obtained a contradiction. Hence (iii)⇒(i), and the Theorem is proved.

Corollary 4.3 (Brodskiĭ [2]). *All torsion-free one-relator groups are locally indicable.*

Corollary 4.4 (Brodskiĭ [2]). *One-relator groups have no non-trivial finitely generated perfect subgroups.*

Proof. If G is a one-relator group, then there is an exact sequence

$$1 \to N \to G \to H \to 1,$$

where N is the normal closure in G of the root of the defining relator, and H is a torsion free one-relator group. Now N is a free product of finite cyclic groups [6], and so has no non-trivial finitely generated perfect subgroups. But neither has H, by Corollary 4.3, and hence neither has G.

Corollary 4.5. *Suppose the group G has a reducible presentation with no proper powers. Then G is locally indicable.*

Proof. Let $\mathscr{P}$ be a such a presentation. Since G is the direct limit of the groups presented by finite subpresentations of $\mathscr{P}$, it is sufficient to consider the case when $\mathscr{P}$ is finite. The proof is by induction on the number of defining relators in $\mathscr{P}$. In the initial case there are no defining relators, so G is free. The inductive step is provided by Theorem 4.2 with B free.

The following consequence of Theorem 4.2 was pointed out to me by S.D. Brodskiĭ. Let us call a group G *equationally closed* if every nontrivial equation over G has a solution in G. That is, if F is a free group and N is the normal closure in $G*F$ of a word w not belonging to any conjugate of G, then the canonical map $G \to (G*F)/N$ is a split injection.

Corollary 4.6. *Every locally indicable group can be embedded in a locally indicable, equationally closed group.*

Proof. Let G be locally indicable, and let F be a fixed free group of countably infinite rank. Any element $w \in G*F$ not in any conjugate of G can be uniquely expressed $w=u^m$ where $m \geqq 1$ is an integer, and u is not a proper power. By Theorem 4.2 and the Freiheitsatz [12], G embeds in a group $G(u)$ in which the equation $u=1$ (and so also $w=1$) has a solution. If G' is the free product of all such $G(u)$, amalgamated over G, then G' is locally indicable, by an inductive argument using Theorem 4.2 and the Freiheitsatz. Also every nontrivial equation over G has a solution in G'. Define $G_0=G$, and inductively $G_i=G'_{i-1}$ for $i \geqq 1$. Then G embeds in the locally indicable, equationally closed group $G'' = \bigcup_i G_i$.

5. Asphericity

We begin this section with a result similar to Lemma 4.1.

Lemma 5.1. *Let L be a connected, non-aspherical 2-complex. Then there exists a tower $g: L' \to L$ such that*

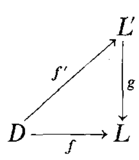
(a) $H^1(L')=0$; *and*

(b) $g_*(\pi_2(L')) \neq 0$ *in* $\pi_2(L)$.

Proof. Since L is non-aspherical, $\pi_2(L) \neq 0$.

Choose $\gamma \neq 0$ in $\pi_2(L) \subset \pi_2(L, L^{(1)})$. Then there exists a 1-connected planar 2-complex D, an element $\alpha \in \pi_2(D, D^{(1)})$, and a combinatorial map $f: D \to K$ such that $f_*(\alpha)=\gamma$. (As in Lemma 4.1, take D to be a wedge of complexes, each consisting of a disc with subdivided boundary, connected to the base point by a subdivided arc.)

Let



be a maximal tower lifting of f ([12], Lemma 3.1), and define $\beta = f'_*(\alpha) \in \pi_2(L', L'^{(1)})$.

Let $\partial: \pi_2(L', L'^{(1)}) \to \pi_1(L'^{(1)})$ denote the boundary map. Then $g_*(\partial\beta)=1$ in $\pi_1(L^{(1)})$. But $g_*: \pi_1(L'^{(1)}) \to \pi_1(L^{(1)})$ is injective, because g is a tower. Hence $\beta \in \operatorname{Ker}\partial = \pi_2(L')$, and so $0 \neq \gamma = g_*(\beta) \in g_*(\pi_2(L'))$ in $\pi_2(L)$, showing (b).

That (a) holds is a consequence of the maximality of f', since $H^1(D)=0$.

Theorem 5.2. *Let L be a connected 2-complex with $H_2(L)=0$ and $\pi_1(L)$ locally indicable. Then L is aspherical.*

Proof. Suppose L is not aspherical. Then by Lemma 5.1 there is a tower $g: L' \to L$ such that $H^1(L')=0$ and $g_*(\pi_2(L')) \neq 0$ in $\pi_2(L)$.

Since $\pi_1(L)$ is locally indicable, $g_*(\pi_1(L'))=1$ in $\pi_1(L)$, so g lifts over the universal cover $q: \tilde{L} \to L$,

$$\begin{array}{ccc} & & \tilde{L} \\ & \overset{\tilde{g}}{\nearrow} & \downarrow{\scriptstyle q} \\ L' & \xrightarrow[g]{} & L. \end{array}$$

Since $H_2(L)=0$, it follows from [12], Lemma 3.2 that $H_2(L')=0$.

From the commutative diagram

$$\begin{array}{ccc} \pi_2(L') & \xrightarrow{h} & H_2(L')=0 \\ \downarrow{\scriptstyle \tilde{g}_*} & & \downarrow{\scriptstyle \tilde{g}_*} \\ \pi_2(\tilde{L}) & \xrightarrow[\cong]{h} & H_2(\tilde{L}) \end{array}$$

it follows that $\tilde{g}_*(\pi_2(L'))=0$ in $\pi_2(\tilde{L})$, so $g_*(\pi_2(L'))=0$ in $\pi_2(L)$, a contradiction.

Hence L is aspherical.

Theorem 5.3. *Suppose (L, K) is a simple reduction with a 2-cell whose attaching map does not represent a proper power in $\pi_1(K \cup L^{(1)})$. Suppose also that each component of K is aspherical with locally indicable fundamental group. Then L is aspherical.*

Proof. Let $(\mathcal{G}, X)$ denote the graph of groups corresponding to the reduction (L, K) (see §2). Up to homotopy, we may assume the 2-cell of $L \setminus K$ is attached along a cyclically reduced path in $L^{(1)}$ which spells out a cyclically reduced closed word w in $(\mathcal{G}, X)$. Clearly w is not a proper power.

Suppose L is not aspherical. Then by Lemma 5.1 there is a tower $g: L' \to L$ such that $H^1(L')=0$ and $g_*(\pi_2(L')) \neq 0$ in $\pi_2(L)$. Since L' is finite, we may assume without loss of generality that L' is minimal with respect to these properties, that is there is no proper connected subcomplex K' of L' such that $H^1(K')=0$ and $g_*(\pi_2(K')) \neq 0$ in $\pi_2(L)$. From this assumption we derive a contradiction.

Since K is aspherical and $g_*(\pi_2(L')) \neq 0$, it follows that $g(L') \nsubseteq K$. Hence, by Lemma, by Lemma 2.1 there is an elementary reduction (L', K') which is carried to (L, K) by g.

First suppose that $L'\backslash K'$ has a 2-cell. Since $H^1(L')=0$, it follows that $H^1(K'')=0$ for some component K'' of K'. Since $\pi_1(L)$ is locally indicable, by Theorem 4.2, it follows that $g_*(\pi_1(K''))=1$ in $\pi_1(L)$. Hence, by Corollary 3.6, $K''=K'$ and (L', K') is an elementary collapse. Thus $H^1(K')=0$ and $g_*(\pi_2(K'))=g_*(\pi_2(L'))\neq 0$, contradicting the minimality of L'.

Now suppose that $L'\backslash K'$ has no 2-cell. Since $H^1(L')=0$, it follows that K' has 2 components, say K'_1 and K'_2, and that $H^1(K'_1)=0=H^1(K'_2)$. Also $\pi_2(L')$ is generated as a $\pi_1(L')$-module by the images of $\pi_2(K'_1)$ and $\pi_2(K'_2)$. Since $g_*(\pi_2(L'))\neq 0$, at least one of $g_*(\pi_2(K'_1))$, $g_*(\pi_2(K'_2))$ is non-zero, contradicting the minimality of L'.

In either case we have obtained a contradiction, so L is aspherical.

Corollary 5.4. *A reducible presentation with no proper powers is aspherical.*

Proof. It is enough to prove the result for finite presentations, and this is done by induction on the number of defining relators. The inductive step is provided by Theorem 5.3 and Corollary 4.5.

The following examples show that the condition on proper powers in Corollary 5.4 is essential for any form of asphericity to hold.

Example 1. $G=SL_2(\mathbb{Z})$ has a reducible presentation $\langle x, y \mid x^4, x^{-2}y^3\rangle$, but $\operatorname{qpd} G=\infty$.

Example 2. $G=\mathbb{Z}\times\mathbb{Z}/m$ has a reducible presentation $\langle x, y \mid x^m, [x, y]\rangle$, but $\operatorname{qpd} G=\infty$.

Example 3. $G=S_3$, the symmetric group on 3 letters, has a reducible presentation $\langle x, y \mid x^2, xyxy^{-2}\rangle$, but $\operatorname{qpd} G=4$.

Example 4. $G=\mathbb{Z}$ has a presentation $\langle x \mid - \rangle$ which is aspherical in any sense, but also a reducible presentation $\mathscr{P}$: $\langle x, y, z \mid x^2, xyxy^{-2}, xzyz^{-1}\rangle$ which is not aspherical in any sense: $K(\mathscr{P})$ has the homotopy type of $S^1 \vee S^2$.

Note also that Chiswell [4] and Sieradski [21] have constructed examples of aspherical presentations of the trivial group which are not *diagrammatically aspherical* (aspherical in the sense of Lyndon and Schupp [17]). Chiswell's example and one of Sieradski's are reducible presentations (with no proper powers).

6. 3-manifold Groups

The following shows that a large class of 3-manifolds have locally indicable fundamental groups.

Theorem 6.1. *Let M be a connected, orientable 3-manifold with $\pi_2(M)=0$. Let G be a non-trivial finitely generated subgroup of $\pi_1(M)$ with $H^1(G)=0$. Then M is a rational homology 3-sphere (in particular M is closed), and G has finite index in $\pi_1(M)$.*

Proof. Let $\bar{M}$ denote the covering of M corresponding to the subgroup G of $\pi_1(M)$. Since $G=\pi_1(\bar{M})$ is finitely generated, it follows [19] that $\bar{M}$ contains a compact submanifold N such that the inclusion-induced map $\pi_1(N) \to \pi_1(\bar{M})$ is an isomorphism. Since $H^1(N) = H^1(G) = 0$, it follows from Poincaré duality that $H^1(\partial N) = 0$, so ∂N consists of a finite number $S_1, \ldots, S_n$ of 2-spheres.

Since $\pi_2(\bar{M}) = \pi_2(M) = 0$, each S_i separates $\bar{M}$ into 2 components, one of which is a (possibly fake) 3-cell D_i. Now $N \nsubseteq D_i$, for otherwise the isomorphism $\pi_1(N) \to \pi_1(\bar{M}) = G$ would factor through $\pi_1(D_i) = 1$, contradicting the hypothesis $G \neq 1$. Hence $\bar{M} = N \cup D_1 \cup \ldots \cup D_n$.

In particular $\bar{M}$ is closed, so M is closed, and G has finite index in $\pi_1(M)$. Since $H^1(G) = 0$, it follows that $H^1(M) = 0$, so $H^1(M; \mathbb{Q}) = 0$, and M is a rational homology 3-sphere, as claimed.

Corollary 6.2. *Let M be a compact orientable 3-manifold. Then either $\pi_1(M)$ is locally indicable, or M has a connected summand which is a rational homology 3-sphere but not a homotopy 3-sphere.*

Proof. Let $M = M_1 \# \ldots \# M_n$ be a prime factorisation of M [18], so that $\pi_1(M) = \pi_1(M_1) * \ldots * \pi_1(M_n)$. If $\pi_1(M)$ is not locally indicable, then some $\pi_1(M_i)$ is not locally indicable. In particular, $\pi_1(M_i) \ncong \mathbb{Z}$, so $\pi_2(M_i) = 0$. Hence M_i is a rational homology 3-sphere, by the Theorem.

The following is an immediate consequence of Corollary 6.2.

Corollary 6.3. *Let ℓ be a tame link in S^3. Then $\pi_1(S^3 \setminus \ell)$ is locally indicable.*

References

1. Baumslag, G.: Some problems on one-relator groups. Proceedings of the Second International Conference on the Theory of Groups (Canberra 1973), pp. 75–81. Lecture Notes in Mathematics **372**. Berlin-Heidelberg-New York: Springer 1974
2. Brodskiĭ, S.D.: Equations over groups and groups with a single defining relator. Uspehi Mat. Nauk **35**, 4, 183 (1980) [Russian]
3. Brown, R., Huebschmann, J.: Identities among relations. In: Low Dimensional Topology vol. 1 (R. Brown and T.L. Thickstun, eds.). London Mathematical Society Lecture Note Series **48**, pp 153–202, Cambridge University Press 1982
4. Chiswell, I.M., Collins, D.J., Huebschmann, J.: Aspherical group presentations. Math. Z. **178**, 1–36 (1981)
5. Dicks, W.: Groups, trees and projective modules. Lecture Notes in Mathematics **790**. Berlin-Heidelberg-New York: Springer 1980
6. Fischer, J., Karrass, A., Solitar, D.: On one-relator groups having elements of finite order. Proc. Amer. Math. Soc. **33**, 297–301 (1972)
7. Gutierrez, M.A., Ratcliffe, J.G.: On the second homotopy group. Quart. J. Math. Oxford (2) **32**, 45–55 (1981)
8. Higgins, P.J.: The fundamental groupoid of a graph of groups. J. London Math. Soc. (2) **13**, 145–149 (1976)
9. Higman, G.: The units of groups rings. Proc. London Math. Soc. (2) **46**, 231–248 (1940)
10. Howie, J.: Aspherical and acyclic 2-complexes. J. London Math. Soc. (2) **20**, 549–558 (1979)
11. Howie, J.: On the fundamental group of an almost-acyclic 2-complex. Proc. Edinburgh Math. Soc. **24**, 119–122 (1981)
12. Howie, J.: On pairs of 2-complexes and systems of equations over groups. J. Reine Angew. Math. **324**, 165–174 (1981)
13. Howie, J., Schneebeli, H.R.: Groups of finite quasi-projective dimension. Comment. Math. Helv. **54**, 615–628 (1979)

14. Karrass, A., Solitar, D.: The subgroups of a free product of two groups with an amalgamated subgroup. Trans. Amer. Math. Soc. **149**, 227–255 (1970)
15. Kirby, R.: Problems in low-dimensional manifold theory. In: Algebraic and Geometric Topology (R.J. Milgram, ed.). Proceedings (Stanford 1976), pp. 273–312. Proceedings of Symposia in Pure Mathematics **32**, part 2. Providence, Rhode Island: American Mathematical Society 1978
16. Lewin, J., Lewin, T.: An embedding of the group algebra of a torsion-free 1-relator group in a field. J. Algebra **52**, 39–74 (1978)
17. Lyndon, R.C., Schupp, P.E.: Combinatorial Group Theory. Ergebnisse der Mathematik und ihrer Grenzgebiete **89**. Berlin-Heidelberg-New York: Springer 1977
18. Milnor, J.: A unique factorisation theorem for 3-manifolds. Amer. J. Math. **84**, 1–7 (1962)
19. Scott, G.P.: Compact submanifolds of 3-manifolds. J. London Math. Soc. (2) **7**, 246–250 (1973)
20. Serre, J.-P.: Arbres, amalgames, SL_2. Astérisque **46**. Paris: Société Mathematique de France 1977
21. Sieradski, A.J.: Framed links for Peiffer identities. Math. Z. **175**, 125–137 (1980)
22. Short, H.: Topological methods in group theory: the adjunction problem. Ph. D. Thesis, University of Warwick 1981
23. Short, H.: Knot groups are locally indicable. Preprint
24. Weinbaum, C.M.: On relators and diagrams for groups with a single defining relator. Illinois J. Math. **16**, 308–322 (1972)

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